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Necessary conditions for the existence of higher order extensions of univalent mappings from the disk to the ball

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ABSTRACT

If $G:\mathbb{C}^{n-1}\to\mathbb{C}$ is a holomorphic function such that $G(0)=0$ and $DG(0)=0$ and f is a normalized univalent mapping of the unit disk $\mathbb{D}\subseteq\mathbb{C}$, we consider the normalized extension of f to the Euclidean unit ball $\mathbb{B}\subseteq\mathbb{C}^n$ given by $\Phi_G(f)(z)=(f(z_1)+G(\sqrt{f'(z_1)}\,\hat z),\sqrt{f'(z_1)}\,\hat z)$, $z\in\mathbb{B}$, $\hat z=(z_2,\dots,z_n)$. While for a given f, $\Phi_G(f)$ will maintain certain geometric properties of f, such as convexity or starlikeness, if G is a polynomial of degree 2 of sufficiently small norm, these properties may be lost whenever G contains a nonzero term of higher degree. By establishing separate necessary and sufficient conditions for the extension of Loewner chains from $\mathbb{D}$ to $\mathbb{B}$ through Φ_G, we are able to completely classify those starlike and convex mappings f on $\mathbb{D}$ for which there exists a G with nonzero higher degree terms such that $\Phi_G(f)$ is a mapping of the same type on $\mathbb{B}$.

1. Introduction

Because of the increased intricacy inherent in the higher-dimensional setting, it is significantly more difficult to construct examples of biholomorphic mappings of the unit ball $\mathbb{B}=\mathbb{B}_n$ of the complex Euclidean space $\mathbb{C}^n$ whose images satisfy a given geometric condition than it is to construct such mappings of the unit disk $\mathbb{D}=\mathbb{B}_1$ of the complex plane $\mathbb{C}$. In 1995, Roper and Suffridge [13] gave the first systematic method of producing such mappings with their introduction of an operator that extends univalent mappings of $\mathbb{D}$ with convex range to such mappings of $\mathbb{B}$, thereby taking advantage of the well established understanding of mappings of $\mathbb{D}$. Subsequently, many other such operators have been considered, as well as numerous other geometric properties. Up to this point, most results have focused on sufficient conditions under which a certain operator successfully extends a particular geometric mapping. Here, we will consider some necessary conditions that must be satisfied by the operator and mapping for such a geometric extension to hold.

Let us consider some notation. The Euclidean norm and inner product of $\mathbb{C}^n$ are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. For $z\in\mathbb{C}^n$, we will find it convenient to write $z=(z_1,\hat z)$ for $z_1\in\mathbb{C}$ and $\hat z\in\mathbb{C}^{n-1}$.

Let $\mathcal{LS}_n$ denote the family of all locally biholomorphic mappings $f:\mathbb{B}_n\to\mathbb{C}^n$, normalized such that $f(0)=0$ and $Df(0)=I$, where Df is the Fréchet derivative of f and $I=I_n$ is the identity operator on $\mathbb{C}^n$. The subfamily $\mathcal{S}_n\subseteq\mathcal{LS}_n$ consists of biholomorphic mappings. We also consider the geometric families

$$\mathcal{S}_n^*=\{f\in\mathcal{S}_n:\ f(\mathbb{B}_n)\text{ is starlike with respect to }0\},$$
$$\mathcal{K}_n=\{f\in\mathcal{S}_n:\ f(\mathbb{B}_n)\text{ is convex}\}.$$

In the case of the unit disk ($n=1$), the family $\mathcal{S}_1$ is the classical family of schlicht mappings and has been well studied, as have the subfamilies $\mathcal{S}_1^*$ and $\mathcal{K}_1$ of starlike and convex mappings, respectively.

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If $\mathcal{F} \subseteq \mathcal{LS}_1$, then we say that $\Phi : \mathcal{F} \to \mathcal{LS}_n$ is an extension operator if Φ is continuous with respect to the compact-open topologies of $\mathcal{F}$ and $\mathcal{LS}_n$ and if

$$\Phi(f)(\zeta, \hat{0}) = \big(f(\zeta), \hat{0}\big), \quad f \in \mathcal{F},\ \zeta \in \mathbb{D}.$$

This is a modification of the definition given in [7].

We will focus our attention on the operator studied in [8], which is a special case of a class of operators introduced in [7]. Before defining this operator, we need to introduce notation concerning the space of Bloch functions. For an analytic function $f : \mathbb{D} \to \mathbb{C}$, the Bloch norm of f is

$$\|f\|_{\mathcal{B}} = \big|f(0)\big| + \sup_{\zeta \in \mathbb{D}} \big|f'(\zeta)\big|\big(1 - |\zeta|^2\big).$$

(We allow for $\|f\|_{\mathcal{B}} = \infty$.) The set of those f for which $\|f\|_{\mathcal{B}} < \infty$ is known as the Bloch space, denoted by $\mathcal{B}$, and is a Banach space under $\|\cdot\|_{\mathcal{B}}$. Because of normalization, $\|f\|_{\mathcal{B}} \in [1, \infty]$ for all $f \in \mathcal{LS}_1$. For $\rho \in [1, \infty]$, let $B(\rho) = \{w \in \mathbb{C}^{n-1}:\ \|w\| < \rho\}$ and $\mathcal{F}(\rho) = \{f \in \mathcal{LS}_1:\ \|f\|_{\mathcal{B}} \leqslant \rho^2\}$. Then for a holomorphic function $G : B(\rho) \to \mathbb{C}$ such that $G(0) = 0$ and $DG(0) = 0$, we define the extension operator $\Phi_G : \mathcal{F}(\rho) \to \mathcal{LS}_n$ by

$$\Phi_G(f)(z) = \Big(f(z_1) + G\big(\sqrt{f'(z_1)}\,\hat{z}\big), \sqrt{f'(z_1)}\,\hat{z}\Big), \quad z \in \mathbb{B}.$$

It is easy to check that Φ_G is well defined and that $\Phi_G(f)$ is biholomorphic whenever f is univalent. In particular, due to the normalization of f, we can always choose the analytic branch of $\sqrt{f'(\cdot)}$ such that $\sqrt{f'(0)} = 1$. Note that the normalization required of G ensures the normalization of $\Phi_G(f)$. In [7], it was shown, in the case $\rho = \infty$, that Φ_G is continuous, and hence is indeed an extension operator. This remains true for general ρ.

We note that when $G = 0$, Φ_0 is the Roper–Suffridge extension operator, defined on all of $\mathcal{LS}_1$, introduced in [13]. A modification of the Roper–Suffridge operator of this type was first considered in [9] in the case such that G is a homogeneous polynomial of degree 2. There, we proved the following.

Theorem A. *Let $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ be a homogeneous polynomial of degree 2. Then $\Phi_Q(\mathcal{K}_1) \subseteq \mathcal{K}_n$ if and only if $\|Q\| \leqslant 1/2$ and $\Phi_Q(\mathcal{S}_1^*) \subseteq \mathcal{S}_n^*$ if and only if $\|Q\| \leqslant 1/4$.*

For this result, we clearly have $\rho = \infty$. Recall that the norm of a homogeneous polynomial $P : \mathbb{C}^{n-1} \to \mathbb{C}$ of degree $k \in \mathbb{N}$ is

$$\|P\| = \sup_{u \in \partial\mathbb{B}_{n-1}} \big|P(u)\big| = \sup_{z \in \mathbb{C}^{n-1} \setminus \{0\}} \frac{|P(z)|}{\|z\|^k}.$$

In [8], we saw that having finite Bloch norm allows for the extension of convex and starlike mappings using Φ_G, for certain holomorphic functions G containing nontrivial terms of degree $\geqslant 3$ in their series expansions about 0.

Theorem B. *Let $f \in \mathcal{S}_1 \cap \mathcal{B}$, and let $G : B(\|f\|_{\mathcal{B}}^{1/2}) \to \mathbb{C}$ be holomorphic with homogeneous polynomial expansion $G = \sum_{k=2}^{\infty} P_k$. If $f \in \mathcal{K}_1$, then $\Phi_G(f) \in \mathcal{K}_n$ provided that*

$$\sum_{k=2}^{\infty} k(k-1)\|P_k\|\|f\|_{\mathcal{B}}^{(k-2)/2} \leqslant 1. \tag{1.1}$$

If $f \in \mathcal{S}_1^*$, then $\Phi_G(f) \in \mathcal{S}_n^*$ provided that

$$\sum_{k=2}^{\infty} (k-1)\|P_k\|\|f\|_{\mathcal{B}}^{(k-2)/2} \leqslant \frac{1}{4}. \tag{1.2}$$

In this article, we are interested in a general converse to Theorem B. That is, is having finite Bloch norm necessary for a given geometric property of a mapping to be preserved by the extension of that mapping using some Φ_G, where G has nonzero terms of degree $\geqslant 3$? It was shown in [8] that particular mappings in $\mathcal{K}_1 \setminus \mathcal{B}$ (specifically, the extreme points of the family of convex mappings of order $\alpha \in [0, 1/2)$) and $\mathcal{S}_1^* \setminus \mathcal{B}$ (specifically, the extreme points of the family of starlike mappings of order $\alpha \in [0, 1)$) do not respectively extend to convex and starlike mappings of $\mathbb{B}$ when G contains nonzero terms of degree $\geqslant 3$. While these special cases were dealt with directly using analytic conditions for convexity and starlikeness in $\mathbb{B}$, our work here will involve the extension of Loewner chains from $\mathbb{D}$ to $\mathbb{B}$. We shall see that the extensions of Loewner chains within the Bloch space behave in a manner similar to those of starlike mappings. Consequently, we will be able show the necessity of a finite Bloch norm for the existence of higher order extensions of starlike and convex mappings that inherit the respective geometric property.

We conclude the article with a note showing that the extension of a Loewner chain on $\mathbb{D}$ using any well-defined Φ_G has range equal to all of $\mathbb{C}^n$, a property not satisfied by all Loewner chains on $\mathbb{B}$.

2. Loewner chains and higher order extensions

We begin this section by reviewing some properties of Loewner chains on the disk and the ball.

For any $n \in \mathbb{N}$, a Loewner chain on $\mathbb{B}_n$ is a family $\{f_t\}_{t\geqslant 0}$ of biholomorphic mappings of $\mathbb{B}_n$ that satisfy $f_t(0)=0$ and $Df_t(0)=e^t I_n$ for all $t\geqslant 0$ and $f_s \prec f_t$ for all $t\geqslant s\geqslant 0$. The last condition is that f_s is subordinate to f_t, which, due to the normalization at 0, is equivalent to the requirement that $f_s(\mathbb{B}_n)\subseteq f_t(\mathbb{B}_n)$.

For a Loewner chain $\{f_t\}$ on $\mathbb{D}$, there is a function $p:\mathbb{D}\times[0,\infty)\to\mathbb{C}$ such that $p(\cdot,t)$ is analytic, $p(\zeta,\cdot)$ is (Lebesgue) measurable, $p(0,t)=1$, and $\operatorname{Re} p(\zeta,t)>0$ for all $\zeta\in\mathbb{D}$ and $t\geqslant 0$ and the Loewner differential equation

$$\frac{\partial}{\partial t} f_t(\zeta)=\zeta f_t'(\zeta)p(\zeta,t), \quad \zeta\in\mathbb{D},\ \text{a.e.}\ t\geqslant 0, \tag{2.1}$$

holds.

We see that the functions $p(\cdot,t)$, $t\geqslant 0$, described above belong to the Carathéodory class of normalized analytic functions on $\mathbb{D}$ with positive real part. In the ball, we consider the closely related family $\mathcal{M}$, consisting of all holomorphic functions $h:\mathbb{B}\to\mathbb{C}^n$ such that $h(0)=0$, $Dh(0)=I$, and $\operatorname{Re}\langle h(z),z\rangle>0$ for all $z\in\mathbb{B}\setminus\{0\}$. This family is compact in the compact-open topology. See [2,3].

The version of the Loewner differential equation on $\mathbb{B}=\mathbb{B}_n$, $n\geqslant 2$, for a Loewner chain $\{f_t\}$ on $\mathbb{B}$ is

$$\frac{\partial}{\partial t} f_t(z)=Df_t(z)h(z,t), \quad z\in\mathbb{B},\ \text{a.e.}\ t\geqslant 0, \tag{2.2}$$

where $h:\mathbb{B}\times[0,\infty)\to\mathbb{C}^n$ is such that $h(\cdot,t)\in\mathcal{M}$ for all $t\geqslant 0$ and $h(z,\cdot)$ is measurable for all $z\in\mathbb{B}$.

The converse to the above is the following theorem, which is a modification of a result of Pfaltzgraff [10].

Theorem C. *Let $\{f_t\}_{t\geqslant 0}$ be a family of holomorphic mappings on $\mathbb{B}$ such that $f_t(0)=0$ and $Df_t(0)=e^t I$ for all $t\geqslant 0$ and $t\mapsto f_t(z)$ is a locally absolutely continuous function of $t\in[0,\infty)$, locally uniformly with respect to $z\in\mathbb{B}$. Furthermore, assume that $h:\mathbb{B}\times[0,\infty)\to\mathbb{C}^n$ is such that $h(\cdot,t)\in\mathcal{M}$ for all $t\geqslant 0$ and $h(z,\cdot)$ is measurable for all $z\in\mathbb{B}$. If (2.2) holds, and if there exists an increasing sequence $\{t_m\}_{m=1}^{\infty}\subseteq[0,\infty)$ and a holomorphic function $g:\mathbb{B}\to\mathbb{C}^n$ such that $e^{-t_m}f_{t_m}\to g$ uniformly on compact subsets of $\mathbb{B}$ as $m\to\infty$, then $\{f_t\}$ is a Loewner chain on $\mathbb{B}$.*

A holomorphic mapping $f\in\mathcal{S}_n$ is said to have parametric representation if there exists a Loewner chain $\{f_t\}_{t\geqslant 0}$ on $\mathbb{B}$ such that $f_0=f$ and $\{e^{-t}f_t\}_{t\geqslant 0}$ is a normal family. The set of such mappings is denoted by $\mathcal{S}_n^0$. Because $\mathcal{S}_n^0=\mathcal{S}_n$ if and only if $n=1$, these mappings are of interest in higher dimensions.

A key theorem relating Loewner chains and starlike mappings is due to Pfaltzgraff and Suffridge [11].

Theorem D. *Let $f\in\mathcal{LS}_n$. Then $f\in\mathcal{S}_n^*$ if and only if $\{e^t f\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$.*

The reader is directed to the monograph of Graham and Kohr [3] or the article [4] for these and other results regarding Loewner chains on $\mathbb{B}$.

Using the definition in [7], we say that an extension operator Φ preserves Loewner chains if $\{e^t\Phi(e^{-t}f_t)\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$ whenever $\{f_t\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{D}$ such that $e^{-t}f_t$ is in the domain of Φ for all $t\geqslant 0$. It is an easy argument, using the continuity of Φ, to show that $\Phi(\mathcal{S}_1\cap\mathcal{F})\subseteq\mathcal{S}_n^0$ for any Loewner chain preserving extension operator $\Phi:\mathcal{F}\to\mathcal{LS}_n$, $\mathcal{F}\subseteq\mathcal{LS}_1$.

Consider the following definition.

Definition 2.1. Let $\{f_t\}_{t\geqslant 0}$ be a Loewner chain on $\mathbb{D}$. We say that $\{f_t\}$ *admits a higher order extension* if there exists a holomorphic function $G:B(\rho)\to\mathbb{C}$, where

$$\rho=\sup_{t\geqslant 0} e^{-t/2}\|f_t\|_{\mathcal{B}}^{1/2}\in[1,\infty],$$

such that $G(0)=0$, $DG(0)=0$, $D^kG(0)\neq 0$ for some $k\geqslant 3$, and $\{e^t\Phi_G(e^{-t}f_t)\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$.

In what follows, we shall see that Loewner chains on $\mathbb{D}$ consisting of Bloch functions admit higher order extensions under a condition similar to that given for starlike mappings in Theorem B. Furthermore, we shall consider conditions under which a converse to this result holds. Theorem D will easily extend this result to starlike mappings, and from that a complete converse to Theorem B will be found.

3. Higher order extensions of Loewner chains in the Bloch space

In this section, we show that a result similar to Theorem B holds for Loewner chains. Before giving the main result of the section, we perform some preliminary calculations involving extensions of Loewner chains.

Suppose that $\{f_t\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{D}$, and set

$$\rho = \sup_{t\geqslant 0} e^{-t/2}\|f_t\|_{\mathcal{B}}^{1/2} \in [1,\infty].$$

Let $G : B(\rho) \to \mathbb{C}$ be holomorphic with homogeneous polynomial expansion $G = \sum_{k=2}^{\infty} P_k$. For each $t \geqslant 0$, let $F_t = e^t \Phi_G(e^{-t} f_t)$. Then

$$F_t(z) = \Big(f_t(z_1) + e^t G\Big(e^{-t/2}\sqrt{f_t'(z_1)}\,\hat{z}\Big), e^{t/2}\sqrt{f_t'(z_1)}\,\hat{z}\Big), \quad z \in \mathbb{B},\ t \geqslant 0.$$

(Here and in what follows, branches of powers of analytic functions may always be chosen taking the principal value at the origin.) Clearly $F_t(0) = 0$ and $DF_t(0) = e^t I$ for all $t \geqslant 0$. For $\rho = \infty$, it was shown in the proof of [7, Theorem 3.2] that $t \mapsto F_t(z)$ is a locally Lipschitz continuous (and hence locally absolutely continuous) function of $t \in [0,\infty)$, locally uniformly with respect to $z \in \mathbb{B}$. That proof is unchanged for $\rho < \infty$.

Because $\mathcal{S}_1$ is compact, there is an increasing sequence $\{t_m\}_{m=1}^{\infty} \subseteq [0,\infty)$ and some $g \in \mathcal{S}_1$ such that $e^{-t_m} f_{t_m} \to g$ uniformly on compact subsets of $\mathbb{D}$. By continuity of Φ_G, it immediately follows that

$$e^{-t_m} F_{t_m} = \Phi_G\big(e^{-t_m} f_{t_m}\big) \to \Phi_G(g)$$

uniformly on compact subsets of $\mathbb{B}$.

A direct calculation (similar to the development of [8, (5.3)]) gives

$$\frac{DG(e^{-t/2}\sqrt{f_t'(z_1)}\,\hat{z})\hat{z}}{e^{-t/2}\sqrt{f_t'(z_1)}} - \frac{G(e^{-t/2}\sqrt{f_t'(z_1)}\,\hat{z})}{e^{-t} f_t'(z_1)} = \sum_{k=2}^{\infty}(k-1)e^{-(k-2)t/2}\big[f_t'(z_1)\big]^{(k-2)/2} P_k(\hat{z}) \tag{3.1}$$

for all $z \in \mathbb{B}$ and $t \geqslant 0$, showing the series on the right-hand side is convergent to a holomorphic function of z for all $t \geqslant 0$. Let $p : \mathbb{D} \times [0,\infty) \to \mathbb{C}$ be the function related to $\{f_t\}$ through the Loewner differential equation (2.1), and define $h : \mathbb{B} \times [0,\infty) \to \mathbb{C}^n$ by

$$\begin{aligned} h(z,t) = \Bigg(&z_1 p(z_1,t) - \sum_{k=2}^{\infty}(k-1)e^{-(k-2)t/2}\big[f_t'(z_1)\big]^{(k-2)/2} P_k(\hat{z}), \\ &\frac{1}{2}\Bigg[1 + p(z_1,t) + z_1 p'(z_1,t) + \sum_{k=2}^{\infty}(k-1)e^{-(k-2)t/2}\big[f_t'(z_1)\big]^{(k-4)/2} f_t''(z_1) P_k(\hat{z})\Bigg]\hat{z}\Bigg). \end{aligned} \tag{3.2}$$

From (3.1), we see that both series appearing in the definition of h are well defined (multiply by $f''(z_1)/f'(z_1)$ to obtain the latter), and thus for all $t \geqslant 0$, $h(\cdot,t)$ is holomorphic, $h(0,t) = 0$, and $Dh(0,t) = I$. In the proof of [7, Theorem 3.2], it was argued that $h(z,\cdot)$ is measurable for all $z \in \mathbb{B}$, and that the Loewner equation

$$\frac{\partial}{\partial t} F_t(z) = DF_t(z) h(z,t), \quad z \in \mathbb{B},\ \text{a.e. } t \geqslant 0, \tag{3.3}$$

is satisfied.

We now see, using Theorem C and the remarks preceding it, that $\{F_t\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$ if and only if $\operatorname{Re}\langle h(z,t), z\rangle > 0$ for all $z \in \mathbb{B}\setminus\{0\}$ and $t \geqslant 0$.

We now give the main result of the section, which states that every Loewner chain $\{f_t\}_{t\geqslant 0}$ on $\mathbb{D}$ such that $\{e^{-t} f_t\}_{t\geqslant 0}$ is bounded in the Bloch norm admits a higher order extension.

Theorem 3.1. *Let $\{f_t\}_{t\geqslant 0}$ be a Loewner chain on $\mathbb{D}$, set*

$$\rho = \sup_{t\geqslant 0} e^{-t/2}\|f_t\|_{\mathcal{B}}^{1/2} \in [1,\infty],$$

and let $G : B(\rho) \to \mathbb{C}$ be holomorphic with homogeneous polynomial expansion $G = \sum_{k=2}^{\infty} P_k$. Then $\{e^t \Phi_G(e^{-t} f_t)\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$ provided that

$$\sum_{k=2}^{\infty}(k-1)\|P_k\| e^{-(k-2)t/2}\|f_t\|_{\mathcal{B}}^{(k-2)/2} \leqslant \frac{1}{4}, \quad t \geqslant 0. \tag{3.4}$$

In particular, $\{f_t\}$ admits a higher order extension whenever $\rho < \infty$.

Note that even if $\{f_t\} \subseteq \mathcal{B}$, it could still be that $\rho = \infty$. Even so, this gives that $\{e^t \Phi_Q(e^{-t} f_t)\}$ is a Loewner chain on $\mathbb{B}$ when Q is homogeneous of degree 2 with $\|Q\| \leqslant 1/4$. This is known to hold for all Loewner chains on $\mathbb{D}$, as proved independently in [5,7]. When $\rho < \infty$, then $\{f_t\} \subseteq \mathcal{B}$, and the condition (3.4) becomes

$$\sum_{k=2}^{\infty}(k-1)\|P_k\|\rho^{k-2} \leqslant \frac{1}{4}.$$

Before beginning the proof, we introduce the function

$$A_f(\zeta) = \frac{1-|\zeta|^2}{2}\frac{f''(\zeta)}{f'(\zeta)} - \bar{\zeta}, \quad \zeta \in \mathbb{D},$$

for a univalent function $f : \mathbb{D} \to \mathbb{C}$. A simple calculation reveals that if one chooses $\zeta \in \mathbb{D}$ and disk automorphism

$$\varphi(w) = \frac{w+\zeta}{1+\bar{\zeta}w}, \quad w \in \mathbb{D},$$

then the Koebe transform of f with respect to φ is

$$g(w) = \frac{f(\varphi(w)) - f(\varphi(0))}{f'(\varphi(0))\varphi'(0)} = w + A_f(\zeta)w^2 + O\big(|w|^3\big).$$

Since $g \in \mathcal{S}_1$, $|A_f(\zeta)| \leqslant 2$ for all $\zeta \in \mathbb{D}$. An immediate consequence of this is the inequality

$$\big(1-|\zeta|^2\big)\left|\frac{f''(\zeta)}{f'(\zeta)}\right| \leqslant 6, \quad \zeta \in \mathbb{D}. \tag{3.5}$$

Proof of Theorem 3.1. Fix $t \geqslant 0$. We must show that $h(\cdot,t) \in \mathcal{M}$. To that end, for each $m = 2,3,\ldots,$ define $h_m : \mathbb{B} \to \mathbb{C}^n$ by

$$\begin{aligned} h_m(z) = \Bigg(& z_1 p(z_1,t) - \sum_{k=2}^{m}(k-1)e^{-(k-2)t/2}\big[f_t'(z_1)\big]^{(k-2)/2}P_k(\hat{z}), \\ & \frac{1}{2}\left[1 + p(z_1,t) + z_1 p'(z_1,t) + \sum_{k=2}^{m}(k-1)e^{-(k-2)t/2}\big[f_t'(z_1)\big]^{(k-4)/2} f_t''(z_1)P_k(\hat{z})\right]\hat{z}\Bigg). \end{aligned}$$

By using the bound $|P_k(\hat{z})| \leqslant \|P_k\|(1-|z_1|^2)^{k/2}$ for $z \in \mathbb{B}$ and the inequalities (3.4) and (3.5), we see that $\|h_m(z)\|$ is bounded by a continuous positive function of z_1 for all $z \in \mathbb{B}$, independent of m, and hence is locally uniformly bounded. Since $h_m(z) \to h(z,t)$ as $m \to \infty$ for all $z \in \mathbb{B}$, Vitali's Theorem in several complex variables [3] then gives that $h_m \to h(\cdot,t)$ uniformly on compact subsets of $\mathbb{B}$. Because $\mathcal{M}$ is a compact family, the result will follow if we show $h_m \in \mathcal{M}$ for an arbitrary m. Since $h_m(0) = 0$ and $Dh_m(0) = I$, it remains to show that $\operatorname{Re}\langle h_m(z), z\rangle > 0$ for all $z \in \mathbb{B} \setminus \{0\}$.

First note that if $z \in \mathbb{B} \setminus \{0\}$ is such that $\hat{z} = 0$, then

$$\operatorname{Re}\big\langle h_m(z), z\big\rangle = |z_1|^2 \operatorname{Re} p(z_1,t) > 0.$$

It therefore remains to consider $z \in \mathbb{B}$ such that $\hat{z} \neq 0$.

Observe that h_m can be holomorphically extended to a neighborhood of any $u \in \partial\mathbb{B}$ such that $\hat{u} \neq 0$. Write $z = \lambda u$ for such a u, $\lambda \in \mathbb{D} \setminus \{0\}$. Then $\operatorname{Re}\langle h_m(z), z\rangle > 0$ if and only if

$$\operatorname{Re}\left\langle \frac{h_m(\lambda u)}{\lambda}, u\right\rangle > 0.$$

The left-hand side of the above is the real part of an analytic function of the complex variable $\lambda \in \overline{\mathbb{D}}$ with value 1 at $\lambda = 0$. By the Minimum Principle for harmonic functions, it is sufficient to consider $\lambda \in \partial\mathbb{D}$, and thus we need only show that $\operatorname{Re}\langle h_m(z), z\rangle \geqslant 0$ for $z \in \partial\mathbb{B}$ with $\hat{z} \neq 0$. Fix such a z.

We calculate

$$\begin{aligned} \operatorname{Re}\big\langle h_m(z), z\big\rangle = {} & \frac{1}{2}\Big(\big(1-|z_1|^2\big)\big(1 + \operatorname{Re}\big[z_1 p'(z_1,t)\big]\big) + \big(1+|z_1|^2\big)\operatorname{Re} p(z_1,t)\Big) \\ & + \operatorname{Re}\left(A_{f_t}(z_1)\sum_{k=2}^{m}(k-1)\big[e^{-t}f_t'(z_1)\big]^{(k-2)/2}P_k(\hat{z})\right). \end{aligned} \tag{3.6}$$

By applying the bound [3, (2.1.6)]

$$\big|p'(z_1,t)\big| \leqslant \frac{2\operatorname{Re} p(z_1,t)}{1-|z_1|^2},$$

we see that to prove $\operatorname{Re}\langle h_m(z), z\rangle \geqslant 0$, it suffices to show

$$\left|A_{f_t}(z_1)\right| \sum_{k=2}^{\infty}(k-1)\left|e^{-t} f_t'(z_1)\right|^{(k-2)/2}\left(1-|z_1|^2\right)^{k/2}\|P_k\| \leqslant \frac{1}{2}\left(1-|z_1|^2\right)+\frac{1}{2}\left(1-|z_1|\right)^2 \operatorname{Re} p(z_1, t).$$

Using that $|A_{f_t}(z_1)| \leqslant 2$, we see that it remains to verify

$$\sum_{k=2}^{\infty}(k-1)\left|e^{-t} f_t'(z_1)\left(1-|z_1|^2\right)\right|^{(k-2)/2}\|P_k\| \leqslant \frac{1}{4}\left(1+\frac{1-|z_1|}{1+|z_1|} \operatorname{Re} p(z_1, t)\right).$$

This is implied by (3.4). □

Note that if $e^{-t/2}\|f_t\|_{\mathcal{B}}^{1/2} = \rho < \infty$ and G does not holomorphically extend to a ball of radius greater than ρ, then $h(\cdot, t)$ may not holomorphically extend to a neighborhood of each $u \in \partial\mathbb{B}$ with $\hat{u} \neq 0$. Passing to h_m therefore enabled our use of the Minimum Principle.

The following corollary gives the natural consequence of Theorem 3.1 for mappings with parametric representation in $\mathbb{B}$ and shows that the extension of starlike mappings given by Theorem B can be alternatively proved using Theorem 3.1.

Corollary 3.2. *Let $\{f_t\}$, ρ, and G be as in Theorem 3.1, and suppose that (3.4) holds. Then $\Phi_G(f_0) \in \mathcal{S}_n^0$. Furthermore, $\Phi_G(f) \in \mathcal{S}_n^*$ for every $f \in \mathcal{S}_1^* \cap \mathcal{B}$ such that (1.2) holds.*

Proof. We see that $\{\Phi_G(e^{-t} f_t)\}_{t\geqslant 0}$ is a normal family because it lies in the compact set $\Phi_G(\mathcal{S}_1)$, and hence $\Phi_G(f_0) \in \mathcal{S}_n^0$.

The result for $f \in \mathcal{S}_1^* \cap \mathcal{B}$ follows from Theorem 3.1 by a simple application of Theorem D. □

4. Necessary conditions for higher order extensions

In this section, we examine some conditions on Loewner chains on $\mathbb{D}$ that prevent the chain from admitting a higher order extension. We begin with a simple lemma.

Lemma 4.1. *Let $g : \mathbb{D} \to \mathbb{C}$ be analytic with $g(0) = 0$ and $\operatorname{Re} g(\zeta) > -c$ for all $\zeta \in \mathbb{D}$ and some constant $c > 0$. If $g(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^k$, $\zeta \in \mathbb{D}$, then $|a_k| \leqslant 2c$ for all $k \in \mathbb{N}$.*

Proof. Let

$$h(\zeta) = \frac{g(\zeta)}{2c} = \sum_{k=1}^{\infty} \frac{a_k}{2c}\zeta^k, \quad \zeta \in \mathbb{D}.$$

Then h is subordinate to the normalized convex mapping $\varphi(\zeta) = \zeta/(1-\zeta)$, $\zeta \in \mathbb{D}$, and hence $|a_k/(2c)| \leqslant 1$ for all k by [1, Theorem 6.4]. □

The first step in our work is noting that if Φ_G extends a Loewner chain from $\mathbb{D}$ to $\mathbb{B}$, then so does $\Phi_{P_k/2}$ for any term P_k in the homogeneous polynomial expansion of the holomorphic function G.

Theorem 4.2. *Let $\{f_t\}_{t\geqslant 0}$ be a Loewner chain on $\mathbb{D}$, set*

$$\rho = \sup_{t\geqslant 0} e^{-t/2}\|f_t\|_{\mathcal{B}}^{1/2} \in [1, \infty],$$

and suppose that $G : B(\rho) \to \mathbb{C}$ is holomorphic with homogeneous polynomial expansion $G = \sum_{k=2}^{\infty} P_k$ and is such that $\{e^t \Phi_G(e^{-t} f_t)\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$. Then $\{e^t \Phi_{P_k/2}(e^{-t} f_t)\}_{t\geqslant 0}$ is also a Loewner chain on $\mathbb{B}$, for each $k = 2, 3, \ldots$.

Proof. From our work in Section 3, we see that (3.3) holds for h given in (3.2) and $F_t = e^t \Phi_G(e^{-t} f_t)$, $t \geqslant 0$. Now

$$\begin{aligned}\operatorname{Re}\langle h(z,t), z\rangle &= |z_1|^2 \operatorname{Re} p(z_1, t) + \frac{\|\hat{z}\|^2}{2}\left(1 + \operatorname{Re} p(z_1, t) + \operatorname{Re}\left[z_1 p'(z_1, t)\right]\right) \\ &\quad + \operatorname{Re}\left(\left(\frac{\|\hat{z}\|^2 f_t''(z_1)}{2 f_t'(z_1)} - \bar{z}_1\right) \sum_{k=2}^{\infty}(k-1)\left[e^{-t} f_t'(z_1)\right]^{(k-2)/2} P_k(\hat{z})\right) > 0 \qquad (4.1)\end{aligned}$$

for all $z \in \mathbb{B} \setminus \{0\}$.

Fix $z \in \mathbb{B} \setminus \{0\}$, and define $g : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$g(\lambda) = \left(\frac{\|\hat{z}\|^2 f_t''(z_1)}{2 f_t'(z_1)} - \bar{z}_1 \right) \sum_{k=2}^{\infty} (k-1) \left[e^{-t} f_t'(z_1) \right]^{(k-2)/2} P_k(\hat{z}) \lambda^k.$$

Clearly g is continuous and analytic in $\mathbb{D}$. (We note that g is well defined, because the series in (4.1) converges for $\lambda\hat{z}$ in place of $\hat{z}$.) Then by replacing z with $(z_1, \lambda\hat{z})$, $|\lambda| = 1$, in (4.1), we have

$$\operatorname{Re} g(\lambda) > -|z_1|^2 \operatorname{Re} p(z_1, t) - \frac{\|\hat{z}\|^2}{2} \left(1 + \operatorname{Re} p(z_1, t) + \operatorname{Re}\left[z_1 p'(z_1, t) \right] \right) \tag{4.2}$$

for all $\lambda \in \partial\mathbb{D}$. The Minimum Principle for harmonic functions then implies that (4.2) holds for all $\lambda \in \mathbb{D}$. Since $g(0) = 0$, the right-hand side of (4.2) is negative. Lemma 4.1 then gives that for all $k = 2, 3, \ldots,$

$$(k-1) \left| \frac{\|\hat{z}\|^2 f_t''(z_1)}{2 f_t'(z_1)} - \bar{z}_1 \right| \left| e^{-t} f_t'(z_1) \right|^{(k-2)/2} \left| P_k(\hat{z}) \right| \leqslant 2|z_1|^2 \operatorname{Re} p(z_1, t) + \|\hat{z}\|^2 \left(1 + \operatorname{Re} p(z_1, t) + \operatorname{Re}\left[z_1 p'(z_1, t) \right] \right).$$

Because z was chosen arbitrarily, the above holds for all $z \in \mathbb{B} \setminus \{0\}$. Furthermore, because both sides extend continuously to $\{z \in \partial\mathbb{B}\colon\ \hat{z} \neq 0\}$, we have

$$(k-1) \left| A_{f_t}(z_1) \right| \left| e^{-t} f_t'(z_1) \right|^{(k-2)/2} \left| \frac{P_k(\hat{z})}{2} \right| \leqslant \frac{1}{2} \left(\left(1 - |z_1|^2 \right) \left(1 + \operatorname{Re}\left[z_1 p'(z_1, t) \right] \right) + \left(1 + |z_1|^2 \right) \operatorname{Re} p(z_1, t) \right) \tag{4.3}$$

for $z \in \partial\mathbb{B}$ such that $\hat{z} \neq 0$. With this in place, we can prove that $\{e^t \Phi_{P_k/2}(e^{-t} f_t)\}$ is a Loewner chain on $\mathbb{B}$ using the proof of Theorem 3.1 with G replaced with $P_k/2$, noting that (4.3) shows that (3.6) is nonnegative in that proof. □

Before giving the principal result of the section, we recall that for nonempty $E \subseteq \mathbb{C}$ and $a \in \mathbb{C}$, the distance between a and E is

$$\operatorname{dist}(a, E) = \inf_{w \in E} |a - w|.$$

The following useful bounds hold for any univalent mapping $f : \mathbb{D} \to \mathbb{C}$ and help to give a geometric interpretation of univalent functions in the Bloch space. (See [12, Corollary 1.4].)

$$\operatorname{dist}\left(f(\zeta), \partial f(\mathbb{D}) \right) \leqslant \left| f'(\zeta) \right| \left(1 - |\zeta|^2 \right) \leqslant 4 \operatorname{dist}\left(f(\zeta), \partial f(\mathbb{D}) \right), \quad \zeta \in \mathbb{D}. \tag{4.4}$$

Indeed, we see that $f \notin \mathcal{B}$ if and only if for all $r > 0$, $f(\mathbb{D})$ contains a disk of radius r.

We now give the following result, which identifies a large class of Loewner chains on $\mathbb{D}$ that do not admit a higher order extension.

Theorem 4.3. *Suppose that $\{f_t\}_{t \geqslant 0}$ is a Loewner chain on $\mathbb{D}$ such that for some $t > s \geqslant 0$, there exists a sequence $\{\alpha_m\}_{m=1}^{\infty} \subseteq f_s(\mathbb{D})$ such that*

$$\lim_{m \to \infty} \operatorname{dist}\left(\alpha_m, \partial f_s(\mathbb{D}) \right) = \infty, \qquad \sup_{m \in \mathbb{N}} \frac{\operatorname{dist}(\alpha_m, \partial f_t(\mathbb{D}))}{\operatorname{dist}(\alpha_m, \partial f_s(\mathbb{D}))} < \infty. \tag{4.5}$$

Let $G : \mathbb{C}^{n-1} \to \mathbb{C}$ be holomorphic such that $G(0) = 0$ and $DG(0) = 0$. If $\{e^t \Phi_G(e^{-t} f_t)\}_{t \geqslant 0}$ is a Loewner chain on $\mathbb{B}$, then G is a homogeneous polynomial of degree 2.

As noted following the statement of Theorem 3.1, Φ_Q extends any Loewner chain when Q is a homogeneous polynomial of degree 2 satisfying $\|Q\| \leqslant 1/4$. Therefore the interesting point of this theorem is that no terms of degree $\geqslant 3$ can appear in the series expansion of G.

Observe that, for s given in the statement of the theorem, the limit on the left side of (4.5) alone is equivalent to the condition $f_s \notin \mathcal{B}$. Due to subordination and the above geometric interpretation of univalent mappings in $\mathcal{B}$, we see that $f_t \notin \mathcal{B}$ for all $t \geqslant s$. For this reason, it is only worth considering G defined on all of $\mathbb{C}^{n-1}$.

Proof. By Theorem 4.2, it suffices to show that for any homogeneous polynomial $P : \mathbb{C}^{n-1} \to \mathbb{C}$ of degree $k \geqslant 3$, for $\{e^t \Phi_P(e^{-t} f_t)\}_{t \geqslant 0}$ to be a Loewner chain on $\mathbb{B}$ requires $P = 0$. For $t \geqslant 0$, we write $F_t = e^t \Phi_P(e^{-t} f_t)$, so that

$$F_t(z) = \left(f_t(z_1) + e^{-(k-2)t/2} \left[f_t'(z_1) \right]^{k/2} P(\hat{z}), e^{t/2} \sqrt{f_t'(z_1)}\, \hat{z} \right), \quad z \in \mathbb{B}.$$

Let $t > s \geqslant 0$ be as given in the statement of the theorem. Choose $w \in F_t(\mathbb{B})$, and write $w = F_t(z)$ for $z \in \mathbb{B}$. Then

$$e^{-(k-1)t} P(\hat{w}) = e^{-(k-2)t/2} \left[f_t'(z_1) \right]^{k/2} P(\hat{z}).$$

It follows that $w_1 = f_t(z_1) + e^{-(k-1)t} P(\hat{w})$. In particular, we find that $w \in F_t(\mathbb{B})$ implies

$$w_1 - e^{-(k-1)t} P(\hat{w}) \in f_t(\mathbb{D}).$$

By subordination, this must hold for $w = F_s(z)$, $z \in \mathbb{B}$, and thus we have

$$f_s(z_1) + e^{-(k-2)s/2}\big(1 - e^{(k-1)(s-t)}\big)\big[f_s'(z_1)\big]^{k/2} P(\hat{z}) \in f_t(\mathbb{D}), \quad z \in \mathbb{B}.$$

For a given $z \in \mathbb{B}$, replace $\hat{z}$ with $e^{i\theta} u$, where $u \in \mathbb{C}^{n-1}$ is such that $\|u\| = \|\hat{z}\|$ and $P(u) = \|P\| \|\hat{z}\|^k$. Then we have

$$f_s(z_1) + e^{ik\theta} e^{-(k-2)s/2}\big(1 - e^{(k-1)(s-t)}\big)\big[f_s'(z_1)\big]^{k/2} \|P\| \|\hat{z}\|^k \in f_t(\mathbb{D}), \quad z \in \mathbb{B},\ \theta \in \mathbb{R}.$$

Now fix $z_1 \in \mathbb{D}$, allow $\theta \in \mathbb{R}$ to vary, and take the limit as $\|\hat{z}\| \to \sqrt{1 - |z_1|^2}$ to see that

$$e^{-(k-2)s/2}\big(1 - e^{(k-1)(s-t)}\big)\big|f_s'(z_1)\big|^{k/2} \|P\| \big(1 - |z_1|^2\big)^{k/2} \leqslant \operatorname{dist}\big(f_s(z_1), \partial f_t(\mathbb{D})\big).$$

It follows that

$$e^{-(k-2)s/2}\big(1 - e^{(k-1)(s-t)}\big)\big[\operatorname{dist}\big(f_s(z_1), \partial f_s(\mathbb{D})\big)\big]^{(k-2)/2} \|P\| \leqslant \frac{\operatorname{dist}(f_s(z_1), \partial f_t(\mathbb{D}))}{\operatorname{dist}(f_s(z_1), \partial f_s(\mathbb{D}))}$$

for all $z_1 \in \mathbb{D}$. Under our hypotheses, the right-hand side is bounded for $f_s(z_1) = \alpha_m$, $m \in \mathbb{N}$, while the left-hand side tends to ∞ as $m \to \infty$ for $f_s(z_1) = \alpha_m$ if $P \neq 0$, because $k \geqslant 3$. Hence $P = 0$. □

Amongst those Loewner chains that do not admit a higher order extension, those that satisfy (4.5) constitute a substantial family. Here we give several corollaries that help to illustrate this.

Corollary 4.4. *Suppose that $\{f_t\}_{t \geqslant 0}$ is a Loewner chain on $\mathbb{D}$ such that for some $s \geqslant 0$, there is a sequence $\{\alpha_m\}_{m=1}^{\infty} \subseteq f_s(\mathbb{D})$ such that $|\alpha_m| \to \infty$ as $m \to \infty$ and*

$$\operatorname{dist}\big(\alpha_m, \partial f_s(\mathbb{D})\big) \geqslant c|\alpha_m|, \quad m \in \mathbb{N},$$

for some constant $c > 0$. Let $G : \mathbb{C}^{n-1} \to \mathbb{C}$ be holomorphic such that $G(0) = 0$ and $DG(0) = 0$. If $\{e^t \Phi_G(e^{-t} f_t)\}_{t \geqslant 0}$ is a Loewner chain on $\mathbb{B}$, then G is a homogeneous polynomial of degree 2.

Proof. Let $t > s$ and choose $w \in \partial f_t(\mathbb{D})$. Then for $m \in \mathbb{N}$,

$$\frac{\operatorname{dist}(\alpha_m, \partial f_t(\mathbb{D}))}{\operatorname{dist}(\alpha_m, \partial f_s(\mathbb{D}))} \leqslant \frac{|\alpha_m - w|}{c|\alpha_m|} \leqslant \frac{1}{c} + \frac{|w|}{c|\alpha_m|}.$$

The right-hand side tends to $1/c$ as $m \to \infty$, showing that the left-hand side is bounded. The result now follows from Theorem 4.3. □

For $a \in \mathbb{C}$, $\theta \in \mathbb{R}$, and $\varepsilon \in (0, \pi)$, define the infinite wedge based at a in the direction of θ with angular radius ε by

$$W(a; \theta, \varepsilon) = \big\{\zeta \in \mathbb{C} \setminus \{a\}:\ \big|\arg(\zeta - a) - \theta\big| < \varepsilon\big\}.$$

We will now show that if the image of any term of a Loewner chain contains such an infinite wedge, then the chain does not admit a higher order extension.

Corollary 4.5. *Suppose that $\{f_t\}_{t \geqslant 0}$ is a Loewner chain on $\mathbb{D}$ such that for some $s \geqslant 0$, $f_s(\mathbb{D})$ contains an infinite wedge, and let $G : \mathbb{C}^{n-1} \to \mathbb{C}$ be holomorphic such that $G(0) = 0$ and $DG(0) = 0$. If $\{e^t \Phi_G(e^{-t} f_t)\}_{t \geqslant 0}$ is a Loewner chain on $\mathbb{B}$, then G is a homogeneous polynomial of degree 2.*

Proof. Let $a \in \mathbb{C}$, $\theta \in \mathbb{R}$, and $\varepsilon \in (0, \pi)$ be such that $W(a; \theta, \varepsilon) \subseteq f_s(\mathbb{D})$. Choose $\alpha_m = a + me^{i\theta}$ for $m \in \mathbb{N}$. Then $\{\alpha_m\}_{m=1}^{\infty} \subseteq W(a; \theta, \varepsilon)$ and $|\alpha_m| \to \infty$ as $m \to \infty$. Since

$$\operatorname{dist}\big(\alpha_m, \partial f_s(\mathbb{D})\big) \geqslant m \sin \varepsilon \geqslant \frac{m + |a|}{1 + |a|} \sin \varepsilon \geqslant |\alpha_m| \frac{\sin \varepsilon}{1 + |a|}$$

for all $m \in \mathbb{N}$, the result follows from Corollary 4.4. □

For our last case, we recall that the original development [6] of Loewner involved the parametric representation of single-slit mappings of $\mathbb{D}$. (See the presentation in [1], for instance.) That is, mappings $f \in \mathcal{S}_1$ such that $\mathbb{C} \setminus f(\mathbb{D})$ consists of a single Jordan arc Γ that extends to ∞. As is the case with all mappings in $\mathcal{S}_1$, f can be embedded as $f = f_0$ in a Loewner chain $\{f_t\}_{t \geqslant 0}$ on $\mathbb{D}$. Furthermore, it is easy to see that for each $t > 0$, $f_t(\mathbb{D}) = \mathbb{C} \setminus \Gamma_t$, where Γ_t is a proper subarc of Γ extending to ∞. We now have the following.

Corollary 4.6. *Let $f \in \mathcal{S}_1 \setminus \mathcal{B}$ be a single-slit mapping with associated Loewner chain $\{f_t\}_{t\geqslant 0}$, and let $G : \mathbb{C}^{n-1} \to \mathbb{C}$ be holomorphic such that $G(0) = 0$ and $DG(0) = 0$. If $\{e^t \Phi_G(e^{-t} f_t)\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$, then G is a homogeneous polynomial of degree 2.*

Proof. For $t > 0$, let Γ and Γ_t be as above. Then $\Gamma \setminus \Gamma_t$ lies in some disk of radius $r > 0$. Choose $\{\alpha_m\}_{m=1}^{\infty} \subseteq f(\mathbb{D})$ such that $\operatorname{dist}(\alpha_m, \partial f(\mathbb{D})) = \operatorname{dist}(\alpha_m, \Gamma) \to \infty$ as $m \to \infty$. It is not hard to see that

$$\frac{\operatorname{dist}(\alpha_m, \partial f_t(\mathbb{D}))}{\operatorname{dist}(\alpha_m, \partial f(\mathbb{D}))} = \frac{\operatorname{dist}(\alpha_m, \Gamma_t)}{\operatorname{dist}(\alpha_m, \Gamma)} \leqslant \frac{\operatorname{dist}(\alpha_m, \Gamma) + 2r}{\operatorname{dist}(\alpha_m, \Gamma)} \to 1$$

as $m \to \infty$, showing that the left-hand side is bounded. The result then follows from Theorem 4.3. □

5. Consequences for the extension of starlike and convex mappings

In this section, we will observe that Theorem D and Theorem 4.3 combine to provide a converse to Theorem B. In other words, we shall see that if a starlike (and hence convex) mapping of $\mathbb{D}$ does not lie in the Bloch space, then it cannot be extended to a starlike mapping of $\mathbb{B}$ using Φ_G if the holomorphic function G has a nonzero term of degree $\geqslant 3$ in its homogeneous polynomial expansion.

Some commonly considered mappings in $\mathcal{S}_1^* \setminus \mathcal{B}$ are the radial (multi-)slit mappings, and for such mappings, the desired converse follows immediately from Corollary 4.5. However, even a normalized univalent mapping of $\mathbb{D}$ onto the convex connected component of the complement of a parabola is such that Corollaries 4.4 and 4.5 fail to apply. Indeed, to help put the converse to Theorem B into context, it is useful to consider the following example of how uncooperative mappings in $\mathcal{S}_1^* \setminus \mathcal{B}$ can be.

Example 5.1. Let $\{\theta_m\}_{m=0}^{\infty} \subseteq [0, 2\pi)$ be an increasing sequence such that $\theta_m \to 2\pi$ as $m \to \infty$. For each $m \in \mathbb{N}$, define

$$W_m = \left\{re^{i\theta}\colon\ 0 < r < R_m,\ \theta_{m-1} < \theta < \theta_m\right\},$$

where $R_m > 0$ is chosen large enough so that W_m contains a disk of radius m. Now let

$$\Omega = \mathbb{D} \cup \bigcup_{m=1}^{\infty} W_m.$$

Then Ω is a simply connected domain, and hence there is a Riemann map g of $\mathbb{D}$ onto Ω such that $g(0) = 0$ and $g'(0) > 0$. Let $f = g/g'(0)$. Then $f \in \mathcal{S}_1$ is a mapping of $\mathbb{D}$ onto a dilation of Ω. Thus $f \in \mathcal{S}_1^* \setminus \mathcal{B}$. Despite being an unbounded starlike domain, $f(\mathbb{D})$ fails to contain a ray, much less an infinite wedge. Furthermore, although $\{|f'(\zeta)|(1 - |\zeta|^2)\colon\ \zeta \in \mathbb{D}\}$ is unbounded, there does not exist a curve $\gamma : [0, \infty) \to \mathbb{D}$ such that

$$\lim_{t\to\infty} \left|f'\big(\gamma(t)\big)\right|\left(1 - \left|\gamma(t)\right|^2\right) = \infty.$$

Indeed, if γ is such that $|f'(\gamma(t_k))|(1 - |\gamma(t_k)|^2) \to \infty$ as $k \to \infty$ for some sequence $\{t_k\}_{k=1}^{\infty} \subseteq [0, \infty)$, then there must also exist a sequence $\{s_k\}_{k=1}^{\infty} \subseteq [0, \infty)$ such that $s_k \to \infty$ as $k \to \infty$ and $|f'(\gamma(s_k))|(1 - |\gamma(s_k)|^2) \leqslant 4/g'(0)$ for all $k \in \mathbb{N}$, using (4.4).

Our key step in obtaining a converse to Theorem B is given in the following lemma. We use the common notation $2A$ for a set $A \subseteq \mathbb{C}$ to mean the set $\{2w\colon\ w \in A\}$.

Lemma 5.2. *Let $f \in \mathcal{S}_1^* \setminus \mathcal{B}$. Then there exists a sequence $\{\alpha_m\}_{m=1}^{\infty} \subseteq f(\mathbb{D})$ such that*

$$\lim_{m\to\infty} \operatorname{dist}\big(\alpha_m, \partial f(\mathbb{D})\big) = \infty, \qquad \sup_{m\in\mathbb{N}} \frac{\operatorname{dist}(\alpha_m, 2\partial f(\mathbb{D}))}{\operatorname{dist}(\alpha_m, \partial f(\mathbb{D}))} \leqslant 2.$$

Proof. Because $f \notin \mathcal{B}$, there is a sequence $\{w_m\}_{m=1}^{\infty} \subseteq f(\mathbb{D})$ such that $\operatorname{dist}(w_m, \partial f(\mathbb{D})) \to \infty$ as $m \to \infty$. For each m, let $[0, w_m]$ denote the compact line segment between 0 and w_m. Of course, $[0, w_m] \subseteq f(\mathbb{D})$. The function $w \mapsto \operatorname{dist}(w, \partial f(\mathbb{D}))$ is continuous from $\mathbb{C}$ into $[0, \infty)$, and hence, for every m, there exists $\alpha_m \in [0, w_m]$ such that

$$\operatorname{dist}\big(\alpha_m, \partial f(\mathbb{D})\big) = \max\left\{\operatorname{dist}\big(w, \partial f(\mathbb{D})\big)\colon\ w \in [0, w_m]\right\}.$$

Clearly, $\operatorname{dist}(\alpha_m, \partial f(\mathbb{D})) \to \infty$ as $m \to \infty$ and

$$\frac{\operatorname{dist}(\alpha_m, 2\partial f(\mathbb{D}))}{\operatorname{dist}(\alpha_m, \partial f(\mathbb{D}))} = \frac{2\operatorname{dist}(\alpha_m/2, \partial f(\mathbb{D}))}{\operatorname{dist}(\alpha_m, \partial f(\mathbb{D}))} \leqslant 2,$$

because $\alpha_m/2 \in [0, w_m]$. □

We now have the following converse to Theorem B for starlike mappings.

Theorem 5.3. *Let $f \in \mathcal{S}_1^* \setminus \mathcal{B}$, and suppose that $G : \mathbb{C}^{n-1} \to \mathbb{C}$ is holomorphic such that $G(0) = 0$ and $DG(0) = 0$. If $\Phi_G(f) \in \mathcal{S}_n^*$, then G is a homogeneous polynomial of degree 2.*

Proof. For $t \geqslant 0$, let $f_t = e^t f$. Then $\{f_t\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{D}$ by Theorem D. If $\Phi_G(f) \in \mathcal{S}_n^*$, then $\{e^t \Phi_G(f)\}_{t\geqslant 0} = \{e^t \Phi_G(e^{-t} f_t)\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$ by the same theorem. The sequence $\{\alpha_m\}_{m=1}^\infty \subseteq f(\mathbb{D})$ guaranteed by Lemma 5.2 meets the hypotheses of Theorem 4.3 for $s = 0$ and $t = \log 2$. It then follows from that theorem that G must be a homogeneous polynomial of degree 2. □

The following converse to Theorem B for convex mappings is essentially a corollary to Theorem 5.3, but we state it as its own theorem because of its significance.

Theorem 5.4. *Let $f \in \mathcal{K}_1 \setminus \mathcal{B}$, and suppose that $G : \mathbb{C}^{n-1} \to \mathbb{C}$ is holomorphic such that $G(0) = 0$ and $DG(0) = 0$. If $\Phi_G(f) \in \mathcal{K}_n$, then $G = Q$ is a homogeneous polynomial of degree 2 such that $\|Q\| \leqslant 1/2$.*

Proof. Everything but the last inequality follows from Theorem 5.3 because $\mathcal{K}_n \subseteq \mathcal{S}_n^*$ for all $n \in \mathbb{N}$. The inequality $\|Q\| \leqslant 1/2$ comes from [8, Theorem 4.2]. □

6. A note on the range of extensions of Loewner chains

A Loewner chain $\{f_t\}_{t\geqslant 0}$ on $\mathbb{D}$ is easily seen to satisfy $\bigcup_{t\geqslant 0} f_t(\mathbb{D}) = \mathbb{C}$ by using the Koebe 1/4-theorem and the normalization of $\{f_t\}$. However, if $\{f_t\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B} = \mathbb{B}_n$, $n \geqslant 2$, then it is possible that $\bigcup_{t\geqslant 0} f_t(\mathbb{B}) \neq \mathbb{C}^n$. Indeed if $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ is a Fatou–Bieberbach map (meaning Ψ is biholomorphic and $\Psi(\mathbb{C}^n) \neq \mathbb{C}^n$, an impossibility when $n = 1$) normalized so that $\Psi(0) = 0$ and $D\Psi(0) = I$, then $\{\Psi \circ f_t\}_{t\geqslant 0}$ is a Loewner chain on $\mathbb{B}$ when $\{f_t\}$ is. (Several interesting examples of Fatou–Bieberbach maps are given in [14].) We now show that extensions of Loewner chains on $\mathbb{D}$ using any Φ_G do indeed have full range, even when the extension is not, itself, a Loewner chain.

Theorem 6.1. *Let $\{f_t\}_{t\geqslant 0}$ be a Loewner chain on $\mathbb{D}$, set*

$$\rho = \sup_{t\geqslant 0} e^{-t/2} \|f_t\|_{\mathcal{B}}^{1/2} \in [1, \infty],$$

and let $G : B(\rho) \to \mathbb{C}$ be holomorphic such that $G(0) = 0$ and $DG(0) = 0$. Then for $F_t = e^t \Phi_G(e^{-t} f_t)$, $t \geqslant 0$, we have

$$\bigcup_{t\geqslant 0} F_t(\mathbb{B}) = \mathbb{C}^n.$$

Proof. Let $w \in \mathbb{C}^n$. Since $\bigcup_{t\geqslant 0} f_t(\mathbb{D}) = \mathbb{C}$, we have that $w_1 \in f_s(\mathbb{D})$ for some $s \geqslant 0$. Using subordination, we may assume s is large enough that $e^{-s}\|\hat{w}\| < \rho$. Continuity of power series yields that

$$\lim_{t\to\infty} e^t G\big(e^{-t}\hat{w}\big) = \lim_{t\to\infty} e^s \sum_{k=2}^{\infty} e^{(k-1)(s-t)} P_k\big(e^{-s}\hat{w}\big) = 0,$$

where $G = \sum_{k=2}^\infty P_k$ is the homogeneous polynomial expansion of G, noting that the expressions within the limits are defined for all $t \geqslant s$. Therefore, choose $T \geqslant s$ such that

$$e^t\big|G\big(e^{-t}\hat{w}\big)\big| < \frac{1}{2}\operatorname{dist}\big(w_1, \partial f_s(\mathbb{D})\big), \quad t \geqslant T.$$

For all $t \geqslant T$, we have $w_1 - e^t G(e^{-t}\hat{w}) \in f_t(\mathbb{D})$ and

$$\begin{aligned} e^t\big|f_t'\big(f_t^{-1}\big(w_1 - e^t G\big(e^{-t}\hat{w}\big)\big)\big)\big|\big(1 - \big|f_t^{-1}\big(w_1 - e^t G\big(e^{-t}\hat{w}\big)\big)\big|^2\big) &\geqslant e^t \operatorname{dist}\big(w_1 - e^t G\big(e^{-t}\hat{w}\big), \partial f_t(\mathbb{D})\big) \\ &\geqslant \frac{e^t}{2}\operatorname{dist}\big(w_1, \partial f_s(\mathbb{D})\big) \to \infty \end{aligned} \tag{6.1}$$

as $t \to \infty$, using (4.4). Therefore, choose $t \geqslant T$ such that the expression on the left-hand side of (6.1) is greater than $\|\hat{w}\|^2$.

Now define

$$z = \left(f_t^{-1}\big(w_1 - e^t G\big(e^{-t}\hat{w}\big)\big), \frac{e^{-t/2}\hat{w}}{\sqrt{f_t'(f_t^{-1}(w_1 - e^t G(e^{-t}\hat{w})))}}\right).$$

Our choice of t gives that $z \in \mathbb{B}$, and a direct calculation verifies that $F_t(z) = w$, which gives the result. □

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