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A SCHOTTKY-TYPE THEOREM FOR STARLIKE DOMAINS IN BANACH SPACES

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Abstract. We show that if U is a starlike domain in a Banach space E and $\mathcal{F}$ is a family of holomorphic functions on U that omit two distinct values and is bounded at the origin, then $\mathcal{F}$ is uniformly bounded on each U-bounded set.

1. Introduction

Let E be a complex Banach space and let $\mathcal{H}(U)$ denote the vector space of all holomorphic functions on an open subset U of E.

Let Δ denote the open unit disc. The classical Schottky Theorem asserts that for each $0 < \alpha < \infty$ and $0 < \beta < 1$, there is a constant $c(\alpha, \beta) > 0$ such that, if $f \in \mathcal{H}(\Delta)$ is a function that omits the values 0 and 1 and satisfies $|f(0)| \leq \alpha$, then $|f(z)| \leq c(\alpha, \beta)$ for every $z \in \beta\Delta$. See Montel ([4], p. 86) or Carathéodory ([1], p. 201).

In a recent paper Paula Takatsuka used the classical Schottky Theorem to prove that, for each $0 < \alpha < \infty$ and $0 < \beta < 1$, there is a constant $c(\alpha, \beta) > 0$ such that, given a Banach space E (or even a locally convex space E), a point $x_0 \in E$ and a balanced open set $U \subset E$, if $f \in \mathcal{H}(x_0 + U)$ is a function that omits the values 0 and 1 and satisfies $|f(x_0)| \leq \alpha$, then $|f(x)| \leq c(\alpha, \beta)$ for every $x \in x_0 + \beta U$. See Takatsuka ([6], Corollary 15).

In Section 2 of this paper we obtain a Schottky-type theorem for starlike domains in Banach spaces. More precisely, we prove that if $U \subset E$ is a starlike domain and $\mathcal{F} \subset \mathcal{H}(U)$ is a family of holomorphic functions that omit the values 0 and 1 and is bounded at the origin, then $\mathcal{F}$ is uniformly bounded on each U-bounded set. In particular, $\mathcal{F}$ is contained and bounded in $\mathcal{H}_b(U)$, the space of all holomorphic functions of bounded type on U.

In Section 3 we obtain a Schottky-type theorem for arbitrary domains in Banach spaces. More precisely, we prove that if $U \subset E$ is a connected open set and $\mathcal{F} \subset \mathcal{H}(U)$ is a family of holomorphic functions that omit the values 0 and 1 and is bounded at some point, then $\mathcal{F}$ is uniformly bounded on each ball which lies strictly inside U. In particular, $\mathcal{F}$ is contained and bounded in $\mathcal{H}_d(U)$, the space of holomorphic functions recently studied by Dineen and Venkova [3].

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We refer to the books of Dineen [2] or Mujica [5] for background information on infinite-dimensional complex analysis.

2. A Schottky-type theorem in starlike domains

A set $U \subset E$ is said to be *starlike* if $\lambda x \in U$ for all $x \in U$ and $\lambda \in [0,1]$. U is said to be x_0-*starlike* if the set $U - x_0$ is starlike, that is, $(1-\lambda)x_0 + \lambda x \in U$ for all $x \in U$ and $\lambda \in [0,1]$. If U is x_0-balanced, then U is x_0-starlike. If U is convex, then U is x_0-starlike for each $x_0 \in U$.

If $U \subset E$ is open and $x \in U$, let $d_U(x)$ denote the distance from $x \in U$ to the boundary of U, that is:

$$d_U(x) = \sup\{r > 0 : B(x,r) \subset U\}.$$

Theorem 1. *Let U be a starlike open subset of a Banach space and let*

$$U_n := \{x \in U : \|x\| < n \quad \textit{and} \quad d_U(x) > 1/n\}$$

for every $n \in \mathbb{N}$. Then for each $0 < \alpha < \infty$, there is a sequence $\{c_n(\alpha)\}_{n=1}^{\infty}$ of positive constants such that, given a function $f \in \mathcal{H}(U)$ that omits the values 0 and 1 and satisfies $|f(0)| \leq \alpha$, then

$$|f(x)| < c_n(\alpha) \textit{ for all } x \in U_n \textit{ and } n \in \mathbb{N}.$$

Proof. For each n let $V_n := \text{st}(U_n) = \{\lambda x : x \in U_n,\ \lambda \in [0,1]\}$. Then $U_n \subset V_n \subset U$ and each V_n is a starlike open set.

Let n_0 be the first $n \in \mathbb{N}$ such that $0 \in U_n$. Fix $n \geq n_0$ and let $\varepsilon := \frac{1}{4n^3}$. We will show that

$$B(y, 2\varepsilon) \subset U \text{ for every } y \in V_n. \tag{1}$$

Indeed let $y = \lambda x$, with $x \in U_n$ and $\lambda \in [0,1]$. We distinguish two cases:

(i) Assume $0 \leq \lambda \leq \frac{1}{2n^2}$. Then, since $\|x\| \leq n$, it follows that $\|\lambda x\| \leq \frac{1}{2n}$, and therefore

$$B(\lambda x, 2\varepsilon) = B(\lambda x, \frac{1}{2n^3}) \subset B\Big(\lambda x, \frac{1}{2n}\Big) \subset B\Big(0, \frac{1}{n}\Big) \subset U.$$

(ii) Assume $\frac{1}{2n^2} \leq \lambda \leq 1$. Then, since $B(x, \frac{1}{n}) \subset U$, it follows that

$$U \supset \lambda\, B\Big(x, \frac{1}{n}\Big) = B\Big(\lambda x, \frac{\lambda}{n}\Big) \supset B\Big(\lambda x, \frac{1}{2n^3}\Big) = B(\lambda x, 2\varepsilon).$$

This proves (1). Note also that $V_n \subset B(0, m\varepsilon)$, where $m = 4n^4$.

Let f and α be as in the statement of the theorem. We will show the existence of constants $\alpha \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$ such that

$$|f(x)| \leq \alpha_k \text{ for all } x \in V_n \cap B(0, k\varepsilon),\ \ 1 \leq k \leq m. \tag{2}$$

Indeed, since $0 \in V_n$, $B(0, 2\varepsilon) \subset U$ and $|f(0)| \leq \alpha$, an application of [6], Corollary 15, yields a constant $\alpha_1 := c(\alpha, \frac{1}{2}) \geq \alpha$ such that

$$|f(x)| \leq \alpha_1 \text{ for all } x \in B(0, \varepsilon).$$

Now assume that the inequality (2) holds for some k, with $1 \leq k \leq m-1$. Let $x \in V_n \cap B(0, (k+1)\varepsilon)$, and let $y := \frac{k}{k+1}\, x$. Then $y \in V_n \cap B(0, k\varepsilon)$ and $x \in B(y, \varepsilon)$. Since $B(y, 2\varepsilon) \subset U$ and $|f(y)| \leq \alpha_k$, an application of [6], Corollary 15, yields a constant $\alpha_{k+1} \geq \alpha_k$ such that $|f(x)| \leq \alpha_{k+1}$. This proves (2).

Since $U_n \subset V_n \subset B(0, m\varepsilon)$, it follows that

$$|f(x)| \leq \alpha_m \text{ for all } x \in U_n.$$

Thus we may take $c_n(\alpha) := \alpha_m$ if $n \geq n_0$. If $n < n_0$, then $U_n \subset U_{n_0}$ and we may take $c_n(\alpha) := c_{n_0}(\alpha)$. This completes the proof. □

Corollary 2. *Let U be an x_0-starlike open subset of a Banach space and let*

$$U_n := \{x \in U : \|x - x_0\| < n \quad \textit{and} \quad d_U(x) > 1/n\}$$

for every $n \in \mathbb{N}$. Then for each $0 < \alpha < \infty$, there is a sequence $\{c_n(\alpha)\}_{n=1}^{\infty}$ of positive constants such that, given a function $f \in \mathcal{H}(U)$ that omits the values 0 and 1 and satisfies $|f(x_0)| \leq \alpha$, then

$$|f(x)| < c_n(\alpha) \textit{ for all } x \in U_n \textit{ and } n \in \mathbb{N}.$$

A set $A \subset U$ is said to be U-*bounded* if A is bounded in E and there exists $\varepsilon > 0$ such that $A + B(0, \varepsilon) \subset U$. Each U_n is U-bounded and each U-bounded set is contained in some U_n. $\mathcal{H}_b(U)$ denotes the space of all $f \in \mathcal{H}(U)$ which are bounded on all U-bounded sets. We endow $\mathcal{H}_b(U)$ with the topology of uniform convergence on all U-bounded sets. Then we easily get the following corollary.

Corollary 3. *Let U be an x_0-starlike open subset of a Banach space and let $\mathcal{F} \subset \mathcal{H}(U)$ be a family of functions that omit two distinct values. If the family $\mathcal{F}$ is bounded at the point $x_0 \in U$, then $\mathcal{F}$ is a bounded subset of $\mathcal{H}_b(U)$.*

3. A Schottky-type theorem in arbitrary domains

Theorem 4. *Let U be a connected open subset of a Banach space and let $x_0 \in U$. Let $0 < \alpha < \infty$. Then for each $a \in U$ and $0 < r < d_U(a)$, there is a constant $c(a, r, \alpha) > 0$ such that, given a function $f \in \mathcal{H}(U)$ that omits the values 0 and 1 and satisfies $|f(x_0)| \leq \alpha$, then*

$$|f(x)| \leq c(a, r, \alpha) \textit{ for all } x \in B(a, r).$$

Proof. If a is a point of U, let γ be a path in U from x_0 to a. Since the trace of γ is compact and connected, we can find a finite number of balls $B(x_0, r_0), B(x_1, r_1), \ldots, B(x_n, r_n) = B(a, d_U(a))$ in U with centers on the trace of γ such that x_{k+1} and x_k are in $B_{k+1} \cap B_k$ and $B(x_k, 2r_k) \subset U$ for $0 \leq k \leq n - 1$.

Applying [6], Corollary 15, on the ball $B(x_0, 2r_0)$ we obtain a constant $\alpha_0 := c(\alpha, \frac{1}{2})$ such that

$$|f(x)| \leq \alpha_0 \text{ for all } x \in B(x_0, r_0).$$

A second application of [6], Corollary 15, yields another constant $\alpha_1 := c(\alpha_0, \frac{1}{2})$ such that

$$|f(x)| \leq \alpha_1 \text{ for all } x \in B(x_1, r_1).$$

Continuing this way, for the last step of this inductive process we first notice that each $0 < r < d_U(a)$ is of the form $r = \beta\, d_U(a)$, with $0 < \beta < 1$, and since we have $|f(a)| \leq \alpha_{n-1}$, [6], Corollary 15, provides the constant $c(a, r, \alpha) := c(\alpha_{n-1}, \beta)$ such that

$$|f(x)| \leq c(a, r, \alpha) \text{ for all } x \in B(a, r).$$ □

Following [3] we denote by $\mathcal{H}_d(U)$ the space of all $f \in \mathcal{H}(U)$ which are bounded on all balls $B(a, r)$, with $a \in U$ and $0 < r < d_U(a)$. We endow $\mathcal{H}_d(U)$ with the topology of uniform convergence on all such balls. Clearly $\mathcal{H}_b(U) \subset \mathcal{H}_d(U) \subset \mathcal{H}(U)$, and Dineen and Venkova [3] have given examples where $\mathcal{H}_b(U) = \mathcal{H}_d(U)$ and where $\mathcal{H}_b(U) \neq \mathcal{H}_d(U)$. We then get the following corollary.

Corollary 5. *Let U be a connected open subset of a Banach space and let $\mathcal{F} \subset \mathcal{H}(U)$ be a family of functions that omit two distinct values. If the family $\mathcal{F}$ is bounded at some point $x_0 \in U$, then $\mathcal{F}$ is a bounded subset of $\mathcal{H}_d(U)$.*

4. Concluding remarks

If E is finite dimensional, then $\mathcal{H}_b(U) = \mathcal{H}_d(U)$ and the conclusions of Corollary 3 and Corollary 5 coincide. But if E is infinite dimensional, then $\mathcal{H}_b(U) \neq \mathcal{H}_d(U)$ in general, as Dineen and Venkova [3] have shown, and thus the conclusion of Corollary 3 is strictly stronger than the conclusion of Corollary 5. It would be important to extend the validity of Corollary 3 to other domains, for instance to polynomially convex domains.

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