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Finitely generated radical ideals in the Sarason algebra

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Abstract

It is shown that a radical ideal in $H^\infty + C$ is finitely generated if and only if it is the zero ideal or a principal ideal generated by an interpolating Blaschke product.

Keywords: Radical ideals; Finitely generated ideals; Sarason algebra; Bounded analytic functions; Interpolating Blaschke products; Canonical generators

1. Introduction

Let H^∞ be the uniform algebra of all bounded holomorphic functions in the open unit disk $\mathbb{D}$. We look upon H^∞ as a closed subalgebra of L^∞, the algebra of equivalence classes of essentially bounded Lebesgue measurable functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$. The smallest closed subalgebra of L^∞ strictly containing H^∞ is called the Sarason algebra and coincides with the set $H^\infty + C$ of sums of (boundary values of) functions in H^∞ and continuous complex-valued functions on $\mathbb{T}$.

In this work we shall be concerned with the ideal structure of $H^\infty + C$ and solve a problem that, apparently, first came up within Professor von Renteln's function theory seminar at the university of Karlsruhe in the 1980s in connection with Hedenmalm's early pioneering work on the ideal structure of various function algebras, including $H^\infty + C$. There are only very few

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papers dealing with the ideal structure of $H^\infty + C$. The main ones being [12,10,11] on the structure of the class of closed, respectively closed prime ideals in $H^\infty + C$.

Recall that a proper ideal P in a uniform algebra A is called prime, if for every $f, g \in A$, $fg \in P$ implies that f or g is in P. Note that for $A = H^\infty$ the zero ideal (0) is a prime ideal, whereas (0) is not a prime ideal in $H^\infty + C$, because $H^\infty + C$ has zero divisors.

A radical ideal in A is, by definition, an arbitrary intersection of prime ideals. It is well known that I is radical if and only if $f \in I$ whenever some power f^n of $f \in A$ belongs to I.

In [5] it was shown that in H^∞ a radical ideal $I \neq (0)$ is finitely generated if and only if I is a principal ideal generated by a Blaschke product having simple zeros. Previously, it was proved in [8] and [21], respectively [22], that the only finitely generated prime ideals in H^∞ different from zero are the principal ideals generated by the polynomial $z - a$, where $a \in \mathbb{D}$.

The situation in $H^\infty + C$ is quite different. Here there do not exist finitely generated prime ideals (see [8]). In the present paper we shall give a complete characterization of the class of finitely generated radical ideals in $H^\infty + C$: they are those principal ideals generated by an interpolating Blaschke product.

One of our major algebraic tools is Nakayama's lemma from commutative algebra (see for instance, [19, Theorem 76]).

Lemma 1.1. *Let R be a commutative ring with identity element 1, I a finitely generated ideal in R and M an arbitrary ideal in R. Suppose that $IM = I$. Then there exists an element $m \in M$ such that $(1 + m)I = 0$.*

Recall that IM is the ideal generated by finite sums of products of the form ιm, $\iota \in I$, $m \in M$.

2. Technical prerequisites

For a uniform algebra A over $\mathbb{C}$, let $M(A)$ denote its maximal ideal space (or spectrum); this is the set of all non-zero, multiplicative linear functionals endowed with the Gelfand-topology. The Gleason part $P(m)$ of a point $m \in M(A)$ is the set of all points $x \in M(A)$ such that $\rho_A(m, x) < 1$, where

$$\rho_A(m, x) = \sup\{|m(f)|:\ f \in A,\ \|f\|_\infty \leqslant 1,\ x(f) = 0\}$$

is the pseudo-hyperbolic distance on $M(A)$. As usual, G is the set of points $m \in M(H^\infty + C)$ whose Gleason part $P(m)$ is nontrivial; if $P(m) = \{m\}$, then m is called a trivial point. Let us mention Budde's result [3], which tells us that the closure of a nontrivial Gleason part always contains trivial points.

As usual, we shall identify $f \in A$ with its Gelfand-transform $\hat{f} : m \mapsto m(f)$, a function in $C(M(A))$. In our setting $A = L^\infty$, $H^\infty + C$ or H^∞. Note that $M(H^\infty + C) = M(H^\infty) \setminus \mathbb{D}$.

Let $X = M(L^\infty)$. Then we may identify X with the Shilov boundary of $H^\infty + C$ (or H^∞). The characteristic function of $E \subseteq X$ is denoted by χ_E. We know that χ_E is continuous on X if and only if E is closed-open. Every function in $f \in L^\infty \simeq C(X)$ has a canonical continuous extension to $M(H^\infty)$ given by

$$\hat{f}(x) = \int_{\operatorname{supp} x} f \, d\mu_x,$$

where $\operatorname{supp} x$ is the support set of the representing measure $\mu_x \in \mathcal{P}(\mathscr{B}(X))$ for the functional $x \in M(H^\infty)$. Note that $\mathcal{P}(\mathscr{B}(X))$ is the set of probability measures on the class $\mathscr{B}(X)$ of Borel-sets on X.

If $f \in H^\infty + C$, then $\hat{f}$, restricted to $M(H^\infty + C)$, equals the Gelfand-transform of f.

If $f \in C(M(H^\infty + C))$, then

$$Z(f) = \{x \in M(H^\infty + C): \ f(x) = 0\}$$

is the zero set of f. Let $m \in M(H^\infty + C)$ and suppose that $f(m) = 0$ for some $f \in H^\infty + C$. Then $n \in \mathbb{N}$ is said to be the order of the zero m of f, denoted by $\operatorname{ord}(f, m) = n$, if the analytic function $f \circ L_m$ has a zero of order n at 0, where L_m is the Hoffman map associated with m. Note that in the (product)-topology of $M(H^\infty)^{\mathbb{D}}$, L_m is given by $L_m(z) = \lim \frac{z+z_\alpha}{1+\bar{z}_\alpha z}$, where (z_α) is a net in $\mathbb{D}$ converging to m. If $f \circ L_m$ is the zero function, that is if f vanishes identically on the part $P(m)$, then let $\operatorname{ord}(f, m) = \infty$. If $f(m) \neq 0$, we say $\operatorname{ord}(f, m) = 0$. The zero set of infinite order of $f \in H^\infty + C$ is denoted by $Z_\infty(f)$. Note that $m \in Z_\infty(f)$ if and only if f vanishes identically on the Gleason part $P(m)$ associated with m. In particular, each $m \in Z(f) \setminus G$ belongs to $Z_\infty(f)$.

If I is an ideal in A (where $A = H^\infty$ or $A = H^\infty + C$) and $m \in M(A)$, then

$$\operatorname{ord}(I, m) = \min\{\operatorname{ord}(f, m): \ f \in I\}.$$

Finally, I is called an ideal of order N ($N = 1, 2, \ldots, \infty$), denoted by $\operatorname{ord} I = N$, if $\sup\{\operatorname{ord}(I, m): \ m \in M(A)\} = N$. The zero set (or hull) of an ideal is given by $Z(I) = \bigcap_{f \in I} Z(f)$. Similarly for $Z_\infty(I)$.

Sets of the form $\{x \in M(H^\infty + C): \ \tau < |f(x)| < \sigma\}$ will be written more shortly by $\{\tau < |f| < \sigma\}$. For a set $Y \subseteq M(H^\infty + C)$, $\overline{Y}$ denotes its closure and Y° the set of interior points of Y. Finally, for $E \subseteq X$, $E^c = X \setminus E$.

Recall that an interpolating Blaschke product b is a Blaschke product having *infinitely* many zeros z_n such that

$$\delta(b) := \inf_n (1 - |z_n|^2) |b'(z_n)| > 0.$$

We refer to the excellent book by Garnett [7] and Hoffman's original paper [16] for this and any other material used here.

Apart from Axler's amazing result [1] that every function in L^∞ can be multiplied by a Blaschke product into $H^\infty + C$, we shall frequently use the following lemmas.

Lemma 2.1. *(See [16].) For every nonvoid open set $U \subseteq M(H^\infty + C)$ there exists an interpolating Blaschke product b such that $Z(b) \subseteq U$.*

Proof. This is an easy consequence of the facts that G is dense in $M(H^\infty + C)$ and that for an interpolating sequence (z_n) the associated Blaschke product b has the property that $Z(b) = \overline{\{z_n: \ n \in \mathbb{N}\}} \setminus \mathbb{D}$. $\square$

Lemma 2.2. *(See [20].) Let E be a nonvoid, proper closed-open subset of $M(L^\infty)$. Then there is an interpolating Blaschke product b_M such that $\{0 < \widehat{\chi}_E < 1\} = \{|b_M| < 1\}$. Moreover,*

- $\widehat{\chi}_E(x) = 0 \Longleftrightarrow \operatorname{supp} x \subseteq E^c$;
- $\widehat{\chi}_E(x) = 1 \Longleftrightarrow \operatorname{supp} x \subseteq E$.

The function b_M above will be called Marshall's interpolating Blaschke product.

Lemma 2.3. *(See [9, Lemma 2.2].) Let $f \in H^\infty + C$ and let E be a closed-open subset of $M(L^\infty)$. Then*

$$f\chi_E \in H^\infty + C \quad \Longleftrightarrow \quad f \equiv 0 \quad \text{on } \{0 < \widehat{\chi}_E < 1\}.$$

Moreover,

$$Z(f) \subseteq Z(f\chi_E), \qquad Z(f)^\circ \subseteq Z(f\chi_E)^\circ,$$

as well as

$$Z_\infty(f) \subseteq Z_\infty(f\chi_E).$$

In [11, Theorem 1.5] it was shown that for $f \in H^\infty + C$, we have $\overline{Z(f)^\circ} = Z_\infty(f)$. We need the following slight generalization.

Corollary 2.4.

(a) *Let I be a finitely generated ideal in $H^\infty + C$. Then $\overline{Z(I)^\circ} = Z_\infty(I)$.*
(b) *There exist ideals in $H^\infty + C$ for which the assertion in* (a) *does not hold.*

Proof. (a) Assume that $I = (f_1, \dots, f_n)$. Then $Z(I)^\circ = \bigcap_{j=1}^n Z(f_j)^\circ$. To show that $Z(I)^\circ$ is dense in $Z_\infty(I)$, choose an open set U in $M(H^\infty + C)$ with $U \cap Z_\infty(I) \neq \emptyset$. Then, exactly as in [11, p. 524], we see that there exists an interpolating Blaschke product b so that $Z(b) \subseteq U \cap \bigcap_{j=1}^n Z_\infty(f_j)$. Hence, by [2], every power of b divides (in $H^\infty + C$) each of the functions f_j. Therefore

$$\{|b| < 1\} \subseteq \bigcap_{j=1}^n Z(f_j)^\circ = \left(\bigcap_{j=1}^n Z(f_j)\right)^\circ = Z(I)^\circ.$$

In particular, $Z(b) \subseteq U \cap Z(I)^\circ$, and so $U \cap Z(I)^\circ \neq \emptyset$.

(b) Just take any maximal ideal $M = \operatorname{Ker} m$ such that the Gleason part $P(m)$ is the singleton $\{m\}$. Here, by Hoffman's theory, $Z(M) = Z_\infty(M) = \{m\}$, but $Z(M)^\circ = \emptyset$. □

Lemma 2.5. *For a proper, nonvoid closed-open set $E \subseteq X$, let B_j be Blaschke products chosen so that $B_j\chi_E \in H^\infty + C$ $(j = 1, \dots, n)$. Then*

$$\bigcap_{j=1}^n Z(B_j)^\circ \setminus \overline{\{0 < \widehat{\chi}_E < 1\}} \neq \emptyset.$$

Moreover, there are an interpolating Blaschke product b and some point $x \in Z(b)$ such that

$$\overline{\{|b| < 1\}} \subseteq \bigcap_{j=1}^n Z(B_j)^\circ$$

and

$$\operatorname{supp} x \subseteq E.$$

Proof. Let b_M be Marshall's interpolating Blaschke product satisfying $\{|b_M| < 1\} = \{0 < \widehat{\chi}_E < 1\}$. By Lemma 2.3, $B_j \chi_E \in H^\infty + C$ implies that $B_j \equiv 0$ on $\{0 < \widehat{\chi}_E < 1\}$; thus

$$\bigl\{|b_M| < 1\bigr\} \subseteq \bigcap_{j=1}^{n} Z(B_j)^\circ.$$

By the proof of Proposition 1.4 in [11], we actually have

$$\overline{\bigl\{|b_M| < 1\bigr\}} \subseteq \bigcap_{j=1}^{n} Z(B_j)^\circ;$$

that is

$$\overline{\{0 < \widehat{\chi}_E < 1\}} \subseteq \bigcap_{j=1}^{n} Z(B_j)^\circ.$$

Since $M(H^\infty + C)$ is connected (see [15, p. 188]), $\emptyset$ and $M(H^\infty + C)$ are the only closed-open sets in $M(H^\infty + C)$. Hence

$$U := \bigcap_{j=1}^{n} Z(B_j)^\circ \setminus \overline{\{0 < \widehat{\chi}_E < 1\}} \neq \emptyset.$$

We claim that U actually meets $\{\widehat{\chi}_E = 0\}^\circ$ **and** $\{\widehat{\chi}_E = 1\}^\circ$. First we note that these two sets are nonvoid, because E and E^c are proper subsets of X. Now assuming that, for instance, $U \cap \{\widehat{\chi}_E = 1\}^\circ = \emptyset$, we would get the following decomposition of the connected set $M(H^\infty + C)$ into two disjoint, nonvoid open sets:

$$M\bigl(H^\infty + C\bigr) = \{\widehat{\chi}_E = 1\}^\circ \cup \left[\{\widehat{\chi}_E = 0\}^\circ \cup \bigcap_{j=1}^{n} Z(B_j)^\circ\right];$$

a contradiction. Hence $U \cap \{\widehat{\chi}_E = 1\}^\circ \neq \emptyset$. The same holds for E replaced by E^c as well.

Since the set G of nontrivial points in $M(H^\infty + C)$ is dense in $M(H^\infty + C)$ (see [16]), there is a point $x \in (U \cap \{\widehat{\chi}_E = 1\}^\circ) \cap G$. By [11, pp. 523–524] there is an interpolating Blaschke product b such that $b(x) = 0$ and $\overline{\{|b| < 1\}} \subseteq \bigcap_{j=1}^{n} Z(B_j)^\circ$. Obviously $\operatorname{supp} x \subseteq E$. □

3. Finitely generated radical ideals

Let A be a uniform algebra. If $f_j \in A$ $(j = 1, \dots, n)$ then

$$I = (f_1, \dots, f_n) = \left\{ \sum_{j=1}^{n} g_j f_j \colon g_j \in A \right\}$$

denotes the ideal generated by the functions f_j. In particular, (f) is the principal ideal $f \cdot A$.

Proposition 3.1. *Let $I \neq (0)$ be a finitely generated radical ideal in $H^\infty + C$. Then $Z(I) \cap X = \emptyset$.*

Proof. Let $I = (f_1, \dots, f_n)$. We view the functions f_j as being defined everywhere on $\mathbb{T}$ (take suitable representatives).

Let $u_j(\lambda) = |f_j(\lambda)|/f_j(\lambda)$ if $f_j(\lambda) \neq 0$, $\lambda \in \mathbb{T}$, and $u_j(\lambda) = 1$ otherwise. Then the u_j are unimodular Borel–Lebesgue measurable functions; hence $u_j \in L^\infty$.

By Axler's theorem [1] there is a Blaschke product B such that Bu_j and $B\sqrt{\sum_{j=1}^n |f_j|} \in H^\infty + C$ for all $j \in \{1, \dots, n\}$. Hence

$$\left(B\sqrt{\sum_{j=1}^n |f_j|}\right)^2 = B^2 \sum_{j=1}^n |f_j| = B \sum_{j=1}^n (Bu_j) f_j \in I.$$

Since I is a radical ideal, $u := B\sqrt{\sum_{j=1}^n |f_j|} \in I$. Now we switch to $X = M(L^\infty)$ and regard all functions appearing here as continuous functions on X. Let $q = \sum_{j=1}^n |f_j|$. Then $Z(I) \cap X = \{x \in M(L^\infty)\colon\ q(x) = 0\}$. Moreover, since $u \in I$, on X,

$$\sqrt{\sum_{j=1}^n |f_j|} = \left|B \cdot \sqrt{\sum_{j=1}^n |f_j|}\right| = |u| \leqslant c \sum_{j=1}^n |f_j|$$

for some constant $c > 0$. Hence, on $X \setminus Z(I)$,

$$0 < \frac{1}{c} \leqslant \sqrt{\sum_{j=1}^n |f_j|}. \tag{3.1}$$

Let E be the closure (in X) of $X \setminus Z(I)$. We will show, by contradiction, that $E = X$.

So suppose that $E \neq X$. Since X is extremely disconnected (see [6, p. 18]), the openness of $X \setminus Z(I)$ implies that E is a closed-open set. Due to continuity, we obtain that $|q| \geqslant (1/c)^2 =: \varepsilon > 0$ on E. Moreover, $q = q\chi_E$. We deduce that for every $f \in I$, we have the representation $f = f\chi_E$.

Choose a Blaschke product φ such that $\varphi\chi_E \in H^\infty + C$. Thus $\varphi\chi_{E^c} \in H^\infty + C$, too. Now consider the following functions (on X)

$$f_0 = 2B^2 q,$$
$$h_j = f_0 + f_j + \varphi\chi_{E^c} \quad (j = 0, \dots, n).$$

We obviously have that for all j, $h_j \in H^\infty + C$, $h_j|_E = (f_0 + f_j)|_E$ and $h_j = \varphi$ on E^c. We claim that $|h_j|$ is bounded away from zero on X. In fact, suppose that $|h_j(x)| = 0$ for some $x \in X$. Then $x \in E$, because on E^c, $|h_j| = |\varphi| = 1$. Hence

$$|f_j(x)| = |f_0(x)| = 2q(x) = 2\sum_{k=1}^n |f_k(x)| = 2|f_j(x)| + r(x)$$

with $r(x) \geqslant 0$. Therefore $r(x) = f_j(x) = 0$ and so $q(x) = 0$. But $q \geqslant \varepsilon > 0$ on E. This contradiction shows that $|h_j| \geqslant \tilde{\varepsilon} > 0$ on X.

We deduce that h_j is invertible in L^∞ and so, by Guillory, Izuchi and Sarason [14], $h_j = B_j v_j$, where B_j is a Blaschke product and v_j is an invertible function in $H^\infty + C$.

Hence

$$f_0 + f_j = (f_0 + f_j)\chi_E = h_j \chi_E = v_j(B_j \chi_E).$$

Since v_j is invertible is $H^\infty + C$ and $f_0 + f_j \in H^\infty + C$, we conclude that $B_j \chi_E \in H^\infty + C$. In particular, by Lemma 2.3, $\{0 < \chi_E < 1\} \subseteq \bigcap_{j=0}^n Z(B_j)^\circ$.

Using that $f_0 \in I$, we get

$$\begin{aligned} I &= (f_1, \dots, f_n) = (f_0, f_1, \dots, f_n) = (f_0 + f_0, f_1 + f_0, \dots, f_n + f_0) \\ &= (B_0\chi_E, B_1\chi_E, \dots, B_n\chi_E). \end{aligned}$$

By Lemma 2.5 (note that by our hypothesis $\emptyset \neq E \neq X$), there is an interpolating Blaschke product b such that for some $x_0 \in G$, $b(x_0) = 0$, $\operatorname{supp} x_0 \subseteq E$ and $Z(b) \subseteq \bigcap_{j=0}^n Z(B_j)^\circ$. By the Axler–Gorkin theorem, for every $k \in \mathbb{N}^*$, b^k divides each B_j. Since

$$Z\big(\bar{b}^k B_j\big)^\circ = Z(B_j)^\circ \supseteq \{0 < \chi_E < 1\}$$

and $Z(b) \subseteq Z(B_j)^\circ \overset{2.3}{\subseteq} Z(B_j\chi_E)^\circ$, b^k divides $B_j\chi_E$, and so by Lemma 2.3,

$$\big(\bar{b}^k B_j\big)\chi_E \in H^\infty + C \quad \text{and} \quad \big(\bar{b}^k B_j\big)\chi_E = \bar{b}^k (B_j \chi_E).$$

In particular we have

$$(\bar{b} B_j \chi_E)^2 = \big(\bar{b}^2 B_j \chi_E\big) B_j \chi_E \in I.$$

I being radical now implies that $q_j := \bar{b} B_j \chi_E \in I$, too. Hence

$$bI \subseteq I = (bq_1, \dots, bq_n) = b(q_1, \dots, q_n) \subseteq bI,$$

and so $I = bI$.

By the Nakayama Lemma 1.1, there is $h \in H^\infty + C$ such that $(1 + hb)I = 0$. Since $\operatorname{supp} x_0 \subseteq E$, $B_1\chi_E \in I$ does not vanish on $\operatorname{supp} x_0$. Thus $1 + hb$ is identically 0 on $\operatorname{supp} x_0$. Hence $b|_{\operatorname{supp} x_0}$ is invertible in $H^\infty|_{\operatorname{supp} x_0}$. This implies that b is a unimodular constant on $\operatorname{supp} x_0$ (see [4]). This is a contradiction to the fact that $b(x_0) = 0$.

We conclude that $E = X$ and so (3.1) holds on X. Thus $Z(I) \cap X = \emptyset$. □

Theorem 3.2. *Let $I \neq (0)$ be a finitely generated radical ideal in $H^\infty + C$. Then I is a principal ideal generated by an interpolating Blaschke product.*

Proof. Step 1 Let $I = (f_1, \dots, f_n)$. By Proposition 3.1, $Z(I) \cap X = \emptyset$. We first show that $Z(I)^\circ = \emptyset$. Indeed, suppose to the contrary that $Z(I)^\circ \neq \emptyset$. Then, by Lemma 2.1, there is an interpolating Blaschke product b such that $\emptyset \neq Z(b) \subseteq Z(I)^\circ$. Hence, by the Axler–Gorkin theorem, for every $n \in \mathbb{N}^*$, b^n divides every element in I. Now let $f \in I$. Then $\bar{b} f \in H^\infty + C$ and $\bar{b}^2 f \in H^\infty + C$. Hence $(\bar{b} f)^2 = (\bar{b}^2 f) f \in I$. Since I is radical, $\bar{b} f \in I$. So we may deduce that $I = bI$. By the Nakayama lemma, there is $h \in H^\infty + C$ such that $(1 + hb)I = 0$. By Suárez's result (see [24, p. 242]), H^∞, and so $H^\infty + C$, is a separating algebra. Hence, by using the fact that $Z(I) \cap X = \emptyset$, there is a function $f \in I$ such that $Z(f) \cap X = \emptyset$. In particular, $(1 + hb) f = 0$. Therefore $1 + hb = 0$ on X. Since the Shilov boundary is a uniqueness set, $1 + hb = 0$ on $M(H^\infty + C)$. Thus b is invertible in $H^\infty + C$; a contradiction to the fact that $Z(b) \neq \emptyset$.

Step 2 Here we show that I is generated by finitely many Carleson–Newman Blaschke products; these are, via definition, finite product of interpolating Blaschke products.

In fact, by Corollary 2.4, $Z_\infty(I) = \overline{Z(I)^\circ}$. Hence, using Step 1, $Z_\infty(I) = \emptyset$. Thus $Z(I) \subseteq G$. In view of Suárez's result (see [24, p. 242]), there is $f \in I$ with $Z(f) \subseteq G$. By [14], $f = Bg$, where $g \in H^\infty + C$ is invertible and B is a Carleson–Newman Blaschke product. In particular, $B \in I$. Now $I = (B, f_1, \ldots, f_n)$. It is now easy to verify, that for every $\varepsilon > 0$, the set of functions $\{B, B + \varepsilon f_1, \ldots, B + \varepsilon f_n\}$ is a generating set for I, too. Now if ε is small, $B + \varepsilon f_j$ does not vanish on the set of trivial points (since $|B| \geqslant \sigma > 0$ there). Thus $Z(B + \varepsilon f_j) \subseteq G$, too and so, by another application of the Guillory–Izuchi–Sarason result, $B + \varepsilon f_j = B_j g_j$ for some $g_j \in H^\infty + C$, g_j invertible. Thus $\{B, B_1, \ldots, B_n\}$ is the desired generating set for I.

Step 3 Next we show that $\operatorname{ord} I = 1$. Suppose, contrariwise, that there is $x \in Z(I)$ such that $N := \operatorname{ord}(I, x) \geqslant 2$. Since by Step 2, $I = (B_0, B_1, \ldots, B_n)$ for some Carleson–Newman Blaschke products B_j, $N < \infty$. Using that for every $m \in M(H^\infty + C)$,

$$\min_{f \in I} \operatorname{ord}(f, m) = \operatorname{ord}(I, m) = \min_{0 \leqslant j \leqslant n} \operatorname{ord}(B_j, m),$$

we see that $\operatorname{ord}(B_j, x) \geqslant N$ for every j and that there is $i_0 \in \{0, \ldots, n\}$ such that $\operatorname{ord}(B_{i_0}, x) = N$.

By Hoffman's factorization theorem [16], we may write $B_j = b_{j,1} \ldots b_{j,N} C_j$, where the $b_{j,k}$ are interpolating Blaschke products vanishing at x and where the C_j are Carleson–Newman Blaschke products.

Since the ideal $\sum_{k=1}^N \sum_{j=0}^n b_{j,k} H^\infty$ is proper, there is by the Corona theorem a sequence (z_n) in $\mathbb{D}$ such that $b_{j,k}(z_n) \to 0$ for every j and k. We may assume that (z_n) is an interpolating sequence. The associated interpolating Blaschke product b now satisfies $Z(b) \subseteq Z(b_{j,k})$. By the Axler–Gorkin theorem [2], b divides each of the $b_{j,k}$ in $H^\infty + C$. Since $\operatorname{ord}(b_{j,k}, \xi) = 1$ for every $\xi \in Z(b_{j,k})$, we have $\operatorname{ord}(\bar{b} b_{j,k}, y) = 0$ for every $y \in Z(b)$. Let $q_j = C_j \prod_{k=1}^N (\bar{b} b_{j,k})$. Note that $q_j \in H^\infty + C$. Then $B_j = b^N q_j$ and so $I = b^N(q_0, \ldots, q_n)$. Now, if J is the ideal generated by the q_j,

$$\operatorname{ord}(I, y) = \operatorname{ord}\big(b^N J, y\big) = N \operatorname{ord}(b, y) + \operatorname{ord}(J, y) \geqslant N \quad \forall y \in Z(b).$$

On the other hand, $(b q_j)^N = B_j q_j^{N-1} \in I$. Since I is radical, $b q_j \in I$ for $j \in \{0, 1, \ldots, n\}$. Hence $bJ \subseteq I$. Using that $N \geqslant 2$, we get

$$\operatorname{ord}(I, y) \leqslant \operatorname{ord}(bJ, y) = \operatorname{ord}(b, y) + \operatorname{ord}(J, y) < N \operatorname{ord}(b, y) + \operatorname{ord}(J, y) = \operatorname{ord}(I, y)$$

whenever $y \in Z(b) \subseteq Z(I)$. This is a contradiction. Thus $\operatorname{ord} I = 1$.

Step 4 Here we will deduce that I is generated by a set of interpolating Blaschke products. In fact, by Step 2, I is generated by a finite number of Carleson–Newman Blaschke products, say $I = (B_0, \dots, B_n)$ with $\operatorname{ord} I = 1$. This implies that there exists $r \in]0, 1[$ such that $\min_{j=0}^n \operatorname{ord}(B_j, a) \leqslant 1$ for every $a \in \mathbb{D}$ with $|a| \geqslant r$. (Otherwise, there is a sequence (a_n) in $\mathbb{D}$ converging to the boundary such that every generator B_j has a zero of order at least 2 at each of the a_n. Hence $\operatorname{ord}(B_j, m) \geqslant 2$ for every cluster point m of the sequence (a_n); in particular $\operatorname{ord}(I, m) \geqslant 2$; a contradiction.) By dividing out all the zeros of B_j in the disk $|z| \leqslant r$, we may henceforth assume that $\min_{j=0}^n \operatorname{ord}(B_j, a) \leqslant 1$ for every $a \in \mathbb{D}$. Note that the new Blaschke products still generate the ideal I, since finite Blaschke products are invertible in $H^\infty + C$. We keep on using the notation B_j. Now consider the ideal $L = I \cap H^\infty$. Obviously $B_j \in L$. Note that, in general, L is much larger than the ideal generated by the B_j in H^∞. Nevertheless, $\operatorname{ord} L = \operatorname{ord} I = 1$. Thus, by [23], L is generated by (uncountably many) interpolating Blaschke products, say $L = (\{b_\lambda\colon \lambda \in \Lambda\})$.

Now $B_j = \sum_{\nu=1}^{N(j)} h_{\nu,j} b_{\lambda_{\nu,j}}$ for some $h_{\nu,j} \in H^\infty$. Thus I is generated by the collection $\{b_{\lambda_{n,j}}\colon j = 0, \dots, n;\, n = 1, \dots, N(j)\}$.

Step 5 In this step we show that I is generated by two interpolating Blaschke products. Let φ be any interpolating Blaschke product contained in $I = (f_1, \dots, f_n)$. Consider a continuous extension of $f_j \in C(M(H^\infty + C))$ to $M(H^\infty)$, denoted by the same symbol. If $\{z_j\colon j \in \mathbb{N}\}$ denotes the zero set of φ in $\mathbb{D}$, we may use Earl's interpolation theorem (see [7]) to get a second interpolating Blaschke product ψ and a constant $M > 0$ such that

$$M\psi(z_j) = \sqrt{\sum_{k=1}^n \bigl|f_k(z_j)\bigr|}, \quad j \in \mathbb{N}.$$

Let $h_k \in H^\infty$ satisfy the interpolation problem $h_k(z_j) = e^{-i \arg(f_k(z_j))}$ if $f_k(z_j) \neq 0$ and $h_k(z_j) = 1$ otherwise. Then

$$M\psi(z_j) = \sqrt{\sum_{k=1}^n h_k(z_j) f_k(z_j)}, \quad j \in \mathbb{N}.$$

Since $Z(\varphi) = \overline{\{z_j\colon j \in \mathbb{N}\}} \setminus \mathbb{D}$, we get that on $Z(\varphi)$

$$M^2\psi^2 - \sum_{k=1}^n h_k f_k \equiv 0.$$

Thus by [2,14], φ divides the $(H^\infty + C)$-function $M^2\psi^2 - \sum_{k=1}^n h_k f_k$. Hence $M^2\psi^2 = \sum_{k=1}^n h_k f_k + h\varphi$ for some $h \in H^\infty + C$ and so $\psi^2 \in I$. Since I is radical, $\psi \in I$. We claim that $I = (\varphi, \psi)$. Let $f \in I$. Then, on $M(H^\infty + C)$, $|f| \leqslant \kappa \sum_{k=1}^n |f_k|$ for some constant $\kappa > 0$. Hence on $Z(\varphi)$ we have

$$|f| \leqslant \kappa \sum_{k=1}^n |f_k| = \kappa M^2 |\psi|^2.$$

Thus the function

$$q(x)=\begin{cases}\frac{f(x)}{\psi(x)} & \text{if } x\in Z(\varphi) \text{ and } \psi(x)\neq 0,\\ 0 & \text{if } x\in Z(\varphi) \text{ and } \psi(x)=0\end{cases}$$

is continuous on $Z(\varphi)$. Since $Z(\varphi)$ is an interpolation set for H^∞, there is $g\in H^\infty$ such that $q=g$ on $Z(\varphi)$. Hence $f-\psi g\equiv 0$ on $Z(\varphi)$ and so, by the usual argument, $f-\psi g=F\varphi$ for some $F\in H^\infty+C$. Thus $f\in(\varphi,\psi)$. This shows that $I\subseteq(\varphi,\psi)$. The reverse inclusion being clear, we have shown that $I=(\varphi,\psi)$ for two interpolating Blaschke products φ and ψ.

Step 6 It remains to show that actually one generator suffices to represent I. Here we consider two cases.

Case 1 If $Z(\varphi)\cap Z(\psi)$ is an open-closed subset of $Z(\varphi)$, then, by [17, pp. 341–342], there is an interpolating Blaschke product b such that $Z(b)=Z(\varphi)\cap Z(\psi)$.

We claim that the ideal generated by b coincides with $I=(\varphi,\psi)$. In fact, $Z(b)\subseteq Z(\psi)$ implies that b divides ψ, say $\psi=q_2b$ for some $q_2\in H^\infty+C$. So $\psi\in(b)$. The same holds for φ; say $\varphi=q_1b$ for $q_1\in H^\infty+C$. Thus $I\subseteq(b)$. But q_1 and q_2 have no zeros in common. (Otherwise $q_1(x)=q_2(x)=0$ implies $\varphi(x)=\psi(x)=0$; hence $x\in Z(b)$ and so $\operatorname{ord}(\varphi,x)\geqslant 2$; a contradiction to the fact that the order of the zeros of interpolating Blaschke products is one.) Thus $1=\phi_1q_1+\phi_2q_2$ for $\phi_j\in H^\infty+C$. Hence

$$b=\phi_1(bq_1)+\phi_2(bq_2)=\phi_1\varphi+\phi_2\psi\in I.$$

To sum up, we have $I=(b)$.

Case 2 $Z(\varphi)\cap Z(\psi)$ is not open-closed in $Z(\varphi)$. Then $Z(\varphi)\setminus Z(\psi)$ is not closed; hence

$$S:=\overline{Z(\varphi)\setminus Z(\psi)}\cap Z(\psi)\neq\emptyset.$$

Let (z_n) be the zero sequence in $\mathbb{D}$ of the interpolating Blaschke product φ. Choose a function $f\in H^\infty$ with $f(z_n)=\sqrt{|\psi(z_n)|}$ for all n and a function $h\in H^\infty$ with

$$h(z_n)=\begin{cases}|\psi(z_n)|/\psi(z_n) & \text{if } \psi(z_n)\neq 0,\\ 1 & \text{if } \psi(z_n)=0.\end{cases}$$

Then $f^2-h\psi=k\varphi$ for some $k\in H^\infty$. In particular, $f^2\in I=I(\varphi,\psi)$. Since I is assumed to be radical, $f\in I$. Hence, for every $x\in M(H^\infty+C)$,

$$\left|f(x)\right|\leqslant\kappa\left(\left|\varphi(x)\right|+\left|\psi(x)\right|\right).$$

But for $x\in Z(\varphi)=\overline{\{z_n\colon n\in\mathbb{N}\}}\setminus\{z_n\colon n\in\mathbb{N}\}$, we have

$$\sqrt{\left|\psi(x)\right|}=\left|f(x)\right|\leqslant\kappa\left|\psi(x)\right|.$$

Henceforth, this holds for $x\in Z(\varphi)\setminus Z(\psi)$; that is

$$1\leqslant\kappa\sqrt{\left|\psi(x)\right|}.$$

Since $Z(\varphi) \setminus Z(\psi)$ is non-empty, we get that $|\psi(x)| \geqslant \kappa^{-2}$ for every $x \in \overline{Z(\varphi) \setminus Z(\psi)}$. In particular, $\psi(x) > 0$ on S; this is a contradiction, since $S \subseteq Z(\psi)$.

Thus we have shown that this second case does not occur. Consequently, I is a principal ideal generated by an interpolating Blaschke product. □

Recall that an invertible function in an algebra is sometimes called "a unit".

Corollary 3.3. *Let $f \in H^\infty + C$. Then the principal ideal (f) is a radical ideal if and only if f is the zero function or a unit times an interpolating Blaschke product.*

Proof. Let $I = (f)$ be a non-zero principal radical ideal in $H^\infty + C$. Then $Z(I) = Z(f)$. By Proposition 3.1, $Z(I) \cap X = \emptyset$. By Theorem 3.2, Step 3, $\operatorname{ord} I = 1$. That is $\operatorname{ord}(f, x) = 1$ for all $x \in Z(f)$. By [18, p. 557], applied to $E = Z(f)$, we get that $f = bg$ for some interpolating Blaschke product b and an invertible function $g \in H^\infty + C$.

Next we prove the converse. Let b be an interpolating Blaschke product. Then, by [2] or [14], (b) equals the ideal of all functions f in $H^\infty + C$ such that f vanishes on $Z(b)$. Hence, if $f^n \in I$ for some n, it trivially follows that $f \in I$.

Finally, since the zero ideal in $H^\infty + C$ is an intersection of maximal (hence prime) ideals, $I = (0)$ is radical, too. □

We find Corollary 3.3 rather surprising when compared with the H^∞-situation. In fact, as was shown in [5], a principal ideal I in H^∞ is radical if and only if I is generated by a generic Blaschke product with simple zeros, not necessarily an interpolating one. We also note that in the $(H^\infty + C)$-setting, if b is an interpolating Blaschke product and π an arbitrary finite Blaschke product, then πb is a generator of $I = (b)$, too. So generators for principal radical ideals in $H^\infty + C$ do not necessarily have simple zeros (in $\mathbb{D}$).

4. Canonical generators

Our proofs above also yield the following general results.

Theorem 4.1.

(1) *Let I be an ideal in $H^\infty + C$ whose hull does not meet the Shilov boundary. Then I is generated by Blaschke products.*
(2) *Let I be an ideal in $H^\infty + C$ whose hull does not meet the set of trivial points. Then I is generated by Carleson–Newman Blaschke products.*
(3) *Let I be an order 1 ideal in $H^\infty + C$. Then I is generated by interpolating Blaschke products.*

Whereas (2) and (3) also hold in the H^∞-setting (two well-known facts), this is not the case for (1). The reason is that in $H^\infty + C$ singular inner functions are codivisible with some Blaschke products [13].

Theorem 4.2. *Let I be an ideal in $H^\infty + C$ for which $Z(I) \cap X$ is closed-open. Then I is generated by functions of the form $B\chi_E$, where $E = X \setminus Z(I)$ and B is a Blaschke product.*

Proof. By our hypothesis, E is closed(-open). Thus, using a compactness argument, there are finitely many functions $f_j \in I$ such that

$$\bigcap_{j=1}^{n} Z(f_j) \cap X = Z(I) \cap X.$$

Choose a Blaschke product B such that $B\overline{f}_j \in H^\infty + C$ for $j = 1, \dots, n$. Then

$$f_0 := B \sum_{j=1}^{n} |f_j|^2 = \sum_{j=1}^{n} (B\overline{f}_j) f_j \in I.$$

Now $|f_0|$ is bounded away from zero on E. Say $|f_0| \geqslant \varepsilon > 0$. Then the rest works as in Proposition 3.1: the functions

$$h_\lambda := 2\varepsilon^{-1} f_0 + f_\lambda / \|f_\lambda\| + \varphi \chi_{E^c},$$

where f_λ runs through all the elements in I, are all zero free on X. Hence $h_\lambda = B_\lambda v_\lambda$ for some Blaschke product B_λ and an invertible function $v_\lambda \in H^\infty + C$.

The set $\{B_\lambda \chi_E \colon \lambda \in \Lambda\}$ is now the desired generating set for I. $\square$

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