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Hyperreflexivity of finite-dimensional subspaces

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Abstract

We show that each reflexive finite-dimensional subspace of operators is hyperreflexive. This gives a positive answer to a problem of Kraus and Larson. We also show that each n -dimensional subspace of Hilbert space operators is $[\sqrt{2n}]$ -hyperreflexive.

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1. Introduction

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X . For an algebra $\mathcal{W} \subset B(X)$ with identity, let $\text{Alg Lat } \mathcal{W}$ denote the set of all operators which leave invariant all (closed) subspaces of X , which are invariant for all operators from \mathcal{W} . The algebra \mathcal{W} is called *reflexive* if $\mathcal{W} = \text{Alg Lat } \mathcal{W}$.

The definition was introduced for the first time in [16] and further studied by a number of authors. The concept of reflexivity is interesting even if the underlying

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space is finite dimensional. For example, the algebra $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus [a] : a, b \in \mathbb{C} \right\}$ is reflexive, but the algebra $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}$ is not reflexive (the former example will be used later).

The definition of reflexivity was extended to subspaces of operators in [13]. Let X, Y be Banach spaces and let \mathcal{M} be a norm-closed subspace of $B(X, Y)$ —the space of all bounded linear operators from X into Y . Write

$$\text{ref } \mathcal{M} = \{T \in B(X, Y) : Tx \in \overline{\mathcal{M}x} \text{ for all } x \in X\},$$

where $\mathcal{M}x = \{Sx : S \in \mathcal{M}\}$. The subspace \mathcal{M} is called *reflexive* if $\mathcal{M} = \text{ref } \mathcal{M}$. For algebras with identity both definition coincide.

A stronger concept of hyperreflexivity was introduced for algebras in [1] and extended for subspaces of operators in [10]. Denote by $\text{dist}(\cdot, \cdot)$ the usual distance in Y ; we use also the same notation for the distance in $B(X, Y)$. Let $\mathcal{M} \subset B(X, Y)$ be a norm-closed subspace and $T \in B(X, Y)$. Write

$$\alpha(T, \mathcal{M}) = \sup\{\text{dist}(Tx, \mathcal{M}x) : x \in X, \|x\| = 1\}. \quad (1)$$

We always have $\alpha(T, \mathcal{M}) \leq \text{dist}(T, \mathcal{M})$. The subspace \mathcal{M} is called *hyperreflexive* if there is a constant $C > 0$ such that for all $T \in B(X, Y)$, we have

$$\text{dist}(T, \mathcal{M}) \leq C\alpha(T, \mathcal{M}). \quad (2)$$

The smallest constant C fulfilling (1) is called the *hyperreflexive constant* and denoted by $\kappa_{\mathcal{M}}$.

Let us observe that if \mathcal{M} is reflexive and $T \in \text{ref } \mathcal{M}$, then $\alpha(T, \mathcal{M}) = 0$. Hence $\text{dist}(T, \mathcal{M}) = 0$ and, since \mathcal{M} is norm closed, we have $T \in \mathcal{M}$. Thus each hyperreflexive subspace is also reflexive. On the other hand there are reflexive non-hyperreflexive subspaces (see [9]). However, if both spaces X and Y are finite dimensional then each reflexive subspace is also hyperreflexive. Namely, as we have observed above the reflexivity of a norm-closed subspace \mathcal{M} is equivalent to the condition:

$$\alpha(T, \mathcal{M}) = 0 \iff \text{dist}(T, \mathcal{M}) = 0.$$

Thus, for the whole conclusion, it is enough to note that all norms on the finite-dimensional space $B(X, Y)/\mathcal{M}$ are equivalent.

In [10, Problem 3.9], Kraus and Larson posed the question whether the concepts of reflexivity and hyperreflexivity are equivalent for finite-dimensional subspaces of operators on infinite-dimensional spaces. The problem was considered also in [6].

In [10] it was shown that each one-dimensional subspace is hyperreflexive. By [14], the hyperreflexive constant is equal to 1.

The aim of this paper is to give a positive answer to the problem of Kraus and Larson. The main result of the paper is

Main Theorem. *Let $\mathcal{M} \subset B(X, Y)$ with $\dim \mathcal{M} < \infty$. If \mathcal{M} is reflexive, then \mathcal{M} is hyperreflexive.*

In [12] it was shown that each n -dimensional subspace of Hilbert space operators is $[\sqrt{2n}]$ -reflexive, where $[\sqrt{2n}]$ is the integer part of $\sqrt{2n}$. Using our main result we show in Section 3 that each n -dimensional subspace is even $[\sqrt{2n}]$ -hyperreflexive (for definitions see Section 3).

Remark. Many authors (including [10]) considered the reflexivity and hyperreflexivity only for subspaces of operators on a Hilbert space. They use a different definition of the distance $\alpha(T, \mathcal{M})$:

$$\alpha(T, \mathcal{M}) = \sup\{\|QTP\| : P, Q \text{ are projections and } Q\mathcal{M}P = 0\}.$$

To see the equivalence of both definitions of the distance $\alpha(\cdot, \cdot)$, note that (see [3, Proposition 58.1]) both distances are equal to

$$\alpha(T, \mathcal{M}) = \sup\{|(Tx, y)| : \|x\| = \|y\| = 1, (Sx, y) = 0 \text{ for all } S \in \mathcal{M}\}. \quad (3)$$

It is easy to see that the definitions of reflexivity and hyperreflexivity used in this paper also agree with the more general definitions introduced in [5].

2. Main theorem

Let X, Y be Banach spaces. Denote by $F(X, Y)$ the set of all finite-rank operators from X to Y and by $F_k(X, Y)$ the set of all operators in $B(X, Y)$ of rank smaller or equal to k . Denote by $S_X = \{x \in X : \|x\| = 1\}$ the unit sphere in X .

Let $n \geq 1$ and let $A_1, \dots, A_n \in B(X, Y)$. Denote by $\text{span}\{A_i : i = 1, \dots, n\}$ the closed linear space generated by A_1, \dots, A_n . Write

$$s_0(A_1, \dots, A_n) = \inf \left\{ \left\| \sum_{i=1}^n \lambda_i A_i \right\| : \lambda_1, \dots, \lambda_n \in \mathbb{C}, \max |\lambda_i| = 1 \right\}.$$

More generally, for $k \in \mathbb{N}$ set

$$s_k(A_1, \dots, A_n) = \inf \{s_0(A_1|_M, \dots, A_n|_M) : M \subset X, \text{codim } M \leq k\}.$$

The following lemma summarizes the properties of the quantities s_k .

Lemma 2.1. *Let $A_1, \dots, A_n \in B(X, Y)$. Then:*

- (1) $s_0(A_1, \dots, A_n) = \inf \{ \|\sum_{i=1}^n \lambda_i A_i\| : \max |\lambda_i| \geq 1 \}$;
- (2) $s_0(A_1) = \|A_1\|$;
- (3) $s_0(A_1, \dots, A_n) \leq \min \{ \|A_i\| : i = 1, \dots, n \}$;
- (4) $s_0(A_1, \dots, A_n) > 0$ if and only if the operators A_1, \dots, A_n are linearly independent;
- (5) $s_k(A_1, \dots, A_n) \geq s_l(A_1, \dots, A_n)$ for $k \leq l$;

- (6) $s_k(A_1, \dots, \widehat{A}_j, \dots, A_n) \geq s_k(A_1, \dots, A_n)$ for $j = 1, \dots, n$, where the hat denotes the omitted term;
- (7) if M is a subspace of X and $\text{codim } M \leq k$ then for any l we have $s_l(A_1|_M, \dots, A_n|_M) \geq s_{l+k}(A_1, \dots, A_n)$;
- (8) $\text{dist}(A_j, \text{span}\{A_i : i \neq j\}) \geq s_0(A_1, \dots, A_n)$ for $j = 1, \dots, n$;
- (9) if $k \in \mathbb{N}$ and no non-trivial linear combination of A_1, \dots, A_n belongs to $F_k(X, Y)$, then $s_k(A_1, \dots, A_n) > 0$.

Proof. Statements (1)–(7) are trivial. To see (8), fix j and observe that

$$\begin{aligned} \text{dist}(A_j, \text{span}\{A_i : i \neq j\}) &= \inf \left\{ \left\| \sum_{i=1}^n \lambda_i A_i \right\| : |\lambda_j| = 1 \right\} \\ &\geq \inf \left\{ \left\| \sum_{i=1}^n \lambda_i A_i \right\| : \max |\lambda_i| \geq 1 \right\} = s_0(A_1, \dots, A_n). \end{aligned}$$

To see (9), let us fix $k \geq 0$. Let $Z = \{\sum_{i=1}^n \lambda_i A_i : \max |\lambda_i| = 1\}$. Since Z is compact and $F_k(X, Y)$ closed, we have $\text{dist}(Z, F_k(X, Y)) > 0$.

Let $M \subset X$, $\text{codim } M \leq k$. Let $P \in B(X)$ be a projection onto M such that $\|P\| \leq k + 2$, see [4, Exercise 5.24]. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $\max |\lambda_i| = 1$. Then

$$\begin{aligned} \text{dist} \left(\sum_{i=1}^n \lambda_i A_i, F_k(X, Y) \right) &\leq \left\| \sum_{i=1}^n \lambda_i A_i - \sum_{i=1}^n \lambda_i A_i (I - P) \right\| \leq \\ &\left\| \sum_{i=1}^n \lambda_i A_i P \right\| \leq \left\| \sum_{i=1}^n \lambda_i A_i|_M \right\| \cdot \|P\| \leq (k + 2) \left\| \sum_{i=1}^n \lambda_i A_i|_M \right\|. \end{aligned}$$

Thus

$$\left\| \sum_{i=1}^n \lambda_i A_i|_M \right\| \geq \frac{1}{k + 2} \text{dist} \left(\sum_{i=1}^n \lambda_i A_i, F_k(X, Y) \right)$$

and so

$$s_k(A_1, \dots, A_n) \geq \frac{1}{k + 2} \text{dist}(Z, F_k(X, Y)) > 0. \quad \square$$

The following lemma is a quantitative version of [15, Lemma 1]. Note that for Hilbert spaces it is possible to take $M = F^\perp$.

Lemma 2.2. *Let $F \subset X$, $\dim F = n < \infty$, let $\varepsilon > 0$. Then there exists a subspace $M \subset X$ such that $\text{codim } M \leq (4n\varepsilon^{-1} + 3)^{2n}$ and*

$$\|f + m\| \geq (1 - \varepsilon) \max\{\|f\|, \|m\|/2\}$$

for all $m \in M$, $f \in F$.

In particular, there is a subspace $M_0 \subset X$ with $\text{codim } M_0 \leq (12n + 3)^{2n}$ such that

$$\|f + m\| \geq \frac{1}{3} \max\{\|f\|, \|m\|\}$$

for all $f \in F$ and $m \in M_0$.

Proof. By the Auerbach lemma there are vectors $x_1, \dots, x_n \in F$ and $x_1^*, \dots, x_n^* \in F^*$ of norm one such that $\langle x_j, x_k^* \rangle = \delta_{j,k}$ (the Kronecker symbol) for all j, k . Thus for all $\gamma_1, \dots, \gamma_n \in \mathbb{C}$ we have

$$\left\| \sum_{j=1}^n \gamma_j x_j \right\| \geq \max_k \left| \left\langle \sum_{j=1}^n \gamma_j x_j, x_k^* \right\rangle \right| = \max_k |\gamma_k|.$$

In particular, the vectors x_1, \dots, x_n are linearly independent and therefore form a basis of F . Let

$$Z = \left\{ \sum_{j=1}^n \left(\frac{k_j \varepsilon}{2n} + i \frac{l_j \varepsilon}{2n} \right) x_j : k_j, l_j \text{ integers, } |k_j|, |l_j| \leq 2n\varepsilon^{-1} + 1 \right\}.$$

Then $\text{card } Z \leq (4n\varepsilon^{-1} + 3)^{2n}$.

Let $u \in F$, $\|u\| = 1$. Write $u = \sum_{j=1}^n (t_j + is_j) x_j$ for real t_j, s_j . Clearly $|t_j|, |s_j| \leq 1$ and we can find integers k_j, l_j such that $|\frac{k_j \varepsilon}{2n} - t_j| \leq \frac{\varepsilon}{4n}$ and $|\frac{l_j \varepsilon}{2n} - s_j| \leq \frac{\varepsilon}{4n}$. Thus

$$\left\| u - \sum_{j=1}^n \left(\frac{k_j \varepsilon}{2n} + i \frac{l_j \varepsilon}{2n} \right) x_j \right\| \leq \sum_{j=1}^n \left(\left| \frac{k_j \varepsilon}{2n} - t_j \right| + \left| \frac{l_j \varepsilon}{2n} - s_j \right| \right) \leq \frac{\varepsilon}{2}.$$

So $\text{dist}(u, Z) \leq \frac{\varepsilon}{2}$. For $z \in Z$ choose $z^* \in X^*$ such that $\|z^*\| = 1$ and $\langle z, z^* \rangle = \|z\|$. Let $M = \bigcap_{z \in Z} \ker z^*$. Clearly $\text{codim } M \leq \text{card } Z \leq (4n\varepsilon^{-1} + 3)^{2n}$.

Let $f \in F$, $\|f\| = 1$ and $m \in M$. Then there exists $z \in Z$ such that $\|z - f\| \leq \frac{\varepsilon}{2}$. Thus $\|z\| \geq 1 - \frac{\varepsilon}{2}$. Let $z^* \in X^*$ be the functional considered above. Then we have

$$\begin{aligned} \|f + m\| &\geq |\langle f + m, z^* \rangle| = |\langle f, z^* \rangle| \\ &\geq |\langle z, z^* \rangle| - |\langle f - z, z^* \rangle| \geq \|z\| - \frac{\varepsilon}{2} \geq 1 - \varepsilon. \end{aligned}$$

Hence $\|f + m\| \geq (1 - \varepsilon)\|f\|$ for all $f \in F$, $m \in M$.

Furthermore,

$$\begin{aligned} \|f + m\| &\geq \frac{1}{2}(1 - \varepsilon) \frac{2 - \varepsilon}{1 - \varepsilon} \|f + m\| = \frac{1}{2}(1 - \varepsilon) (\|f + m\| + \frac{1}{1 - \varepsilon} \|f + m\|) \\ &\geq \frac{1}{2}(1 - \varepsilon) (\|m\| - \|f\| + \|f\|) = \frac{1}{2}(1 - \varepsilon) \|m\|. \end{aligned}$$

In particular, for $\varepsilon = \frac{1}{3}$ we get that there exists a subspace $M_0 \subset X$ with $\text{codim } M_0 \leq (12n + 3)^{2n}$ such that

$$\|f + m\| \geq \frac{1}{3} \max\{\|f\|, \|m\|\}$$

for all $f \in F$ and $m \in M_0$. \square

For simplicity we write $r(n) = (12n + 3)^{2n}$ for the codimension of the space M_0 in the previous lemma.

Theorem 2.3. *There are increasing sequences of nonnegative integers $h(n)$, $g(n)$ and sequences of positive numbers c_n , c'_n with the following properties:*

- (a) *if $A_1, \dots, A_n \in B(X, Y)$ satisfy $\|A_j\| \leq 1$ for $j = 1, \dots, n$ and no non-trivial linear combination of A_1, \dots, A_n belongs to $F(X, Y)$, then there exists a unit vector $u \in X$ such that*

$$\left\| \sum_{i=1}^n \lambda_i A_i u \right\| \geq c_n s_{h(n)}^n(A_1, \dots, A_n) \cdot \max\{|\lambda_i| : i = 1, \dots, n\}$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$;

- (b) *if $T, A_1, \dots, A_n \in B(X, Y)$ satisfy $\|A_j\| \leq 1$ for $j = 1, \dots, n$ and no non-trivial linear combination of A_1, \dots, A_n belongs to $F(X, Y)$, then*

$$\begin{aligned} &\alpha(T, \text{span}\{A_1, \dots, A_n\}) \\ &\geq c'_n s_{g(n)}^n(A_1, \dots, A_n) \cdot \text{dist}(T, \text{span}\{A_1, \dots, A_n\}). \end{aligned}$$

Proof. We prove both statements by induction on n .

Let $n = 1$ and let $A_1 \in B(X, Y)$ satisfy $\|A_1\| \leq 1$. Set $c_1 = \frac{1}{2}$ and $h(1) = 0$. There is a vector $u \in X$ such that $\|u\| = 1$ and $\|A_1 u\| \geq \frac{1}{2} \|A_1\|$. Thus $\|\lambda_1 A_1 u\| \geq \frac{1}{2} |\lambda_1| \cdot \|A_1\| = \frac{1}{2} |\lambda_1| s_0(A_1)$ for all $\lambda_1 \in \mathbb{C}$. This proves statement (a) for $n = 1$.

$(a)_n \Rightarrow (b)_n$: Let $g(n) = h(n) + 2n + 2 + (n + 1)r((2n + 2)(n + 1))$ and $c'_n = (\frac{12n}{c_n} + 6)^{-1}$.

Let $T \in B(X, Y)$. Write for short $\varepsilon = \alpha(T, \text{span}\{A_1, \dots, A_n\})$ and $\varepsilon' = \frac{\varepsilon}{c_n s_{g(n)}^n(A_1, \dots, A_n)}$.

For $x \in X$ with $\|x\| = 1$ and $\delta > 0$ set

$$D_{x,\delta} = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \left\| Tx - \sum_{j=1}^n \lambda_j A_j x \right\| \leq \delta \right\}.$$

Clearly $D_{x,\delta}$ is a closed convex set. By the definition of the distance α , $D_{x,\varepsilon} \neq \emptyset$ for all $x \in X$, $\|x\| = 1$.

To show property $(b)_n$, we must prove that

$$\bigcap_{x \in S_X} D_{x, \varepsilon'} \neq \emptyset. \tag{4}$$

Indeed, for $(\gamma_1, \dots, \gamma_n) \in \bigcap_{x \in S_X} D_{x, \varepsilon'}$ we have $\|Tx - \sum_{j=1}^n \gamma_j A_j x\| \leq \varepsilon'$ for all $x \in X$, $\|x\| = 1$, and so $\|T - \sum_{j=1}^n \gamma_j A_j\| \leq \varepsilon'$. Therefore $\text{dist}(T, \text{span}\{A_1, \dots, A_n\}) \leq \varepsilon'$, and so statement (b) for n is fulfilled.

By $(a)_n$ and Lemma 2.1(9), there exists a vector $x_0 \in X$ with $\|x_0\| = 1$ and a constant $c > 0$ such that $\|\sum_{i=1}^n \lambda_i A_i x_0\| \geq c \max |\lambda_i|$ for all $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. Therefore, the set $D_{x_0, \varepsilon'}$ is bounded. Thus (4) is equivalent to

$$\bigcap_{x \in S_X} (D_{x, \varepsilon'} \cap D_{x_0, \varepsilon'}) \neq \emptyset, \tag{5}$$

where the sets $D_{x, \varepsilon'} \cap D_{x_0, \varepsilon'}$ are convex compact subsets of $\mathbb{C}^n \sim \mathbb{R}^{2n}$. By the classical Helly theorem (see [7]), it is sufficient to show that

$$\bigcap_{i=0}^{2n+1} D_{x_i, \varepsilon'} \neq \emptyset$$

for all $(2n + 1)$ -tuples of unit vectors $x_1, \dots, x_{2n+1} \in X$.

Fix $x_1, \dots, x_{2n+1} \in X$ of norm one. Let $F_1 = \text{span}\{x_i : i = 0, \dots, 2n + 1\}$ and let $M_1 \subset X$ be a subspace such that $X = F_1 \oplus M_1$. Then $\text{codim } M_1 \leq 2n + 2$ and $F_1 \cap M_1 = \emptyset$. Let

$$F_2 = \text{span}\{Tx_i, A_j x_i : i = 0, \dots, 2n + 1, j = 1, \dots, n\}.$$

Then $\dim F_2 \leq (2n + 2)(n + 1)$. By Lemma 2.2, there is a subspace $M_2 \subset Y$ with $\text{codim } M_2 \leq r((2n + 2)(n + 1))$ such that $\|f + m\| \geq \frac{1}{3} \max\{\|f\|, \|m\|\}$ for all $f \in F_2$, $m \in M_2$.

Let $M = M_1 \cap T^{-1}M_2 \cap \bigcap_{j=1}^n A_j^{-1}M_2$. Then $\text{codim } M \leq \text{codim } M_1 + (n + 1) \text{codim } M_2$, and so $h(n) + \text{codim } M \leq g(n)$. By the induction assumption $(a)_n$ and by Lemma 2.1(7), (5), there exists a vector $u \in M$, $\|u\| = 1$ such that

$$\begin{aligned} \left\| \sum_{j=1}^n \lambda_j A_j u \right\| &\geq c_n s_{h(n)}^n(A_1|_M, \dots, A_n|_M) \cdot \max_j |\lambda_j| \\ &\geq c_n s_{g(n)}^n(A_1, \dots, A_n) \cdot \max_j |\lambda_j| \end{aligned} \tag{6}$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$.

Claim 1. $D_{x_i, 6\varepsilon} \cap D_{u, 6\varepsilon} \neq \emptyset$ for $i = 0, 1, \dots, 2n + 1$.

Proof. Fix $i \in \{0, 1, \dots, 2n + 1\}$. Note that $x_i \in F_1$, $u \in M \subset M_1$, and so $x_i + u \neq 0$. Set $v = \frac{x_i + u}{\|x_i + u\|}$. Suppose on the contrary that $D_{x_i, 6\epsilon} \cap D_{u, 6\epsilon} = \emptyset$. For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ we have

$$\begin{aligned} \left\| Tv - \sum_{j=1}^n \lambda_j A_j v \right\| &= \frac{1}{\|x_i + u\|} \left\| Tx_i - \sum_{j=1}^n \lambda_j A_j x_i + Tu - \sum_{j=1}^n \lambda_j A_j u \right\| \\ &\geq \frac{1}{6} \max \left\{ \left\| Tx_i - \sum_{j=1}^n \lambda_j A_j x_i \right\|, \left\| Tu - \sum_{j=1}^n \lambda_j A_j u \right\| \right\}, \end{aligned}$$

since $Tx_i - \sum_{j=1}^n \lambda_j A_j x_i \in F_2$ and $Tu - \sum_{j=1}^n \lambda_j A_j u \in M_2$.

Since either $(\lambda_1, \dots, \lambda_n) \notin D_{x_i, 6\epsilon}$ or $(\lambda_1, \dots, \lambda_n) \notin D_{u, 6\epsilon}$, at least one of the two terms is greater than 6ϵ . Thus

$$\left\| Tv - \sum_{j=1}^n \lambda_j A_j v \right\| > \epsilon$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Hence $D_{v, \epsilon} = \emptyset$, a contradiction.

Claim 2. Let $(\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n) \in D_{u, 6\epsilon}$. Then

$$\max_j |\lambda_j - \mu_j| \leq \frac{12\epsilon}{c_n S_{g(n)}^n(A_1, \dots, A_n)}.$$

Proof. Let $(\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n) \in D_{u, 6\epsilon}$. Then

$$\left\| Tu - \sum_{j=1}^n \lambda_j A_j u \right\| \leq 6\epsilon \quad \text{and} \quad \left\| Tu - \sum_{j=1}^n \mu_j A_j u \right\| \leq 6\epsilon.$$

Hence

$$\left\| \sum_{j=1}^n (\lambda_j - \mu_j) A_j u \right\| \leq 12\epsilon$$

and so, by (6) we have

$$\max_j |\lambda_j - \mu_j| \leq \frac{\left\| \sum_{j=1}^n (\lambda_j - \mu_j) A_j u \right\|}{c_n S_{g(n)}^n(A_1, \dots, A_n)} \leq \frac{12\epsilon}{c_n S_{g(n)}^n(A_1, \dots, A_n)}.$$

Claim 3. Let $i \in \{0, 1, \dots, 2n + 1\}$. Then $D_{x_i, \epsilon'} \supset D_{u, 6\epsilon}$.

Proof. Let $(\lambda_1, \dots, \lambda_n) \in D_{u,6\varepsilon} \cap D_{x_i,6\varepsilon}$. Let $(\mu_1, \dots, \mu_n) \in D_{u,6\varepsilon}$ be arbitrary. Then $\max_j |\lambda_j - \mu_j| \leq \frac{12\varepsilon}{c_n s_{g(n)}^n(A_1, \dots, A_n)}$ and

$$\begin{aligned} \left\| Tx_i - \sum_{j=1}^n \mu_j A_j x_i \right\| &\leq \left\| Tx_i - \sum_{j=1}^n \lambda_j A_j x_i \right\| + \left\| \sum_{j=1}^n (\lambda_j - \mu_j) A_j x_i \right\| \\ &\leq 6\varepsilon + \frac{12\varepsilon n}{c_n s_{g(n)}^n(A_1, \dots, A_n)} \leq \frac{\varepsilon}{s_{g(n)}^n(A_1, \dots, A_n)} \left(6 + \frac{12n}{c_n} \right) = \varepsilon'. \end{aligned}$$

Thus $(\mu_1, \dots, \mu_n) \in D_{x_i, \varepsilon'}$.

Hence $\bigcap_{i=0}^{2n+1} D_{x_i, \varepsilon'} \supset D_{u, 6\varepsilon} \neq \emptyset$, and (4) is fulfilled. This proves statement (b) for n .

(b)_{n-1} \Rightarrow (a)_n: Let $n \geq 2$ and suppose that property (b) holds for $n - 1$. Set $c_n = \frac{c'_{n-1}}{18n}$ and $h(n) = g(n - 1) + n^2 \cdot r(n(n - 1))$.

We construct inductively vectors $u_1, \dots, u_n \in X$ of norm one in the following way. Let $k \in \{1, \dots, n\}$ and suppose that the vectors $u_j, j = 1, \dots, k - 1$ have already been constructed. Let

$$F_k = \text{span}\{A_i u_j : i = 1, \dots, n, j = 1, \dots, k - 1\}.$$

Then $\dim F_k \leq n(k - 1) \leq n(n - 1)$. By Lemma 2.2, there is a subspace $M_k \subset Y$ such that $\text{codim } M_k \leq r(n(n - 1))$ and $\|f + m\| \geq \frac{1}{3} \max\{\|f\|, \|m\|\}$ for all $f \in F_k, m \in M_k$. Let $M_k' = \bigcap_{j=1}^k \bigcap_{i=1}^n A_i^{-1} M_j$. Then $\text{codim } M_k' \leq n^2 \cdot \text{codim } M_k$, and so $g(n - 1) + \text{codim } M_k' \leq h(n)$. By property (b)_{n-1}, there is a vector $u_k \in M_k'$ of norm one such that

$$\begin{aligned} &\text{dist}(A_k u_k, \text{span}\{A_i u_k : i \neq k\}) \\ &\geq \frac{1}{2} c'_{n-1} s_{g(n-1)}^{n-1} (A_1|_{M_k'}, \dots, \widehat{A_k|_{M_k'}}, \dots, A_n|_{M_k'}) \text{dist}(A_k|_{M_k'}, \text{span}\{A_i|_{M_k'} : i \neq k\}) \\ &\geq \frac{1}{2} c'_{n-1} s_{h(n)}^{n-1} (A_1, \dots, A_n) \cdot s_0(A_1|_{M_k'}, \dots, A_n|_{M_k'}) \\ &\geq \frac{1}{2} c'_{n-1} s_{h(n)}^n (A_1, \dots, A_n), \end{aligned}$$

where the hat denotes the omitted term; in the estimates we used Lemma 2.1(6),(8) and (7).

Let $u_1, \dots, u_n \in S_X$ be constructed in the above described way. Set $v = \sum_{j=1}^n u_j$. For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $k \in \{1, \dots, n\}$, since $\sum_{j=1}^k \sum_{i=1}^n \lambda_i A_i u_j \in F_{k+1}$, $\sum_{j=k+1}^n \sum_{i=1}^n \lambda_i A_i u_j \in M_{k+1}$, $\sum_{j=1}^{k-1} \sum_{i=1}^n \lambda_i A_i u_j \in F_k$ and $\sum_{i=1}^n \lambda_i A_i u_k \in M_k$, we have

$$\left\| \sum_{i=1}^n \lambda_i A_i v \right\| = \left\| \sum_{j=1}^n \sum_{i=1}^n \lambda_i A_i u_j \right\| \geq \frac{1}{3} \left\| \sum_{j=1}^k \sum_{i=1}^n \lambda_i A_i u_j \right\|$$

$$\begin{aligned} &\geq \frac{1}{9} \left\| \sum_{i=1}^n \lambda_i A_i u_k \right\| \geq \frac{1}{9} |\lambda_k| \cdot \text{dist}(A_k u_k, \text{span}\{A_i u_k : i \neq k\}) \\ &\geq \frac{1}{18} c'_{n-1} s^n_{h(n)}(A_1, \dots, A_n) \cdot |\lambda_k|, \end{aligned}$$

(if $k = n$ then the first inequality is trivial). In particular, $v \neq 0$, by Lemma 2.1(9). Hence the vector $u = \frac{v}{\|v\|}$ satisfies $\|u\| = 1$ and

$$\begin{aligned} \left\| \sum_{i=1}^n \lambda_i A_i u \right\| &\geq \frac{1}{18 \|v\|} c_{n-1} s^n_{h(n)}(A_1, \dots, A_n) \cdot \max_k |\lambda_k| \\ &\geq c_n s^n_{h(n)}(A_1, \dots, A_n) \cdot \max_k |\lambda_k|. \end{aligned}$$

This finishes the proof. \square

Corollary 2.4. *Let $\mathcal{M} \subset B(X, Y)$ be a finite-dimensional subspace which contains no non-zero finite rank operators. Then \mathcal{M} is hyperreflexive.*

Proof. Choose a basis A_1, \dots, A_n of \mathcal{M} . The proof follows from the previous theorem, property (b). \square

Now we are ready to prove the main theorem.

Theorem 2.5. *Let $\mathcal{M} \subset B(X, Y)$, $\dim \mathcal{M} < \infty$. Then \mathcal{M} is hyperreflexive if and only if \mathcal{M} is reflexive.*

Proof. If \mathcal{M} is hyperreflexive then \mathcal{M} is clearly reflexive. Conversely, let \mathcal{M} be reflexive. Let $\mathcal{M}_1 = \mathcal{M} \cap F(X, Y)$ and let \mathcal{M}_2 be any subspace of \mathcal{M} such that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Choose a basis A_1, \dots, A_k of \mathcal{M}_1 and a basis B_1, \dots, B_l of \mathcal{M}_2 .

Let $M = \bigcap_{i=1}^k \ker A_i$. Then $\text{codim } M < \infty$. By the previous result for the operators $B_i|_M$, there is a constant $d_1 > 0$ such that

$$\text{dist}(T|_M, \text{span}\{B_1|_M, \dots, B_l|_M\}) \leq d_1 \cdot \alpha(T|_M, \text{span}\{B_1|_M, \dots, B_l|_M\}). \tag{7}$$

Let $P \in B(X)$ be a projection onto M and $F = \ker P$. Let $F' = \text{span}\{Sf : S \in \mathcal{M}, f \in F\}$. Clearly $\dim F' < \infty$. By Lemma 2.2, there is a subspace $M' \subset Y$ such that $\text{codim } M' < \infty$ and $\|f' + m'\| \geq \frac{1}{3} \max\{\|f'\|, \|m'\|\}$ for all $f' \in F'$ and $m' \in M'$. Set $M'' = M \cap \bigcap_{i=1}^l B_i^{-1} M'$. Clearly $\text{codim } M'' < \infty$.

Let $u \in M''$ be a “separating vector” for the operators $B_i|_{M''}$, i.e., $\|u\| = 1$ and there is a constant $d_2 > 0$ such that $\|\sum_{i=1}^l \gamma_i B_i u\| \geq d_2 \max |\gamma_i|$ for all $\gamma_1, \dots, \gamma_l \in \mathbb{C}$. Such a vector exists by Theorem 2.3 and Lemma 2.1(9).

It follows from [11, Corollary 2.8] that since \mathcal{M} is reflexive, \mathcal{M}_1 is also reflexive. For the sake of completeness we include the proof of this here. Since $\mathcal{M}_1 \subset \mathcal{M}$, we

have $\text{ref } \mathcal{M}_1 \subset \text{ref } \mathcal{M}$. By reflexivity of \mathcal{M} , we have $\text{ref } \mathcal{M}_1 \subset \mathcal{M}$. To show the reflexivity of $\mathcal{M}_1 = \mathcal{M} \cap F(X, Y)$, it is enough to show that $\text{ref } \mathcal{M}_1 \subset F(X, Y)$. Let $B \in \text{ref } \mathcal{M}_1$. Then, for all $u \in X$, $Bu \in \text{span}\{A_i x : i = 1, \dots, k, x \in X\}$. Hence $\text{rank } B < \infty$ and $B \in F(X, Y)$.

Now, for $i = 1, \dots, k$ consider the operators $\tilde{A}_i : F \rightarrow \text{span}\{A_1 x, \dots, A_k x : x \in F\}$ induced by A_i . Since the operators A_1, \dots, A_k are equal to zero on M , it is easy to see that $\tilde{\mathcal{M}}_1 = \text{span}\{\tilde{A}_1, \dots, \tilde{A}_k\}$ is reflexive. As it was observed in the introduction, $\tilde{\mathcal{M}}_1$ is hyperreflexive. Thus there exists a constant $d_3 > 0$ with the following property: if $\varepsilon > 0$ and $T : F \rightarrow \text{span}\{A_1 x, \dots, A_k x : x \in F\}$ satisfies $\text{dist}(Tx, \text{span}\{A_1 x, \dots, A_k x\}) \leq \varepsilon$ for all $x \in F$, $\|x\| = 1$ then there exist numbers $\gamma_1, \dots, \gamma_k \in \mathbb{C}$ such that $\|T - \sum_{i=1}^k \gamma_i A_i|_F\| \leq d_3 \varepsilon$.

We show now that \mathcal{M} is hyperreflexive. Let $\varepsilon > 0$, $T \in B(X, Y)$ and let

$$\text{dist}(Tx, \mathcal{M}x) \leq \varepsilon$$

for all $x \in X$, $\|x\| = 1$. By (7) there exist numbers $\beta_1, \dots, \beta_l \in \mathbb{C}$ such that

$$\left\| T|_M - \sum_{j=1}^l \beta_j B_j|_M \right\| \leq d_1 \varepsilon. \tag{8}$$

Set $S = T - \sum_{j=1}^l \beta_j B_j$. Thus $\|S|_M\| \leq d_1 \varepsilon$ and S satisfies $\text{dist}(Sx, \mathcal{M}x) \leq \varepsilon$ for all $x \in X$, $\|x\| = 1$.

Let $x \in F$, $\|x\| = 1$. Then there are numbers $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_l \in \mathbb{C}$ such that

$$\left\| Sx - \sum_{i=1}^k \lambda_i A_i x - \sum_{j=1}^l \mu_j B_j x \right\| \leq \varepsilon.$$

Similarly, there are numbers $\lambda'_1, \dots, \lambda'_k, \mu'_1, \dots, \mu'_l \in \mathbb{C}$ such that

$$\left\| S(x + u) - \sum_{i=1}^k \lambda'_i A_i(x + u) - \sum_{j=1}^l \mu'_j B_j(x + u) \right\| \leq \varepsilon \|x + u\| \leq 2\varepsilon.$$

By subtracting we have

$$\left\| Su + \sum_{i=1}^k (\lambda_i - \lambda'_i) A_i x - \sum_{j=1}^l (\mu_j - \mu'_j) B_j x - \sum_{j=1}^l \mu'_j B_j u \right\| \leq 3\varepsilon,$$

since $A_i u = 0$ for all i . By the definitions of M'' and F' and by (8), we have

$$\begin{aligned} \left\| \sum_{j=1}^l \mu'_j B_j u \right\| &\leq 3 \left\| \sum_{i=1}^k (\lambda_i - \lambda'_i) A_i x + \sum_{j=1}^l (\mu_j - \mu'_j) B_j x - \sum_{j=1}^l \mu'_j B_j u \right\| \\ &\leq 3(3\varepsilon + \|Su\|) \leq 3\varepsilon(3 + d_1). \end{aligned}$$

Since $\|\sum_{j=1}^l \mu_j' B_j u\| \geq d_2 \max |\mu_j'|$, we have $\max |\mu_j'| \leq 3\varepsilon \frac{d_1+3}{d_2}$. Thus we have

$$\begin{aligned} \left\| Sx - \sum_{i=1}^n \lambda_i' A_i x \right\| &\leq \|Sx - S(x+u)\| \\ &\quad + \left\| S(x+u) - \sum_{i=1}^n \lambda_i' A_i x - \sum_{j=1}^l \mu_j' B_j(x+u) \right\| \\ &\quad + \left\| \sum_{j=1}^l \mu_j' B_j(x+u) \right\| \\ &\leq \|Su\| + 2\varepsilon + \sum_{j=1}^l |\mu_j'| \cdot \|B_j\| \cdot 2 \leq d_4 \varepsilon, \end{aligned}$$

where $d_4 = d_1 + 2 + \frac{3d_1+9}{d_2} \cdot 2 \sum_{j=1}^l \|B_j\|$. Thus there exist numbers $\gamma_1, \dots, \gamma_k \in \mathbb{C}$ such that $\|S|_F - \sum_{i=1}^k \gamma_i A_i|_F\| \leq d_3 d_4 \varepsilon$.

Let $f \in F$, $m \in M$ and $\|f+m\| = 1$. Then $\|m\| = \|P(f+m)\| \leq \|P\|$ and $\|f\| \leq \|f+m\| + \|m\| \leq 1 + \|P\|$. Since $A|_M = 0$, we have

$$\begin{aligned} &\left\| T(f+m) - \sum_{i=1}^k \gamma_i A_i(f+m) - \sum_{j=1}^l \beta_j B_j(f+m) \right\| \\ &= \left\| S(f+m) - \sum_{i=1}^k \gamma_i A_i f \right\| \leq \|Sm\| + \left\| Sf - \sum_{i=1}^k \gamma_i A_i f \right\| \\ &\leq d_1 \varepsilon \|m\| + d_3 d_4 \varepsilon \|f\|. \end{aligned}$$

Thus $\|T - \sum_{i=1}^k \gamma_i A_i - \sum_{j=1}^l \beta_j B_j\| \leq \varepsilon (d_1 \|P\| + d_3 d_4 (\|P\| + 1))$, and so \mathcal{M} is hyper-reflexive. \square

3. Examples and corollaries

The example from [9], mentioned in the introduction shows also that there is no constant in condition (2) for the hyperreflexivity of a finite-dimensional subspace depending only on the dimension of the subspace. Below we give another example of this kind.

Example 3.1. Let $H = \mathbb{C}^3$ with the Hilbert norm. For $\varepsilon > 0$ consider the operators $A_{1,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus [\varepsilon]$ and $A_{2,\varepsilon} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus [0]$. Let $\mathcal{M}_\varepsilon = \text{span}\{A_{1,\varepsilon}, A_{2,\varepsilon}\}$. Clearly $\dim \mathcal{M}_\varepsilon = 2$. It is easy to verify that \mathcal{M}_ε is reflexive for all ε .

Let $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus [0]$. For $\beta, \gamma \in \mathbb{C}$ we have

$$\|\beta A_{1,\varepsilon} + \gamma A_{2,\varepsilon} - T\| = \left\| \begin{bmatrix} \beta - 1 & \gamma \\ 0 & \beta \end{bmatrix} \oplus [\varepsilon\beta] \right\| \geq \max\{|\beta - 1|, |\beta|\} \geq \frac{1}{2}.$$

Thus $\text{dist}(T, \mathcal{M}_\varepsilon) \geq \frac{1}{2}$ for all $\varepsilon > 0$.

Let $x = \begin{bmatrix} a \\ b \end{bmatrix} \oplus [c] \in H$, $\|x\| = 1$. If $b \neq 0$ then $\text{dist}(Tx, \mathcal{M}_\varepsilon x) \leq \|ab^{-1}A_{2,\varepsilon}x - Tx\| = 0$.

If $b = 0$ then $\text{dist}(Tx, \mathcal{M}_\varepsilon x) \leq \|A_{1,\varepsilon}x - Tx\| = \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus [\varepsilon c] \right\| \leq \varepsilon$. Thus $\alpha(T, \mathcal{M}_\varepsilon) \leq \varepsilon$ and there is no constant $C > 0$ such that

$$\text{dist}(T, \mathcal{M}_\varepsilon) \leq C \cdot \alpha(T, \mathcal{M}_\varepsilon)$$

for all $\varepsilon > 0$.

Now we consider finite-dimensional subspaces of $B(H)$, where H is a Hilbert space. It is well known that $B(H)$ is the dual of the trace class operators. If \mathcal{M} is a w^* -closed subspace of $B(H)$, in particular if $\dim \mathcal{M} < \infty$, then \mathcal{M} is reflexive if and only if $\mathcal{M}_\perp \cap F_1(H)$ is total in \mathcal{M}_\perp (see for example [2]). According to [2], a subspace \mathcal{M} is called k -reflexive if $\mathcal{M}_\perp \cap F_k(H)$ is total in \mathcal{M}_\perp . In [12] it was shown that each n -dimensional subspace is $[\sqrt{2n}]$ -reflexive ($[\cdot]$ denotes the integer part). For any subspace $\mathcal{M} \subset B(H)$ and $T \in B(H)$, as it was suggested in [8], we can consider

$$\alpha_k(T, \mathcal{M}) = \sup\{|\langle T, t \rangle| : t \in \mathcal{M}_\perp, \|t\| \leq 1, \text{rank } t \leq k\}$$

(compare with (3)). As in [8] we can call the subspace $\mathcal{M} \subset B(H)$ k -hyperreflexive if there is a constant C such that

$$\text{dist}(T, \mathcal{M}) \leq C\alpha_k(T, \mathcal{M})$$

for each operator $T \in B(H)$. We will show the following

Corollary 3.2. *Let $\mathcal{M} \subset B(H)$ and $\dim \mathcal{M} = n$. Then \mathcal{M} is $[\sqrt{2n}]$ -hyperreflexive.*

Proof. Let $k = [\sqrt{2n}]$. By $\mathcal{M}^{(k)}$ we denote the k -th amplification of \mathcal{M}

$$\mathcal{M}^{(k)} = \left\{ \underbrace{S \oplus \dots \oplus S}_k : S \in \mathcal{M} \right\} \subset B(H^{(k)}),$$

where $H^{(k)}$ is the direct sum of k -copies of H , $H^{(k)} = \underbrace{H \oplus \dots \oplus H}_k$. Since

$\dim \mathcal{M} = n$, \mathcal{M} is k -reflexive by [12, Theorem 12]. By [2, Corollary 2.8], $\mathcal{M}^{(k)}$ is

reflexive. Since $\dim \mathcal{M}^{(k)} = n$, it is also hyperreflexive. Hence [8, Theorem 3.5] implies that \mathcal{M} is k -hyperreflexive. \square

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