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The Cauchy problem of Lorentzian minimal surfaces in globally hyperbolic manifolds

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Abstract In this note a proof is given for global existence and uniqueness of minimal Lorentzian surface maps from a cylinder into a large class of globally hyperbolic Lorentzian manifolds for given initial values up to the first derivatives.

Keywords Minimal surfaces · String theory · Globally hyperbolic

1 Introduction

In the search for a unified field theory of all interactions it is widely believed that a perturbative expansion of such a theory could be provided by String Theory. Its classical action, at least in the case of Closed Bosonic String Theory, is the area of a surface mapped into spacetime such that the induced metric on the surface is Lorentzian (which is only possible in the case of a cylinder, i.e., in the free sector of the theory). While harmonic mappings of Lorentzian surfaces into Riemannian manifolds are a well-examined object since the pioneering work of Gu [6], there seems to be no comparable global result in the double-Lorentzian case in the literature. All gradient estimates (a main tool in Gu's paper) fail in this case because of the presence of null vectors, so one has to apply different methods. The result given by Shatah and Struwe in [12] is a local one without time estimates. The main result of this section is that the minimal surface problem is globally well-posed if we make a slightly stronger assumption than global hyperbolicity for the target manifold (cf. Theorem 16). The solution will exist *locally* on the string worldsheet, but *globally* in the target space in

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the sense that the image will intersect *any* time slice in a standard way. The results extend earlier results of Thomas Deck [3] for static spacetimes. The minimal surface problem for a *Riemannian* target space was solved by J. Douglas in the case of a flat target space [4] and by C.B. Morrey in the case of a general Riemannian manifold as target space [9]. The problem for flat Lorentzian target space is trivial as it consists basically in solving the linear wave equation. But in the case of non-flat Lorentzian target manifolds, left- and right-movers in the language of physics do not decouple any more such that the problem is more involved. We are now going to adress this remaining question in the case of a large class of globally hyperbolic target manifold giving an affirmative answer.

2 Well-known facts

Let (M, g) be a Lorentzian manifold of dimension n. We first recall some well-known facts about (cylindrical) Lorentzian minimal surfaces. The restriction to cylindrical domains of definition is unavoidable if one considers compact Lorentzian surfaces with Riemannian boundary (spacelike boundary curves with timelike normal vector fields) as by taking two copies of such a surface, rearranging the metric to a standard form at the boundaries by diffeomorphisms, and pasting them together one can produce a smooth Lorentzian metric on the Schottky double which leaves us with the torus as the only possible Schottky double, that means, the surface we began with was necessarily a cylinder, and on other surfaces with boundary there is no Lorentz metric with spacelike boundary. Thus our domain of definition will be $\mathbb{I} \times \mathbb{S}^1$ where $\mathbb{I}$ is an arbitrary interval.

Definition 1 The *area* of a map $y : \mathbb{I} \times \mathbb{S}^1 \to M$ for which the pulled-back metric is Lorentzian everywhere is defined to be

$$A(y) := \int_{(\mathbb{I}\times\mathbb{S}^1, y^*g)} 1.$$

where the used volume form on $\mathbb{I}\times\mathbb{S}^1$ is defined by $\sqrt{-det(y^*g)}dv \wedge dw$ in an arbitrary basis (v, w). If y is a critical point of the area functional A in the space of smooth maps it is called *minimal surface*.

This functional has a lot of symmetries: it is invariant under diffeomorphisms of $\mathbb{I}\times\mathbb{S}^1$ (and under isometries of (M, g)).

Definition 2 Let a Lorentzian metric h on $\mathbb{I}\times\mathbb{S}^1$ be given, then a map $y : (\mathbb{I}\times\mathbb{S}^1, h) \to M$ is called *wave map* w.r.t. h iff it satisfies the following differential equation:

$$\mathrm{tr}_h(\nabla \mathrm{d}y) = 0$$

where $\nabla \mathrm{d}y$ is understood as a section in $\tau^*_{\mathbb{I}\times\mathbb{S}^1} \otimes \tau^*_{\mathbb{I}\times\mathbb{S}^1} \otimes y^*\tau_M$ (here, for a manifold L, τ_L resp. τ^*_L means the tangent resp. cotangent bundle of L, and ∇ is the pulled-back Levi–Civita covariant derivative), and the trace is understood as a section of $y^*\tau_M$. In a pseudo-orthogonal frame (V_1, V_2) with $h(V_i, V_j)|_p = (-1)^i\delta_{ij}$ at a point $p \in \mathbb{I} \times \mathbb{S}^1$ the defining equation reads:

$$(V_1(V_1(y^\alpha))(p) - V_2(V_2(y^\alpha))(p) + \Gamma^\alpha_{\beta\gamma}(y(p))(V_1(y^\beta)(p)V_1(y^\gamma)(p)$$
$$- V_2(y^\beta)(p)V_2(y^\gamma))(p) = 0$$

where the $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols in a chart around $y(p)$.

Equivalently, we can use lightlike vectors $X_1 := V_1 + V_2$, $X_2 := V_1 - V_2$ at p and consider the equation

$$\nabla_{X_1} X_2(y) = 0$$

at p, or in coordinates of the target manifold:

$$X_1(X_2(y)) + \Gamma^{\alpha}_{\beta\gamma}(y) X_1(y^{\beta}) X_2(y^{\gamma}).$$

Of course we can interchange the roles of X_1 and X_2.

Now by examining the Euler-Lagrange equations we get:

Theorem 3 *y is a minimal surface iff it is a wave map with respect to y^*g.*

Definition 4 For a map $y\colon (\mathbb{I} \times \mathbb{S}^1, h) \to (M, g)$ the *energy $E(y)$* is defined as

$$E(y) := \int_{(\mathbb{I}\times\mathbb{S}^1, h)} \langle \mathrm{d}y, \mathrm{d}y \rangle$$

where the scalar product is the natural one on the bundle $\tau^*_{\mathbb{I}\times\mathbb{S}^1} \otimes y^*\tau_M$. If (y, h) is a critical point of E in the space of smooth maps and smooth metrics, it is called *minimal-energy surface*.

The energy is invariant under conformal changes of h. The next theorem is an immediate consequence of the Euler-Lagrange equations:

Theorem 5 *y is a minimal energy surface iff*

(i) *the map y is a wave map, and*
(ii) *the metric h is conformally equivalent to y^*g.*

Thus, minimal energy surfaces are minimal surfaces, and while area and energy differ as functionals, they have the same critical points. By diffeomorphism invariance and by the result of Kulkarni [7] that every Lorentzian metric on a cylinder with two spacelike boundary components can be linked by a diffeomorphism to a conformal multiple of the standard metric, we get

Theorem 6 *$y\colon \mathbb{I} \times \mathbb{S}^1$ is a minimal surface if it is a wave map w.r.t. the standard metric and if y^*g is conformally equivalent to the standard metric. On the other hand, every Lorentzian minimal surface map from the cylinder to M with Riemannian boundaries is linked by a diffeomorphism of the cylinder to a wave map w.r.t. the standard metric whose pulled-back metric is conformally equivalent to the standard metric.*

On a cylinder with the standard Lorentzian metric we can write the wave map equation in global characteristic coordinates $\xi := t + s$, $\eta := t - s$ on Δ (where $\mathbb{I}$ is parametrized by t and $\mathbb{S}^1$ by s) as

$$\nabla_{\xi} \partial_{\eta} y = 0$$

or even more in coordinates as

$$\partial_{\xi}\partial_{\eta} y^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}(y) \partial_{\xi} y^{\beta} \partial_{\eta} y^{\gamma} = 0.$$

This is an equation of parallel transport w.r.t. the (metric) pulled-back covariant derivative which will be relevant in the fourth section.

We will first focus on the first property of Theorem 6: that y be a wave map. For this purpose, we will consider the domain of the maps as located in the upper half Minkowski plane. Later on we will return to the case of $\mathbb{I} \times \mathbb{S}^1$ by requiring periodicity of y.

3 Wave maps. Local existence and uniqueness

We will look for wave maps y from some subsets of the upper half Minkowski plane $H := \{(t,s)|t \geq 0\}$ to M with some initial values on the s-axis $\{0\} \times \mathbb{R}$. Throughout the proof we will work in characteristic coordinates $\xi := \frac{t+s}{2}$, $\eta := \frac{t-s}{2}$ on $\mathbb{R}^{1,1}$. Let $\frac{\partial y^\alpha}{\partial \xi} = u^\alpha$, $\frac{\partial y^\alpha}{\partial \eta} = v^\alpha$. We will need an arbitrary auxiliary *Riemannian* metric h and an associated norm $||.||$ on M. Define, for every $p \in M$, $R(p)$ as the supremum of radii $r > 0$ such that the exponential map $exp_p^g|_{B_r^h(0)}$ is a diffeomorphism onto its image. The trivializations of these balls fix coordinate patches at every p. Let $\Gamma^\alpha_{\beta\gamma}$ be the Christoffel symbols of the Levi–Civita connection of g in these coordinates, let $G(p)$ be the maximum of the operator norm (w.r.t. h) of $\Gamma^\gamma_{\alpha\beta}$ and their first derivatives in the coordinate patch around p, similarly $\underline{u}, \underline{v} := ||u||_{C^1(\{0\}\times\mathbb{R})}, ||v||_{C^1(\{0\}\times\mathbb{R})}$, respectively.

Theorem 7 *Let M be a Lorentzian manifold with tangent bundle $\tau_M \colon TM \to M$, let $k \colon \mathbb{R} \to TM$ be a smooth curve, let $k_0 := \tau_M \circ k$, assume that $\langle \partial_s k_0, \partial_s k_0 \rangle \geq 0 \geq \langle k, k \rangle$ everywhere. Let $G(p)$ be bounded by $G(\delta)$ in a ball $B_\delta(k_0)$ of radius δ (w.r.t. h) around $im(k_0)$. Then there is a unique smooth map from a neighbourhood $U \subset H$ of $\{0\} \times \mathbb{R}$ to M which is a wave map and whose restriction and normal derivative at $\mathbb{R} \times \{0\}$ correspond to the given curve, i.e., $y(s,0) = k_0(s)$, $\partial_t y(s,0) = k(s)$. Let $R_k := \inf_{p \in im(k_0)} R(p)$, $L_k := \min\{\frac{R_k}{5}, \frac{1}{11G(\delta)}, \frac{\delta}{5}\}$. Then the neighbourhood can be chosen as a small strip $y \colon [0, \frac{1}{\sqrt{2}} l] \times \mathbb{R} \to M$ with $l := \min\{\frac{L_k}{(\underline{u}+\underline{v})}, 1\}$. Note that $\underline{u}, \underline{v}$ only depend on k, and that if k is a closed curve in TM then $l > 0$.*

The theorem will be proved as an easy corollary of the same statement for small characteristic triangles based on the boundary curve $\{0\} \times \mathbb{R}$:

Proposition 8 *Let M be a Lorentzian manifold, let $k \colon \mathbb{R} \to TM$ be as above, l as above. Then for each characteristic triangle Δ of base length $\leq l$ there is a unique smooth map $y \colon \Delta \to M$ which is a wave map and whose restriction and normal derivative at $\{0\} \times \mathbb{R}$ correspond to the given curve, i.e., $y(0,s) = k_0(s)$, $\partial_t y(0,s) = k(s)$.*

Proof First note that the choice of l as above assures that the image of each interval of length l is contained in some coordinate patch as $l \leq \frac{R_k}{2||\partial_s k_0||}$. Let Δ be the triangular region spanned by the interval of length $\leq l$ (which we call the base side a of the triangle) and the characteristic lines beginning at the endpoints A and B of the interval. For every point $p = (t,s)$ in Δ we define four special lines: $c_{p,\eta}(\tau) := p + \tau(1/\sqrt{2}, -1/\sqrt{2})$, $c_{p,\xi}(\tau) := p + \tau(1/\sqrt{2}, 1/\sqrt{2})$. The points where $c_{p,\eta}$ resp. $c_{p,\xi}$ hit the basis of the triangle at points we call $p' = (0, s+t)$ resp. $p'' = (0, s-t)$ (Fig. 1).

We use a way of splitting the differential equation similar to the one in Gu's paper [6] and consider the following set of first-order ordinary differential equations of maps

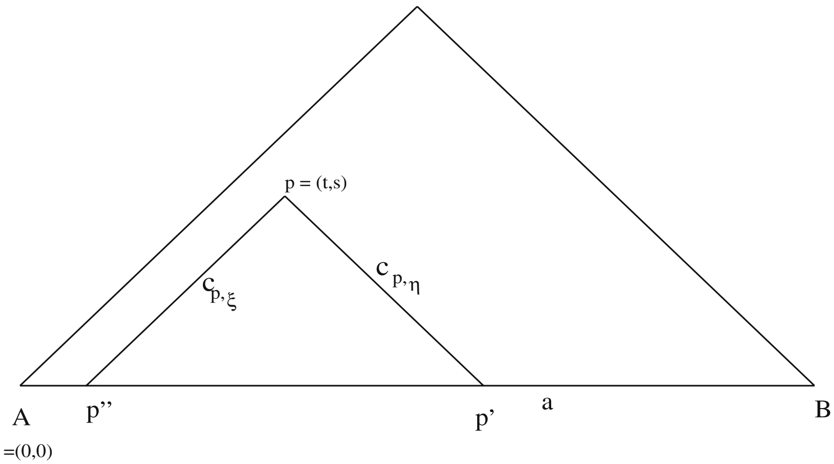

Fig. 1 Distinguished lines in a characteristic triangle

$y, z, u, v, \hat{u}, \hat{v}$ from Δ to $\mathbb{R}^n$:

$$\begin{aligned}
&\frac{\partial u^\alpha}{\partial \eta} + \Gamma^\alpha_{\beta\gamma}(z)u^\beta \hat{v}^\gamma = 0, \quad &&\frac{\partial \hat{u}^\alpha}{\partial \eta} + \Gamma^\alpha_{\beta\gamma}(y)\hat{u}^\beta v^\gamma = 0,\\
&\frac{\partial v^\alpha}{\partial \xi} + \Gamma^\alpha_{\beta\gamma}(z)\hat{v}^\beta u^\gamma = 0, \quad &&\frac{\partial \hat{v}^\alpha}{\partial \xi} + \Gamma^\alpha_{\beta\gamma}(y)v^\beta \hat{u}^\gamma = 0,\\
&\frac{\partial y^\alpha}{\partial \xi} = u^\alpha, \frac{\partial y^\alpha}{\partial \eta} = v^\alpha, \quad &&\frac{\partial z^\alpha}{\partial \xi} = \hat{u}^\alpha, \frac{\partial z^\alpha}{\partial \eta} = \hat{v}^\alpha
\end{aligned} \tag{1}$$

with initial conditions

$$\begin{aligned}
y^\alpha(0,s) = z^\alpha(0,s) &= k_0^\alpha(s),\\
u^\alpha(0,s) = \hat{u}^\alpha(0,s) &= \frac{\partial k_0^\alpha(s)}{\partial s} + k^\alpha(s),\\
v^\alpha(0,s) = \hat{v}^\alpha(0,s) &= -\frac{\partial k_0^\alpha(s)}{\partial s} + k^\alpha(s).
\end{aligned} \tag{2}$$

Lemma 9 *C^2-solutions (y, z) of (1) are uniquely determined by their initial values on the base a of the triangle. The solution depends smoothly on the initial values.*

Proof Without restriction of generality, consider the triangle with base $|s| \leq \frac{l}{2}$. For every two solutions $r_i = (y_i, z_i, u_i, v_i, \hat{u}_i, \hat{v}_i)$, $i = 1, 2$ define

$$f(t) = \sup_{|s| \leq \frac{l}{2} - t} (||y_1 - y_2|| + \cdots + ||\hat{v}_1 - \hat{v}_2||)(t, s).$$

Then the standard ODE estimates for the system (1) (cf. [13], 16.VI, eq. (8)) imply

$$f(t) \leq e^{2GK} f(0) + e^{2GK} \cdot 2GK \cdot \int_0^t Cf(\tau)\mathrm{d}\tau$$

for a C^0-bound K of the r_i. For the case that the initial data coincide, Gronwall's lemma implies $f(t) = 0$ for all t. Moreover, the equation above immediately implies continuous dependence of the solution on the initial values. For the higher derivatives one can proceed inductively, and if one defines Fréchet structures as usual by

$$\mathrm{d}(f,f') := \sum 2^{-i} \frac{||f-f'||_{C^i}}{1+||f-f'||_{C^i}},$$

the dependence is smooth. □

Note that for example the first equation of the system (1) corresponds to the invariant equation $\nabla_\eta u = 0$ (with ∇ denoting the covariant derivative of the Levi–Civita connection on M pulled back by y), an equation of parallel transport along η-lines. A solution of this system will give $y = z$ (because of uniqueness and symmetry of the equations under $y \leftrightarrow z, u \leftrightarrow \hat{u}, v \leftrightarrow \hat{v}$) and therefore be a solution of the original problem. At this point we can forget about the Lorentzian metric g that does not appear at all in the equations and use the Riemannian metric h as long as we still keep the Christoffel symbols of g.

The system (1) has twice the size of its analogue in Gu's work. This is mainly due to our intent to include a special solution $\tilde{r}$ defined below and to the structure of the corresponding inequalities.

Now we will define a map whose fix points are exactly the solutions of (1). Of course we should admit only maps which satisfy the basic compatibility condition

$$\frac{\partial y^\alpha}{\partial \xi} = u^\alpha, \quad \frac{\partial z^\alpha}{\partial \eta} = \hat{v}^\alpha. \tag{3}$$

Now choose the coordinates in such a way that $k_0(A) = 0$ in the local coordinate system. Then we define a subspace of maps $\Delta \to M$ which map the point A to the origin in our coordinates:

$$N := \{(y,z,u,v,\hat{u},\hat{v}) \in C^1(\Delta, (\mathbb{R}^n)^6)\text{: Equations (3) hold and y(A)} = 0 = \text{z(A)}\}$$

and a subspace $N' \subset N$ by

$$N' := \{r \in N\text{: } r \text{ satisfies equations (2)}\}.$$

Restricting ourselves to N, we can forget about y, z by inserting

$$y_{(u,v,\hat{u},\hat{v})}(p) = \int_{-\sqrt{2}\cdot t}^{0} u(c_{p,\xi}(\tau))\mathrm{d}\tau + \int_0^{s-\frac{1}{2}\sqrt{2}t} (u-\hat{v})(a(\tau))\mathrm{d}\tau$$

$$z_{(u,v,\hat{u},\hat{v})}(p) = \int_{-\sqrt{2}\cdot t}^{0} \hat{v}(c_{p,\eta}(\tau))\mathrm{d}\tau + \int_0^{s+\frac{1}{2}\sqrt{2}t} (u-\hat{v})(a(\tau))\mathrm{d}\tau$$

where $a(\tau) := A + (0,\tau)$. Therefore there is a canonical embedding of N to $C^1(\Delta, (\mathbb{R}^n)^4)$. Elements of the latter one will be denoted by $r = (r_1, r_2, r_3, r_4) = (u, v, \hat{u}, \hat{v})$.

Now we define the map

$$\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4)\colon C^1(\Delta, (\mathbb{R}^n)^4) \to C^1(\Delta, (\mathbb{R}^n)^4)$$
$$r = (u, v, \hat{u}, \hat{v}) \mapsto (\Phi_1(r) \ldots \Phi_4(r)) = (u_\Phi, v_\Phi, \hat{u}_\Phi, \hat{v}_\Phi)$$

by the following system of linear first-order ODEs (with initial condition (2)):

$$\begin{aligned} \frac{\partial u_\Phi^\alpha}{\partial \eta} + \Gamma^\alpha_{\beta\gamma}(z_r) u_\Phi^\beta \hat{v}^\gamma = 0, \quad \frac{\partial \hat{u}_\Phi^\alpha}{\partial \eta} + \Gamma^\alpha_{\beta\gamma}(y_r) \hat{u}_\Phi^\beta v^\gamma = 0 \\ \frac{\partial v_\Phi^\alpha}{\partial \xi} + \Gamma^\alpha_{\beta\gamma}(z_r) \hat{v}_\Phi^\beta u^\gamma = 0, \quad \frac{\partial \hat{v}_\Phi^\alpha}{\partial \xi} + \Gamma^\alpha_{\beta\gamma}(y_r) v_\Phi^\beta \hat{u}^\gamma = 0 \end{aligned} \tag{4}$$

Note that a fix point of this map in the space N' will also be $(y \leftrightarrow z, u \leftrightarrow \hat{u}, v \leftrightarrow \hat{v})$-symmetric because of the initial conditions (2) and

$$\frac{\partial v}{\partial \xi} = -\Gamma(z)\hat{v}u = \frac{\partial u}{\partial \eta} = \frac{\partial y}{\partial \eta \partial \xi}$$

which yields $v = \partial y / \partial \eta$ and the other missing equation of system (1). It will therefore be a solution of the original problem. As stated above, the map Φ is well-defined in the space N of maps which map A to zero, but whose restrictions on the base side can be different. Nevertheless, the restrictions of $y, z, u, v, \hat{u}, \hat{v}$ on the base side remain fixed under Φ, i.e., $u_\Phi|_{\{0\}\times\mathbb{R}} = u|_{\{0\}\times\mathbb{R}}$ etc. The fact that Φ maps C^1 maps to C^1 maps is seen trivially from the defining equation for half of the coordinates, for the other half of coordinates use the usual theorems about differentiable dependence of ODE systems on parameters as in [13], II.13.V.

Now let $r = (u, v, \hat{u}, \hat{v}) \in C^1(\Delta, (\mathbb{R}^n)^4)$ be given. For the proof of the theorem we will make use of an associated special solution $\tilde{r}$ of the system (1) to compare with, namely the one with $\partial_t y|_a = -\partial_s y|_a$, i.e., with $u = 0 = \hat{u}$ everywhere, a map which is constant along ξ-lines. It is easy to see that this is a solution (for a different initial value curve $\tilde{k}$, but still $\tilde{k}_0 = k_0$). That means, $\tilde{r}$ is in N but not in N'. Thus define $\tilde{r}(p) := \tilde{r}(p'') := (0, V, 0, V)$ (with $V(q) := (v - u)(q)$ for all $q \in a$). The image of the corresponding y_r, z_r is one-dimensional, namely the chosen interval of the curve k_0; as it is a solution, it is a fix point of Φ: $\Phi(\tilde{r}) = \tilde{r}$. Moreover, as we assumed $l \leq R_k / (2||u - v||_{\{0\}\times\mathbb{R}})$, l is so small that the whole image of the base of the triangle is contained in some coordinate patch with distance $\geq R_k/2$ from its boundary, thus we have a ball-shaped coordinate patch B containing the image of $\tilde{r}$ with

$$\mathrm{dist}_h(\tilde{r}(\Delta), \partial B) \geq \frac{R_k}{2} \tag{5}$$

Now consider the affine subspace $M(r) := \{f \in C^1(\Delta, (\mathbb{R}^n)^4) : \ f|_a = r|_a\}$, define seminorms $||.||_j$ of maps from Δ to $(\mathbb{R}^n)^4$ by

$$||f||_0 := \max_{i=1,\dots,4} \{||f_i||_{C^0(\Delta)}\}$$

$$||f||_j := \max_{i=1,\dots,4} \{||d^j f_i||_{C^0(\Delta)}\}$$

for $j = 1, 2, \dots$. Now, for given $K \in \mathbb{R}$ let us define

$$M_K(r) := \{f \in M(r) | \ ||f - \tilde{r}||_1 \leq K\} = M(r) \cap B_K^{||.||_1}(\tilde{r}).$$

Note that in general $\tilde{r}$ does not need to be in $M(r)$, hence for small K the set $M_K(r)$ will be empty. Secondly, $M_K(r)$ only depends on $r|_a$ because $\tilde{r}$ does. Moreover, on $M_K(r)$ the seminorm $||.||_1$ induces a metric as $||f - g||_1 = 0$ means that f and g differ by an additive constant while $f, g \in M(r)$ implies that f and g coincide on a. Now, to get an easier formulation of the following lemma, we consider the reflection σ at the symmetry axis of the characteristic triangle Δ and the map $S : C^1(\Delta, (\mathbb{R}^n)^4) \to C^1(\Delta, (\mathbb{R}^n)^4)$ with $S((f_1, f_2, f_3, f_4)) = (f_1, f_2 \circ \sigma, f_3, f_4 \circ \sigma)$. Finally, we define $\hat{\Phi} := S \circ \Phi \circ S$.

Lemma 10 *Recall* $\underline{u} := ||u||_{C^1(a)}, \underline{v} := ||v||_{C^1(a)}$. *Define* $K := \frac{3}{l}(\underline{u} + \underline{v})$, *then*

(i) *If* $f \in C^1(\Delta, (\mathbb{R}^n)^4)$, *then* $\partial_\eta \partial_\eta(\hat{\Phi}(f))$ *and* $\partial_\eta \partial_\xi(\hat{\Phi}(f))$ *exist and are continuous.*
(ii) $\hat{\Phi}(f)|_a = f|_a$, *i.e., the values of the maps on the base side remain fixed.*
(iii) *The set* $M_K(r)$ *is not empty.*
(iv) $f \in M_K(r)$ *implies that* $||\hat{\Phi}(f) - \tilde{r}||_1 \leq K$.

Proof of the lemma (i) and (ii) follow easily from the defining equations. For (iii) consider $\bar{r}(p) := \bar{r}(p'')$ and $\bar{r}|_a := r|_a$. As $l \leq 1$, we have $||\bar{r} - \tilde{r}||_1 \leq \underline{u} + \underline{v} < K$. For (iv) let $f = (u, v, \hat{u}, \hat{v}) \in M_K(r)$; then use again that on the base side a, $\tilde{r}$ and r (and thus $\tilde{r}$ and f) differ at most by the length $(\underline{u} + \underline{v})$, so $||f - \tilde{r}||_1 \leq K$ implies $||f - \tilde{r}||_0 \leq Kl + (\underline{u} + \underline{v}) = 4(\underline{u} + \underline{v}) =: K'$.

As $||f - \tilde{r}||_0 < K'$, we have the following estimates:

$$||\Gamma^\alpha_{\beta\gamma}(y_m) \cdot v^\gamma|| \leq G(\delta) \cdot (\underline{u} + \underline{v} + ||f - \tilde{r}||_0) \leq 5G(\delta)(\underline{u} + \underline{v})$$

where $G(\delta)$ was defined in the theorem. Thus standard ODE estimates ([13], 12.V.) give

$$||\hat{\Phi}_i(f) - \tilde{r}_i||_{C^0} \leq (\underline{u} + \underline{v})e^{5G(\delta)l(\underline{u}+\underline{v})} \leq K'$$

Now we want to get the same result for K instead of K' and on the C^1 level. For half of the coordinates this follows easily from the defining equations and the C^0 estimates, as e.g.,

$$\left\| \frac{\partial u_\Phi}{\partial \eta} \right\|_{C^0} = ||\Gamma(z) \cdot u_\Phi \hat{v}||_{C^0} \leq (K')^2 G \leq \frac{3}{8} \frac{K'}{l} = \frac{K}{2}$$

as $l \leq \frac{3}{32(\underline{u}+\underline{v})G}$. The other half we get by applying again standard ODE theorems ([13], 13.V. and 12.V.) to $\partial_s f$ (f considered to be fixed on the corresponding characteristic line), where the norm of the Lipschitz coefficient is again $\leq G \cdot K'$. Then the comparison with the fix point $\tilde{r}$ of $\hat{\Phi}$ yields $||\partial_s(\hat{\Phi}(f)) - \partial_s \tilde{r}||_0 \leq \gamma e^{Ll}$, where γ is the distance $||\partial_s f - \partial_s \tilde{r}||_{C^0(a)}$ of the maps restricted on a which is less or equal to $K/4$ while L is the Lipschitz coefficient which can be taken to be $G \cdot K'$. Thus we get

$$||\partial_s f - \partial_s \tilde{r}||_0 \leq \frac{K}{4} e^{GK'l} < \frac{K}{2}$$

which completes the proof of the lemma. □

In short, the lemma above tells us that $\hat{\Phi}(M_K(r)) \subset M_K(r)$ (and that differentiability improves a little bit) if we choose K as above.

Lemma 11 *Let* conv *be the convex closure in* $C^1(\Delta, \mathbb{R}^{4n})$, *i.e., for a set* M, conv(M) *denotes the smallest closed convex set in* $C^1(\Delta, \mathbb{R}^{4n})$ *that contains* M. *Then the set* $A := \hat{\Phi}(\text{conv}(\hat{\Phi}(M_K(r))))$ *has compact closure in* $C^1(\Delta, \mathbb{R}^{4n})$.

Proof According to the Arzela–Ascoli theorem, a set in $C^1(\Delta, \mathbb{R}^{4n})$ is precompact if and only if it is C^1-bounded and all of its first derivatives are equicontinuous. But following the lemma above, $\hat{\Phi}(M_K(r)) \subset M_K(r)$, $\text{conv}(\hat{\Phi}(M_K(r))) \subset M_K(r)$ as the norms are convex, and $\hat{\Phi}(\mathrm{conv}(\hat{\Phi}(M_K(r))) \subset M_K(r)$ following the lemma again. Therefore A is C^1-bounded as $M_K(r)$ is.

Now let $r_2 = (u_2, v_2, \hat{u}_2, \hat{v}_2) \in A$, and $r_1 = (u_1, v_1, \hat{u}_1, \hat{v}_1) \in \mathrm{conv}(\hat{\Phi}(M_K(r)))$ with $\Phi(r_1) = r_2$. Then $u_2, v_2, \hat{u}_2, \hat{v}_2$ are equicontinuous as they are C^1-bounded. Analogously for $\partial_\eta u_2, \partial_\xi v_2, \partial_\eta \hat{u}_2, \partial_\xi \hat{v}_2$ (look at the defining equations). Thus it remains to be shown that $\partial_\xi u_2, \partial_\eta v_2, \partial_\xi \hat{u}_2, \partial_\eta \hat{v}_2$ are equicontinuous in A. First, one can establish equicontinuity in one half of the directions, i.e., show that $\partial_\xi u_2(p) - \partial_\xi u_2(p + \tau \cdot (1, -1) \le C \cdot \tau$. This can be proven by the usual ODE theorem about differential dependence on the parameter τ and then performing the admissible formal differentiation

$$\partial_\eta(\partial_\xi u_2) = \partial_\xi(\partial_\eta u_2) = \partial_\xi(-\Gamma(y_1)u_2 v_1)$$

which shows equicontinuity w.r.t. one half of the variables. The other half one can get by differentiating the defining equation as above

$$\begin{aligned}\frac{\partial}{\partial \eta}\frac{\partial u_2^\alpha}{\partial \xi} &= -\left(\frac{\partial}{\partial \xi}\Gamma^\alpha_{\beta\gamma}(z_1)\right)\hat{v}_1^\gamma u_2^\beta - \Gamma^\alpha_{\beta\gamma}(z_1)\left(\frac{\partial}{\partial \xi}\hat{v}_1^\gamma\right)u_2^\beta - \Gamma^\alpha_{\beta\gamma}(z_1)\hat{v}_1^\gamma\left(\frac{\partial}{\partial \xi}u_2^\beta\right)\\ &=: f\left(\eta, \frac{\partial u_2}{\partial \xi}\right),\end{aligned}$$

understood as the defining ODE for $\partial u/\partial \xi$ along a fixed η-line $c_{(T,S),\eta}$ for given u. This functional f satisfies the Lipschitz condition

$$|f(\eta, V) - f(\eta, W)| \le L|V - W|$$

for all η in the interval and for a real constant L. Now we compare $c_{(T,S),\eta}$ with a second η-line $c_{(T,S+\epsilon),\eta}$, parallel to the first one with distance ϵ for which all quantities are denoted by $u^{(\epsilon)}, \hat{v}^{(\epsilon)}, z^{(\epsilon)}$:

$$\begin{aligned}f^{(\epsilon)}\left(\eta, \frac{\partial u_2}{\partial \xi}\right) := &-\left(\frac{\partial}{\partial \xi}\Gamma^\alpha_{\beta\gamma}(z_1^{(\epsilon)})\right)\hat{v}_1^{(\epsilon)\gamma}u_2^{(\epsilon)\beta} - \Gamma^\alpha_{\beta\gamma}(z_1^{(\epsilon)})\left(\frac{\partial}{\partial \xi}\hat{v}_1^{(\epsilon)\gamma}\right)u_2^{(\epsilon)\beta}\\ &- \Gamma^\alpha_{\beta\gamma}(z_1^{(\epsilon)})\hat{v}_1^{(\epsilon)\gamma}\left(\frac{\partial}{\partial \xi}u_2^{(\epsilon)\beta}\right)\end{aligned}$$

The equicontinuity will be shown by the theorem about continuous dependence of ODEs on a real parameter ([13], 12.VI). To do this, we have to prove that for any $\delta > 0$ there is an $\epsilon > 0$ such that

$$f^{(\epsilon)}(\eta, V) - f(\eta, V) \le \delta.$$

Hence we have to consider

$$\begin{aligned}&-\left(\frac{\partial}{\partial \xi}\Gamma^\alpha_{\beta\gamma}(z_1)\right)\hat{v}_1^\gamma V^\beta + \left(\frac{\partial}{\partial \xi}\Gamma^\alpha_{\beta\gamma}(z_1^{(\epsilon)})\right)\hat{v}_1^{(\epsilon)\gamma}V^\beta\\ &- \Gamma^\alpha_{\beta\gamma}(z_1)\left(\frac{\partial}{\partial \xi}\hat{v}_1^\gamma\right)V^\beta + \Gamma^\alpha_{\beta\gamma}(z_1^{(\epsilon)})\left(\frac{\partial}{\partial \xi}\hat{v}_1^{(\epsilon)\gamma}\right)V^\beta\\ &- \Gamma^\alpha_{\beta\gamma}(z_1)\hat{v}_1^\gamma\left(\frac{\partial}{\partial \xi}V^\beta\right) + \Gamma^\alpha_{\beta\gamma}(z_1^{(\epsilon)})\hat{v}_1^{(\epsilon)\gamma}\left(\frac{\partial}{\partial \xi}V^\beta\right).\end{aligned}$$

The first line is quite easily shown to be less than $3GK^3\epsilon$ as every difference between terms with (ϵ) and such without is less than $K\epsilon$ or $G\epsilon$, equally the last line. Most subtle is the second one as it seems we have not enough control over $\frac{\partial}{\partial \xi}\hat{v}_1^\gamma - \frac{\partial}{\partial \xi}\hat{v}_1^{(\epsilon)\gamma}$.

But on $\hat{\Phi}(M_K(r))$ this term is bounded linearly in ϵ as $\partial \hat{v}_1^\alpha / \partial \xi = -\Gamma^\alpha_{\beta\gamma}(y_0) v_1^\beta \hat{u}_0^\gamma$. This Lipschitz bound survives the convex closure as the C^1 Lipschitz norm is continuous in C^1 and convex. Thus the second line is equally bounded in linear terms of ϵ and we are done. □

Now, due to Lemma 11 we can apply the Schauder–Tychonoff fix point theorem ([5], p. 148) to the operator $\hat{\Phi}$ restricted to the closed convex, set conv$(\hat{\Phi}(M_K(r)))$ and prove the existence of a fix point in C^1 in $(u, v, \hat{u}, \hat{v})$, i.e., of a C^2-fix point in (y, z). Below we will prove uniqueness of such fix points. Smoothness of the solution is easily seen by successive application of the quoted ODE theorem ([13], 13.V. and 12.V.) to the solution (the single equations of the system (1) considered as solutions to ODEs). □

4 Minimal surfaces. Local existence, uniqueness

A harmonic mapping is a minimal surface, i.e., a critical point of the area functional, if and only if it is conformal. In Riemannian signature the solution of the PDE's corresponding to conformality is an additional difficulty, in Lorentzian signature it's a matter of initial data:

Theorem 12 *Assumptions as in Theorem 7 and additionally assume that the curve k is such that $\langle \partial_s k_0, \partial_s k_0 \rangle = -\langle k, k \rangle$ and $\partial_s k_0 \perp k$ (note that the first requirement can be satisfied by reparametrization of the curve). Then the unique wave map y extending k fulfills $y^* g = U \cdot (-dt^2 + ds^2)$ with a smooth function U (which might change its sign).*

Proof This is just a consequence of the facts that for such a curve $\langle u, u \rangle = 0 = \langle v, v \rangle$ on the s-axis, therefore $\langle u, u \rangle = 0 = \langle v, v \rangle$ everywhere on the whole domain of definition because of metricity of the parallel transport, so the pulled-back metric is a multiple of the Minkowski metric (with a possibly not overall-positive conformal factor). □

Now we want to exclude sign changes of the conformal factor which is positive on the s-axis.

Theorem 13 *Let M be time-oriented. If the curve $k : \mathbb{R} \to M$ satisfies $\langle \partial_s k_0, \partial_s k_0 \rangle = -\langle k, k \rangle \geq 0$, $\partial_s k_0 \perp k$ and $k \pm \partial_s k_0 \neq 0$ at every point, then the corresponding wave map has always*

$$\langle \partial_t y, \partial_t y \rangle \leq 0, \quad \langle \partial_s y, \partial_s y \rangle \geq 0$$

and thus is a Lorentzian minimal surface, i.e., a critical point of the area functional (whose pulled-back metric is conformally equivalent to the standard Minkowski metric).

Proof By assumption, on the x-axis always $u, v \neq 0$. As u, v are parallel-transported over the surface along characteristic lines, they vanish nowhere and stay therefore on the forward light cone because of time-orientation. So $\partial_t y = \frac{1}{2}(u + v)$ is (being a convex combination) contained in the solid forward light cone. Therefore $\partial_t y$ is always causal and future-directed while $\partial_s y$ is spacelike or lightlike. □

Note, however, that it *can* happen that u, v touch the light cone somewhere (and thus coincide there): Consider e.g., the case of a k_0 being a circle of radius r on the x^1x^2-plane in 3-dimensional Minkowski space, $k = \partial/\partial x^0$ constant, the rotational-symmetric solution $y(t,s) = (t, r\cos t \sin s, r \cos t \cos s)$ is one with cosine-shaped radius, so if the length of the string is 2π the circle $\{\pi\} \times \mathbb{R}$ is mapped onto a single point in $\mathbb{R}^{1,2}$, $\partial_s y = 0$ there.

Note that the resulting minimal surfaces will in general neither maximize nor minimize locally the area functional (cf. [8]).

5 Global existence

In the following we consider only globally hyperbolic Lorentzian manifolds, i.e., locally causally convex Lorentzian manifolds with compact diamonds $J^+(p) \cap J^-(q)$. Bernal and Sanchez showed recently (cf. [2]) that any globally hyperbolic manifold M is isometric to a product $(\mathbb{R} \times N, -f(p)dt^2 \oplus \hat{g}_t)$, where f is a smooth positive function on M and $\hat{g}$ a smooth x^0-dependent family of Riemannian metrics on N, and every $\{t\} \times N$ is a Cauchy hypersurface. Such a parametrization is called *orthogonal Cauchy foliation*. Conversely, of course every manifold possessing an orthogonal Cauchy foliation is globally hyperbolic. Now additionally we assume that M is *bbc*:

Definition 14 A Lorentzian manifold (M, g) is said to be *bbc* (bounded–bounded-complete) iff it admits a foliation $(M, g) \cong (\mathbb{R} \times N, g(p) = -f(p)dt^2 \oplus g_t)$ where f is a smooth positive function on M which is *bounded* on every $\{t\} \times N$, and g_t is a family of *complete* Riemannian metrics on N, and with the additional property that the eigenvalues of $\dot{g}_t \circ g_t^{-1}$ (here g_t is interpreted as endomorphism $TN \to T^*N$ and dotted quantities are derivatives by t) are *bounded* on N for all t.

Another way to express the last condition is to say that $\frac{\mathrm{d}}{\mathrm{d}t} g_t \in \Gamma(\tau_N^* \otimes_s \tau_N^*)$ has bounded eigenvalues w.r.t. g_t (here $\otimes_s$ denotes the symmetric tensor product). The following proposition links the notion of bbc manifolds to the one of globally hyperbolic manifolds:

Proposition 15 *The $t =$ constant hypersurfaces of a bbc manifold are Cauchy hypersurfaces. In particular, bbc manifolds are globally hyperbolic.*

Proof Assume that for an inextendible curve c the function c_0 is bounded. Thus we denote its supremum (which is its $s \to \infty$ limit by monotonicity) by T. Now we will show $c_N(s) := proj_N \circ c(s)$ to converge to a point p in N.

For simplicity let c be parametrized by its x_0-component c_0 with respect to the Cauchy foliation. We have to show that for every positive real number ϵ there is a point $p_\epsilon \in N$ and a positive number σ such that $c(s) \subset B_\epsilon^{g_T}(p_\epsilon)$ for all $s \geq T - \sigma$. First we choose any $\underline{T} < T$, and as we can find a joint constant K with $|\dot{g}_t(X,Y)| \leq K|g_t(X,Y)|$ for all $t \in [\underline{T}, T]$, we have the estimate

$$|g_t(X,Y)| \leq |g_T(X,Y)| \cdot e^{K(t-T)}$$

and therefore for the geodesic distance d^{g_T} w.r.t. the metric g_T and for $S, S' \geq T - \sigma$ with $\sigma < T - \underline{T}$ we get

$$d^{g_T}(c(S), c(S')) \leq \int_S^{S'} \sqrt{g_T(\dot{c}_N(s), \dot{c}_N(s))}\mathrm{d}s \leq \int_S^{S'} \sqrt{e^{K\sigma} g_{c_0(s)}(\dot{c}_N(s), \dot{c}_N(s))}\mathrm{d}s$$
$$\leq \sqrt{e^{K\sigma}} \int_S^{S'} \dot{c}_0(s)\mathrm{d}s \leq e^{K\sigma} \cdot \sigma$$

This shows that c is C^0-extendible contrary to the assumption and completes the proof. □

Examples If g_t is a compact perturbation (e.g., described by pullback by a symmetric endomorphism of the tangent bundle, which is the identity outside of a compactum on every N_t) of a fixed complete Riemannian metric for all values of t, and f is bounded, then $(N, -fdt^2 \oplus g_t)$ is bbc, even without assuming a priori that all hypersurfaces $t = constant$ are Cauchy hypersurfaces (cf. [1]). On the contrary, any open precompact subset of the Minkowski space is not bbc as it does not contain any complete spacelike hypersurface. The set of isometry classes of bbc manifolds is geometrically quite rich, e.g., for every Lorentzian holonomy representation with a fixed spinor (a parallel spinor field), there is an explicit bbc manifold, which has this holonomy representation (cf. [1]).

As auxiliary Riemannian metric h we take now the *flip metric to* g, i.e., $+fdt^2 \oplus g_t$. Of course, this depends on the choice of coordinates, but this does not matter for the proof.

Moreover, we shall restrict ourselves to the case of *closed* curves as initial data. These give rise to smooth maps from subsets of flat cylinders into spacetime because periodic initial data produce periodic solutions (as horizontal translations are conformal transformations). As one can expect, we do not get a *worldsheet* notion of globality. Every statement about global existence on the world-sheet in a reparametrization-invariant theory would be a pure artefact of the chosen gauge as global existence has no gauge-invariant meaning. Instead, we have a *target space* global existence theorem, saying that the preimage of any spacelike Cauchy hypersurface of M under y is a one-sphere on the cylindrical worldsheet.

Theorem 16 *Let* (M, g) *be a bbc manifold, let* $k \colon \mathbb{R} \to TM$ *be a smooth* 2π*-periodic curve,* $k_0 := \tau_M \circ k$*. Assume* $k \pm \partial_s k_0 \neq 0$ *at every point,* $\langle \partial_s k_0, \partial_s k_0 \rangle = -\langle k, k \rangle \geq 0$ *and* $\langle \partial_s k_0, k \rangle = 0$*. Then there is a unique smooth (and* 2π*-periodic) map from an open subset* Ω *of the upper half plane* $y \colon \Omega \to M$ *which is a minimal surface and whose restriction and normal derivative at* $\{0\} \times \mathbb{R}$ *correspond to the given curve, i.e.,* $y(0,s) = k_0(s)$*,* $\partial_t y(0,s) = k(s)$*, and with the property that for all* $p \in \partial\Omega \setminus (\{0\} \times \mathbb{R})$

$$\lim_{q \to p} y^0(q) = \infty,$$

i.e., for any spacelike Cauchy hypersurface $S \subset M$*, its preimage* $y^{-1}(S)$ *is always a smooth* 2π*-periodic graph over the s-axis in* $\mathbb{R}^{1,1}$.

Proof First we switch back to the description of the 2π-periodic map by a cylindrical domain $\mathbb{I} \times \mathbb{S}^1$ and define the maximal domain of existence Ω of y, as the largest open and connected region in $\mathbb{I} \times \mathbb{S}^1$ containing $\{0\} \times \mathbb{S}^1$ on which a solution is defined. Now assume that there is a spacelike Cauchy hypersurface S with $y^{-1}(S) \not\cong \mathbb{S}^1$. Then, according to [11], we can find an orthogonal Cauchy foliation with $S = t^{-1}(\{0\})$. Now consider the set $\{t \in \mathbb{R} | y^{-1}(\{t\} \times N) \not\cong \mathbb{S}^1\}$. As this set is a closed interval and nonempty as it contains 0, there is a minimum we call T. Now observe the curves $y^{-1}(\{\tau\} \times N)$ for $\tau < T$ on the cylinder. These are smooth closed spacelike curves on the cylinder which are therefore graphs over the s-axis such that we can write them as $(f_\tau(s), s)$ with real functions f_τ. Now from conformality of y and from the proof of Theorem 13 one sees that y always maps the backward null cone to the backward null cone in spacetime; thus $\partial_\xi y^0, \partial_\eta y^0 > 0$ (y^0 is increasing along upward characteristic lines). Thus the f_τ are Lipschitz functions with Lipschitz constant 1, and their limit curve corresponding to $f_T := lim_{\tau \to T} f_\tau$ either is situated at infinity (then Ω is the whole cylinder, *Case B*) or is a proper curve on the cylinder, which is not necessarily smooth anymore but still Lipschitz with Lipschitz constant 1 (*Case A*). Let us consider the second case first.

Case A. The limit curve contains points of Ω and points of its complement, which is a closed subset of $\mathbb{S}^1$ and therefore compact, thus we can choose an $\underline{x}$ such that $(\underline{t}, \underline{x}) := (f_T(\underline{x}), \underline{x}) \notin \Omega$ and is one of the values x where $f_T(x)$ is minimal among all x for which $(f_T(x), x) \notin \Omega$. Then because of the twofold minimal choice of this point, the backward causal cone of $(\underline{t}, \underline{x})$ in $\mathbb{R}^{1,1}$ belongs entirely to the domain of existence with the exception of $(\underline{t}, \underline{x})$ itself, and in this cone we have $y^0 \leq T$. Within this cone we choose now an arbitrary characteristic triangle Δ_0 with $(\underline{t}, \underline{x})$ as top point. The midpoint of the base side of this triangle we call X. This gives us suitable L^1-bounds for $||u||$ and $||v||$ in this triangle in the following way: Let $T_0 := \min\{y^0(p) | p \in a\}$. Then for every point $p \in \Delta_0 \backslash (\underline{t}, \underline{x})$ we have the following estimate along a lightlike piecewise smooth curve c in the triangle linking p with X:

$$\int_c ||\dot{c}|| = 2\sqrt{2}C \int_c u^0 \leq 2\sqrt{2}C(T - T_0)$$

where C is a bound for f on $[T_0, T] \times N$. Therefore, $y(\Delta_0) \subset B_{4C(T-T_0)}(y(X))$. Choose the triangle Δ_0 so small that this ball is contained in some h-ball B' of half the injectivity radius at its midpoint (this is feasible as the curve of midpoints is timelike and with bounded t-value and therefore C^0-extendible to a point $\overline{X}$, so the injectivity radius along this curve has a nonzero minimum). Let G be an upper bound for the operator norm of the Christoffel symbols in this ball B'. Now, the global upper estimate $4C(T - T_0)$ for all integrals of the form $\int_{c_\xi} ||u||$ and $\int_{c_\eta} ||v||$ together with G imply global C^0-bounds for $||u||, ||v||$ in $\Delta_0 \backslash (\underline{t}, \underline{x})$ because of the defining differential equation

$$0 = D_\eta u = \partial_\eta u^\gamma(x) + \Gamma^\gamma_{\alpha\beta}(y) u^\alpha(x) v^\beta(x).$$

(cf. [13]). Now, once established upper bounds for G and R^{-1} in the backward cone, the length of the triangles one can add to the domain of definition of y is proportional to $\frac{1}{||u||+||v||}$, but this quantity is bounded from zero now in $\Delta_0 \backslash (\underline{x}, \underline{t})$. Thus, somewhere at the very top of the triangle there is a horizontal line from the left to the right side of the triangle which one can prolong still a bit over the endpoints such that it is still in Ω (which is possible since Ω is open), still in the injectivity radius ball and such that it still satisfies the (open) conditions of the local statement. Thus, one can add a triangle containing the point $(\underline{t}, \underline{x})$ in its interior, a contradiction.

Case B. In this case we have $\underline{t} = \infty$. As above, from the definition of $(\underline{x}, \underline{t})$ it can be easily seen that in this case y is defined globally on the cylinder. But then by estimating $\langle u, \partial_{x^0}\rangle$ from above it is easy to show that the image of the whole halfplane C^1-converges to one point P in M for $t \to \infty$ (which again enables us to get an a priori bound on operator norm of the Christoffel symbols): As y^0 is bounded and monotone increasing in t and along lightlike lines, y^0 converges on the halfplane to T. We want to show that for every sequence of points $p_n = (x_n, t_n)$ on the halfplane with $lim_{n\to\infty} t_n = \infty$, the sequence $\pi \circ y \circ p_n$ of points in M, where π is the projection on N, is a Cauchy sequence for the metric g_T. Now define $\rho_n \in]0, \frac{1}{n}[$, such that for all $\tau \in]T - \rho_n, T]$ and for all $p \in N$, $g_T(V, W) < 2g_\tau(V, W)$ for all $V, W \in T_pN$. This is possible because the eigenvalues of $\dot{g}_t \circ g_t^{-1}$ are uniformally bounded for $t \in [T - \epsilon, T] \times N$ by assumption. Now we want to compute the g_T-distance between the images of two points p_{n+q} and p_{n+m}, both in the subset $y^{-1}(]T - \rho_n, T[\times N)$ of the halfplane (cylinder). To this purpose, consider a piecewise smooth lightlike curve c in the cylinder linking them, parametrized over the unit interval, and the curve $k := y \circ c$. Then

$$\begin{aligned} d_{g_T}(\pi(y(p_{n+q})), \pi(y(p_{n+m}))) &\leq \int_{[0,1]} \sqrt{g_T(d\pi \circ \dot{k}(t), d\pi \circ \dot{k}(t))}\mathrm{d}t \\ &\leq \int_{[0,1]} \sqrt{2g_{y^0(k(t))}(d\pi \circ \dot{k}(t), d\pi \circ \dot{k}(t))}\mathrm{d}t \\ &\leq \int_{[0,1]} \sqrt{2}\sqrt{f(y(k(t)))}y^0(k(t))\mathrm{d}t \\ &\leq \sqrt{2}\sqrt{C} \cdot \frac{1}{n} \end{aligned}$$

where C is a common bound for f in $[\underline{T}, T] \times N$. Thus the sequence is a Cauchy sequence w.r.t. the metric g_T, and because of completeness of g_T the halfplane converges to a point $P \in M$.

Now, having established the convergence of the solution, the proof can be given by using Lorentzian normal coordinates at this point. Choose a Lorentzian normal neighborhood U of P, let the corresponding chart onto a region in $\mathbb{R}^{1,n-1}$ with coordinates $e_0, e_1, \ldots, e_{n-1}$ be denoted by κ. Then without restriction of generality $d\kappa(\frac{\partial}{\partial x^0})|_P = e_0$. Now consider the corresponding *Euclidean* scalar product $\langle \cdot, \cdot \rangle_e$ and the associated norm $||.||_e$ on $\mathbb{R}^n$ where $||e_0||_e = 1$. Via the map $d\kappa$ we consider $h = \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_e$ as metrics on U as well as on $\kappa(U)$. Let $\rho > 0, \rho < \frac{1}{10}$ be given. Choose $\gamma_0 < 1$ such that the following conditions are satisfied in the Euclidean ball $B_0 := B_{\gamma_0}(0)$:

(i) The supremum of the $||\Gamma^\gamma_{\alpha\beta}||_e$ is bounded by $G < \frac{1}{10}$.

(ii) For every $q \in \kappa^{-1}(B_0)$ and $v, w \in T_qM$ we have that

$$|\langle v, w\rangle_e - \langle v, w\rangle| \leq |2\langle v, e_0\rangle_e \langle w, e_0\rangle_e|(1 + \rho)$$

(iii) For every $q \in \kappa^{-1}(B_0)$ and $v \in T_qM$ we have

$$|\langle v, \partial_{x^0}\rangle| \in [(1 - \rho)\langle v, e_0\rangle_e, (1 + \rho)\langle v, e_0\rangle_e]$$

The requirement (i) can be satisfied as in Lorentzian normal coordinates at P the Christoffel symbols vanish at P. The requirements (ii) and (iii) can be satisfied as they are satisfied at $P = \kappa^{-1}(\{0\})$ with $\rho = 0$.

Choose T_0 such that $||u(t, s)||, ||v(t, s)|| < 1$ for $t > T_0$ and such that $y([T_0, \infty[\times[0, 2\pi]) \subset B_{\gamma_0}(0)$. The plan of the proof is the following:

1. We first show that $||du||_e, ||dv||_e$ are bounded on $[T_0, \infty[\times[0, 2\pi]$.
2. Then we show that $||d^2u||_e, ||d^2v||_e$ are bounded on $[T_0, \infty[\times[0, 2\pi]$ as well.
3. We will conclude that $||du||_e, ||dv||_e \to 0$ for $t \to \infty$.
4. Then we will prove by a backward estimate that after a possible modification of T_0, on $[T_0, \infty[\times[0, 2\pi]$ we can estimate from above $||du||_e, ||dv||_e$ linearly in $sup_{t\geq T_0} \int_{\{t\}\times[0,2\pi]} ||u||_e + ||v||_e$.
5. Finally, we will consider the curve of spacelike mean values and show that its zero coordinate would transgress 0, a contradiction.

Now let us give the details of the proof.

Steps 1 and 2. We want to have a C^0-bound of $||du||_e, ||dv||_e$ on $C_0 := C_{T_0} := [T_0, \infty[\times[0, 2\pi]$. The terms $||\partial_\eta u||_e, ||\partial_\xi v||_e$ are bounded on C_{T_0} anyway as $||u||_e, ||v||_e$ are bounded on C_{T_0} and because of the defining equation. Thus we consider $\partial_\xi u$ along a line $c(\tau) := c_{p,\eta}(\tau) = p + \tau(\sqrt{2}, -\sqrt{2})$ with $p \in \{T_0\} \times [0, 2\pi]$. Then we have

$$\partial_\eta(\partial_\xi u) = \partial_\xi \partial_\eta u = \partial_\xi(\Gamma(y)uv) = (\partial_\xi \Gamma(y))uv + \Gamma u(\partial_\xi v) + \Gamma v(\partial_\xi u)$$

Thus we define

$$a := (\partial_\xi \Gamma(y))uv + \Gamma u(\partial_\xi v), \quad A := \Gamma v.$$

Now we want to get estimates for these quantities. First we get an estimate for A: consider $H(\tau) := \int_0^\tau ||A||_e(c(s))ds$. We know that

$$\lim_{\tau\to\infty} H(\tau) \leq G \int_0^\infty ||v(c(\tau))||_e d\tau \leq 2(1+\rho)G\gamma_0 =: K < \infty.$$

The second inequality holds because of (ii) and of $\langle v, v\rangle = 0$.

To get an estimate for a, note that for $R := \sup_{q\in B_0} ||d\Gamma^\gamma_{\alpha\beta}(q)||_e$ we have

$$||a||_e \leq R||u||_e \cdot ||v||_e + G||u||_e \cdot G||u||_e \cdot ||v||_e \leq ||v||_e,$$

therefore $\int_0^\infty ||a(c(\tau))||_e \leq 2(1+\rho)\gamma_0 < \infty$. In this situation we can use the L^1 estimates of C^0 solutions of ordinary differential equations (cf. [13], 16.IV, eq. (7)). This gives

$$\begin{aligned} ||\partial_\xi u(t)||_e &\leq ||\partial_\xi u|_{\{T_0\}\times[0,2\pi]}||_e e^{H(t)} + e^{H(t)} \int_0^t e^{-H(s)}||a(c(s))||_e \mathrm{d}s \\ &\leq ||\partial_\xi u(p)||_e \cdot e^K + e^K 2(1+\rho)\gamma_0 < \infty \end{aligned}$$

A global bound for $||\partial_\eta v||_e$ can be found in exactly the same way, so let us denote the bound for $||du||_e, ||dv||_e$ by $K_1(T_0)$. Analogously, one can get a global bound $K_2(T_0)$ for $||d^2u||_e, ||d^2v||_e$ by calculating $\partial_\eta(\partial_\xi \partial_\xi u)$ as above and inserting the defining equation for $\partial_\xi v = \Gamma uv$ successively until there is a factor v in every term.

Step 3. First we want to show that $\int_0^\infty \langle \partial_\eta v(c(\tau))\rangle_e d\tau < \infty$ (recall that $c(\tau) := c_{p,\eta}(\tau) = p + \tau(\sqrt{2}, -\sqrt{2})$). For this purpose, we first note that $||\nabla_\eta v||_e, ||\nabla^2_\eta v||_e$ are bounded in

C_0 by, say, S_1 and S_2 as $\nabla_\eta w = \partial_\eta w + \Gamma v w$. Then we prove a global bound for

$$\begin{aligned}\int_c \langle \nabla_\eta v, \nabla_\eta v\rangle &= \int_c \partial_\eta \langle v, \nabla_\eta v\rangle - \int_c \langle v, \nabla_\eta^2 v\rangle \\ &\le 2 \sup_{C_0}(\langle v, \nabla_\eta v\rangle) - \int_c \langle v, \nabla_\eta^2 v\rangle \\ &\le 2(3+\rho)S_1 \cdot \sup_{C_0}(||v||_e) + (3+\rho)S_2 \int_c ||v||_e \\ &\le 2(3+\rho)S_1 \cdot \sup_{C_0}(||v||_e) + (3+\rho)S_2 \cdot 2(1+\rho)\gamma_0 =: M\end{aligned}$$

Now we get

$$\begin{aligned}\int_0^\infty \langle \partial_\eta v(c(\tau)), \partial_\eta v(c(\tau))\rangle_e d\tau &\le \int_0^\infty (\langle \nabla_\eta v(c(\tau)), \nabla_\eta v(c(\tau))\rangle_e + ||\Gamma u v||_e(c(\tau)))\mathrm{d}\tau \\ &\le (1+\rho)\int_c \langle \nabla_\eta v, e_0\rangle_e^2 + M + 2G(1+\rho)\gamma_0 \\ &\le (1+\rho)\int_c \langle \partial_\eta v, e_0\rangle_e^2 + M + 4G(1+\rho)\gamma_0 \\ &\le S_1 \cdot (1+\rho)\int_c \partial_\eta v^0 + M + 4G(1+\rho)\gamma_0 \\ &\le S_1 \cdot (1+\rho)\sup_{C_0} v^0 + M + 4G(1+\rho)\gamma_0 < \infty\end{aligned}$$

Therefore

$$\int_{C_0} ||\partial_\eta v||_e^2 = \frac{1}{\sqrt{2}} \int_{[0,2\pi]} \mathrm{d}s \int_0^\infty ||\partial_\eta v(c_{-\eta,(s,T_0)}(t)||_e \mathrm{d}t < \infty$$

and analogously we conclude $\int_{C_0} ||\partial_\xi u||_e^2 < \infty$. Thus

$$\int_{\{t\}\times[0,2\pi]} ||\partial_\eta v||_e^2 \to_{t\to\infty} 0.$$

Now we use the following elementary lemma about a bound on the supremum of a function on $\mathbb{R}$ by its L^1-norm and the supremum of its first derivative:

Lemma 17 *Let $\mathbb{I}$ be a connected subset of $\mathbb{R}$, let $f\colon \mathbb{I} \to [0,\infty[$ be differentiable. Then*

$$(\sup_{x\in\mathbb{I}}(f(x)))^2 \le 2\sup_{x\in\mathbb{I}}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\cdot \int_{\mathbb{I}} f + (\inf_{x\in\mathbb{I}}(f(x)))^2$$

Proof of the lemma by the equation $f^2(x\pm t) = 2\int_x^{x\pm t} f \cdot \frac{\mathrm{d}f}{\mathrm{d}s}\mathrm{d}s + f^2(x)$ where one can choose $f^2(x)$ arbitrarily close to the infimum. □

Now we set $f := ||\partial_\eta v||_e^2$. The upper estimates $K_1(T_0)$ and $K_2(T_0)$ for $||d^2u||_e, ||d^2v||_e$ in C_0 imply the upper estimate $2\sqrt{2}K_1(T_0)K_2(T_0)$ for $\partial_s f$; and as one can estimate the infimum by the integral, the lemma implies that

$$\sup_{\{t\}\times[0,2\pi]}(||\partial_\eta v||_e^2) \le \left(2\sqrt{2}K_1(T_0)\cdot K_2(T_0) + \frac{1}{2\pi}\right)\cdot \int_{\{t\}\times[0,2\pi]} ||\partial_\eta v||^2 \to 0.$$

Thus $||du||_e, ||dv||_e \to 0$ for $t \to \infty$.

Step 4. Define

$$\gamma(U) := \max\{r \in \mathbb{R} : y([U,\infty] \times \mathbb{R}) \in B_r(0)\},$$

$$\epsilon(U) := \sup_{t \geq T} \int_{\{t\}\times[0,2\pi]} ||u||_e + ||v||_e.$$

We want to show that on each C_U one can estimate $||du||_e$, $||dv||_e$ from above linearly in $\epsilon(U)$. To this purpose, consider a point $(t,s) \in C_U$ and an η-line in backward time direction, $c_{p,-\eta}(\tau) := p - \tau(-\sqrt{2}, \sqrt{2})$ where we choose p such that $(t,s) \in c_{p,\eta}([0,\infty[)$ and that $||\partial_\xi u||_e < \epsilon(U)$. Now the ODE estimate we used above gives

$$||\partial_\xi u(s,t)||_e = e^{H(\tau)} \cdot (||\partial_\xi u(p)||_e + \int_0^\tau e^{-H(\sigma)} ||a(c_{p,-\eta}(\sigma))||_e d\sigma)$$

where $(t,s) = c_{p,-\eta}(\tau)$, $H(\sigma) = \int_0^\sigma ||A(c(r))||_e dr < G\gamma_0$ as above and

$$\begin{aligned} ||a||_e = ||(\partial_\xi \Gamma)uv + \Gamma u(\partial_\xi v)||_e &\leq (R+G^2)||u||_e^2 \cdot ||v||_e \\ &\leq (R+G^2)\left(2K_1(T_0) + \frac{1}{2\pi}\right)\epsilon ||v||_e \end{aligned}$$

where $K_1(T_0)$ is the upper bound for $||du||_e, ||dv||_e$ in C_{T_0} (the second line again because of the lemma as $||u||_e^2 \leq 2K_1(T_0)\epsilon + \frac{\epsilon}{2\pi}$ in C_T). Thus we get

$$\int_0^\tau ||a(\sigma)||_e \mathrm{d}s \leq (R+G^2)\left(2K_1(T_0) + \frac{1}{2\pi}\right)\epsilon\gamma_0$$

and finally

$$||\partial_\xi u(s,t)||_e \leq e^{G\gamma_0} \cdot \left(1 + (R+G^2)\left(2K_1(T_0) + \frac{1}{2\pi}\right)\gamma_0\right)\epsilon =: \overline{K}(T_0) \cdot \epsilon$$

to stress that the first two factors do only depend on the choice of T_0, not on the one of T. This is the desired upper bound for $||\partial_\xi u||_e$. For $||\partial_\eta u||_e$ we just have to insert the defining equation and use the lemma to show that

$$||\partial_\eta u||_e \leq G||u||_e \cdot ||v||_e \leq GK_1(T_0) \cdot \epsilon + \left(\frac{\epsilon}{2\pi}\right)^2.$$

These upper estimates and the lemma above enable us to get better estimates for $||u||_e, ||v||_e$ which are linear in ϵ:

$$\begin{aligned} \left(\sup_{\{t\}\times[0,2\pi]}(||u||_e)\right)^2 &\leq 2\left\|\frac{\mathrm{d}u}{\mathrm{d}s}\right\|_e \int_{\{t\}\times[0,2\pi]} ||u||_e + (inf_{\{t\}\times[0,2\pi]}(||u||_e))^2 \\ &\leq \overline{K}(T_0) \cdot \epsilon \cdot \epsilon + \frac{\epsilon^2}{2\pi} \end{aligned}$$

so there is a constant $\underline{K}(T_0)$ such that $||u||_e, ||v||_e \leq \underline{K}(T_0) \cdot \epsilon$.

Step 5. Now we use the vector space structure of $(\mathbb{R}^n, \kappa^* g)$ to define the *mean value curve* $\underline{y}(t)$ *of* y by

$$\underline{y}(t) = \int_{\{t\}\times[0,2\pi]} y(t,s)\mathrm{d}s.$$

Lemma 18 *Let y be a wave map from the cylinder into $(\mathbb{R}^n, \nabla, h)$ where ∇ is a connection on $\mathbb{R}^n$ and h a Riemannian metric on $\mathbb{R}^n$ with norm $||.||$. Let $G(t)$ be an upper bound of the Christoffel symbols in the metric h on $y(\{t\} \times [0, 2\pi])$. Let $\underline{y}$ be the curve of spacelike mean values defined above. Define $A(t) := \max_{\{t\}\times[0,2\pi]}(||u||, ||v||)$, then*

$$\left\| \frac{\partial^2}{\partial t^2}\underline{y}(t) \right\| \leq 2G(t)A^2(t) \cdot 2\pi.$$

Proof of the lemma

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\underline{y}(t) &= \frac{\partial^2}{\partial t^2}\int_{\{t\}\times[0,2\pi]} y(t,s)\mathrm{d}s \\
&= \int_{\{t\}\times[0,2\pi]} \frac{\partial^2}{\partial t^2} y(t,s)\mathrm{d}s \\
&= \int_{\{t\}\times[0,2\pi]} \frac{\partial^2}{\partial s^2} y(t,s)\mathrm{d}s + \int_{\{t\}\times[0,2\pi]} \Gamma(y(t,s))(\partial_s\partial_s y - \partial_t\partial_t y)
\end{aligned}$$

Now the first integral in the last line vanishes as it equals $\partial_s y(t, 2\pi) - \partial_s y(t, 0)$. The second one can be estimated by $\int_{\{t\}\times[0,2\pi]} G||u|| \cdot ||v||$. □

This second lemma implies (with $l = \hat{T} - T$):

$$\underline{y}^0(\hat{T}) - \underline{y}^0(T) \geq \epsilon l - \frac{1}{2}G(\overline{K}(T_0))^2\epsilon^2 l^2 = \epsilon l(1 - G(\overline{K}(T_0))^2\epsilon l)$$

which can be made larger than 2γ (and produces a contradiction) if, for example, $\epsilon l \geq 4\gamma$ and $G(\overline{K}(T_0))^2\epsilon l < \frac{1}{2}$. This can be arranged by appropriate choice of l if $\frac{4\gamma}{\epsilon} < \frac{1}{2G(\overline{K}(T_0))^2\epsilon}$ which is equivalent to $\gamma < \frac{1}{8G(\overline{K}(T_0))^2}$. □

Of course, one can reverse the direction of the process and solve the minimal surface equations backwards in time. Then one ends with a map y from an open set of a cylinder into spacetime, and as $y^0(x,t)$ is a function strictly monotone in t with limits $\pm\infty$, y crosses every spacelike Cauchy surface in M in the image of a one-sphere (which can be a point, e.g., in the example of the previous section).

6 Summary

Now let us summarize the main result: For any given curve with the properties described above there is exactly one minimal surface parametrized by conformal gauge, i.e., such that the pulled-back metric is conformally equivalent to the two-dimensional Minkowski metric on the cylinder. As mentionned before, every metric on the cylinder with Lorentzian ends can be brought into this gauge (cf. [7]).

Therefore these initial curves are in one-two-one correspondence to *unparametrized* minimal surfaces. On the other hand we can give up parametrizations of the initial curves and consider them as special one-dimensional submanifolds of the total space of TM. In summary, for M being bbc, we have established the following one-to-one correspondences:

closed spacelike curves in M with an orthogonal timelike vector field along it, modulo reparametrizations

$\leftrightarrow$ closed spacelike curves c in M with an orthogonal timelike vector field V along it, with $||V|| = -||\dot{c}||$

$\leftrightarrow$ minimal Lorentzian surface maps from a cylindrical open subset O of a cylinder (with $\mathbb{S}^1 \times \{0\}$ as the lower boundary of $\overline{O}$) into M in conformal gauge (i.e., such that the pulled-back metric is conformally equivalent to the standard Lorentzian metric on the cylinder), modulo rigid rotations

$\leftrightarrow$ unparametrized minimal Lorentzian surface maps from a cylinder into M

where the arrows from below in inverse direction stand for the choice of the conformal parametrization, the restriction to the boundary, and finally forgetting the parametrization. Note that despite of the somewhat unusual notion of global solution, the notion is covariant in the sense that it does not depend on the choice of a time slice.

Note that as we have proved smooth dependence of the solution, intersecting with any spacelike Cauchy submanifold is not only a one-to-one correspondence but a diffeomorphism between Fréchet manifolds.

References

1. Baum, H., Müller, O.: Codazzi spinors and globally hyperbolic manifolds with special holonomy. Mathematische Zeitschrift (2005) (Submitted)
2. Bernal, A.N., Sánchez, M.: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. Comm. Math. Phys. **257**(1), 43–50 (2005)
3. Deck, T.: A geometric Cauchy problem for timelike minimal surfaces. Ann. Glob. Ana. Geom. **12**, 1–14 (1994)
4. Douglas, J.: The most general form of the problem of Plateau. Am. J. Math. **61**, 590–608 (1939)
5. Dugundji, J., Granas, A.: Fix Point Theorems. Springer-Verlag, New York (2003)
6. Gu, C.-H.: On the Cauchy problem for Harmonic maps defined on two-dimensional Minkowski space. Commun. Pure Appl. Math. **33**(6), 727–737 (1980)
7. Kulkarni, R.S.: An analogue of the Riemann mapping theorem for Lorentz metrics. Proc. R. Soc. Lon. A **401**, 117–130 (1985)
8. Larsen, J.C.: Flat maximal immersions. J. Geom. Phys. **18**, 118–130 (1996)
9. Morrey, C.B. Jr.: The problem of Plateau on a Riemannian Manifold. Ann. Math. **49**, 807–851 (1948)
10. O'Neill, B.: Semi-Riemannian Geometry. Academic Press, Inc, New York (1983)
11. Sanchez, M.: Further results on the smoothing of Cauchy hypersurfaces and Cauchy time functions. Lett. Math. Phys. **77**, 183–197 (2006)
12. Shatah, J., Struwe M.: Geometric Wave Equations. Courant Lecture Notes in Mathematics, chapter 5 (1998)
13. Walter, W.: Ordinary Differential Equations. Springer-Verlag, Berlin Heidelberg, New York (1998)