The Routh and Routh-Hurwitz Stability Criteria

Their Derivation by a Novel Method Using Comparatively Elementary Algebra

By J. Morris

THE classic Routh stability criteria[1] are applicable only to polynomial characteristic equations with real coefficients. The writer recently came across co-related criteria in a Russian treatise[2] on the stability of rotating shafts which criteria are applicable to polynomial equations with complex coefficients. These criteria were apparently devised by Hurwitz as extensions of the Routh criteria and are thus known as Routh-Hurwitz stability criteria. They are, however, not well known to British students of stability analysis nor are they readily accessible in British publications.

Such proofs as the writer has encountered of these criteria are complicated and difficult to follow.

The purpose of this paper is to present an apparently novel and relatively simple algebraic method for deriving these criteria and for verifying their validity.

The Routh Stability Criteria

If for a co-ordinate Z, say, in the differential equations of motion we write $Z=Z_0e^{\lambda t}$ where Z_0 is constant, we obtain after elimination a polynomial equation of the form:

$$F_6(\lambda)=\lambda^6+b_1\lambda^5+a_1\lambda^4+b_2\lambda^3+a_2\lambda^2+b_3\lambda+a_3=0$$

or

$$F_6(\lambda)=(\lambda^6+a_1\lambda^4+a_2\lambda^2+a_3)+\lambda(b_1\lambda^4+b_2\lambda^2+b_3)=0 \quad \text{........} \quad (1)$$

for which the sextic has been chosen for exemplification.

The coefficients a_r, b_r $(r=1, 2, 3)$ are supposed real and the conditions for the disturbed motion to be stable are that the three quadratic factors into which the sextic polynomial F_6 can be resolved are positive, that is if

$$F_6(\lambda)=(\lambda^2+p_1\lambda+q_1)(\lambda^2+p_2\lambda+q_2)(\lambda^2+p_3\lambda+q_3) \quad \text{............} \quad (2)$$

where p_r's, q_r's $(r=1, 2, 3)$ are all positive.

The Routh criteria to ensure that these requirements are as follows:

In the determinantal discriminant

$$\Delta_5=\begin{vmatrix} b_1 & b_2 & b_3 & 0 & 0 \\ 1 & a_1 & a_2 & a_3 & 0 \\ 0 & b_1 & b_2 & b_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 & b_3 \end{vmatrix} \quad \text{..........................} \quad (3)$$

the leading diagonal minors, viz.

$$\Delta_1=b_1,\quad \Delta_2=\begin{vmatrix} b_1 & b_2 \\ 1 & a_1 \end{vmatrix},\quad \Delta_3=\begin{vmatrix} b_1 & b_2 & b_3 \\ 1 & a_1 & a_2 \\ 0 & b_1 & b_2 \end{vmatrix}$$

$$\Delta_4=\begin{vmatrix} b_1 & b_2 & b_3 & 0 \\ 1 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 \end{vmatrix} \quad \text{.........................} \quad (4)$$

as well as Δ_5 itself, must all be positive. It can be shown[3] that these conditions will be fulfilled provided:

(i) All the a_r and b_r coefficients are positive.

(ii) The roots in λ^2 of the equations

$$f_3(\lambda^2)=\lambda^6+a_1\lambda^4+a_2\lambda^2+a_3=0 \quad \text{..........................} \quad (5)$$

$$f_2(\lambda^2)=b_1\lambda^4+b_2\lambda^2+b_3=0 \quad \text{.............................} \quad (6)$$

are real and negative and in addition separate one another. (For a simplified derivation of conditions for this see Appendix.)

REFERENCES TO LITERATURE

(1) E. J. Routh, *Advanced Rigid Dynamics.*

(2) F. M. Dimentberg, *Flexural Vibrations of Rotating Shafts*, Moscow, 1959, English translation, Butterworths, 1961.

(3) J. Morris and J. W. Head, 'Characteristic Polynomial Equations', AIRCRAFT ENGINEERING, December, 1955.

(4) F. R. Gantmacher, *A Theory of Matrices*, Vol. 2, p. 248. Translated and revised by J. L. Brenner, Interscience Publishers, New York and London, 1959.

The Routh-Hurwitz Criteria for Complex Characteristic Equations

It may, however, be convenient in certain cases to adopt complex co-ordinates, in which cases the consequential characteristic polynomial equation in λ may be of the form, for the cubic, for example:

$$F_3(\lambda)=\lambda^3+(a_1+ib_1)\lambda^2+(a_2+ib_2)\lambda+(a_3+ib_3)=0 \quad \text{............} \quad (7)$$

where for a complex co-ordinate in the differential equations of motion Z, say, we write $Z=Z_0e^{i\lambda t}$, where Z_0 is also complex.

In these circumstances, it is necessary for stability that all the coefficients of the imaginary parts of the roots in λ of Eq. (7) should be positive.

For example, the stability requirements for the complex Eq. (7) are such that the β_r's $(r=1, 2, 3)$ in the three appropriate linear complex factors, namely

$$(\lambda+\alpha_1-i\beta_1)(\lambda+\alpha_2-i\beta_2)(\lambda+\alpha_3-i\beta_3)$$

shall be positive.

The Routh-Hurwitz criteria[4] to ensure that these requirements are fulfilled are as follows: In the determinantal discriminant

$$\Delta_6=\begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & b_1 & b_2 & b_3 \end{vmatrix} \quad \text{........................} \quad (8)$$

the even leading diagonal minors must have alternate signs. Thus

$$\Delta_2=\begin{vmatrix} 1 & a_1 \\ 0 & b_1 \end{vmatrix},\quad \Delta_4=\begin{vmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & b_1 & b_2 \end{vmatrix} \quad \text{..............} \quad (9)$$

and Δ_6 must have alternate signs.

It may be noticed that if the foregoing requirements for stability are to be fulfilled b_1 must always be negative for it represents $-(\beta_1+\beta_2+\beta_3)$ where the β's are essentially positive. Thus if we write Eq. (7) in the form

$$\lambda^3+(a_1-ib_1)\lambda^2+(a_2-ib_2)\lambda+(a_3-ib_3)=0 \quad \text{...............} \quad (10)$$

the Routh-Hurwitz criteria may be written

(i) b_1 must be positive.

(ii) The 1st, 3rd, principal diagonal minors Δ_1, Δ_3, as well as the determinantal discriminant Δ_5 itself, namely

$$\Delta_5=\begin{vmatrix} b_1 & b_2 & b_3 & 0 & 0 \\ 1 & a_1 & a_2 & a_3 & 0 \\ 0 & b_1 & b_2 & b_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 & b_3 \end{vmatrix} \quad \text{...........................} \quad (11)$$

must all be positive.

Theorem Related to the Routh-Hurwitz Criteria

The following theorem is implicit in the Routh-Hurwitz criteria. Let us first consider the product of the two complex factors:

$$\Phi_2(\lambda)=(\lambda+\alpha_1-i\beta_1)(\lambda+\alpha_2-i\beta_2) \quad \ldots\ldots \quad (12)$$

in which β_1, β_2 are specifically positive.

Thus writing

$$\Phi_2(\lambda)=\lambda^2+(a_1-ib_1)\lambda+(a_2-ib_2) \quad \ldots\ldots \quad (13)$$

we have

$$a_1=(\alpha_1+\alpha_2), \qquad b_1=(\beta_1+\beta_2)$$
$$a_2=(\alpha_1\alpha_2-\beta_1\beta_2), \quad b_2=(\alpha_1\beta_2+\alpha_2\beta_1)$$

Let us now write

$$\Phi_2(\lambda)=F_2(\lambda)-iF_1(\lambda) \quad \ldots\ldots \quad (14)$$

so that

$$F_2(\lambda)=\lambda^2+a_1\lambda+a_2 \quad \ldots\ldots \quad (15)$$
$$F_1(\lambda)=b_1\lambda+b_2 \quad \ldots\ldots \quad (16)$$

Inserting the root of $F_1(\lambda)=0$ in the expression for $F_2(\lambda)$ we find that

$$F_2=(b_2^2-a_1b_1b_2+a_2b_1^2)/b_1^2 \quad \ldots\ldots \quad (17)$$

which having regard to foregoing relations may be written

$$-\beta_1\beta_2[(\alpha_1-\alpha_2)^2+(\beta_1+\beta_2)^2]/(\beta_1+\beta_2)^2 \quad \ldots\ldots \quad (18)$$

and since the β's are essentially positive this expression is negative.

Thus the Routh-Hurwitz criteria for the cubic characteristic polynomial may be expressed as follows.

Let

$$\Delta_3=\begin{vmatrix} b_1 & b_2 & 0 \\ 1 & a_1 & a_2 \\ 0 & b_1 & b_2 \end{vmatrix}$$

then $\Delta_1=b_1$ and $\Delta_3=(a_1b_1b_2-b_2^2-a_2b_1^2)$ must be positive.

Treatment of the General Case

For the general case let

$$\Phi_n(\lambda)=F_n(\lambda)-iF_{n-1}(\lambda)$$

be a polynomial characteristic equation with complex coefficients such that

$$F_n(\lambda)=\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\ldots\ldots+a_{n-1}\lambda+a_n \quad \ldots\ldots \quad (20)$$
$$F_{n-1}(\lambda)=b_1\lambda^{n-1}+b_2\lambda^{n-2}+\ldots\ldots+b_{n-1}\lambda+b_n \quad \ldots\ldots \quad (21)$$

Let us write for the product

$$(\lambda+\alpha-i\beta)(F_n-iF_{n-1})=(\lambda+\alpha)F_n-\beta F_{n-1}-i[\beta F_n+(\lambda+\alpha)F_{n-1}]$$
$$=f_{n+1}(\lambda)-if_n(\lambda) \quad \ldots\ldots \quad (22)$$

so that

$$f_{n+1}(\lambda)=(\lambda+\alpha)F_n-\beta F_{n-1} \quad \ldots\ldots \quad (23)$$

and

$$f_n(\lambda)=\beta F_n+(\lambda+\alpha)F_{n-1} \quad \ldots\ldots \quad (24)$$

Eliminating $(\lambda+\alpha)$ from Eqs. (23) and (24) we obtain

$$\beta=\frac{(F_nf_n-F_{n-1}f_{n+1})}{F_n^2+F_{n-1}^2} \quad \ldots\ldots \quad (25)$$

Suppose now that the roots of $F_{n-1}=0$ are real and separate those of $F_n=0$, also supposed real. Then for the roots of $F_{n-1}=0$, β as given by Eq. (25) will be positive if these roots also separate those of $f_n=0$, which thus must also have real roots; and as the roots of $F_n=0$ straddle those of $F_{n-1}=0$, then also the roots of $F_n=0$ must separate those of $f_{n+1}=0$, which must thus also have real roots. Hence also the roots of $F_n=0$ and the roots of $f_n=0$, since they straddle those of $F_{n-1}=0$, must also separate the roots of $f_{n+1}=0$.

All these conditions are necessary and sufficient for β, as given by Eq. (25), to be positive.

Reciprocally if the roots of $f_n=0$ separate those of $f_{n+1}=0$ then reasoning on the same lines the roots of $F_{n-1}=0$, $F_n=0$ will be real and the former will separate the latter.

Hence we may express the Routh-Hurwitz stability criteria as follows: Let the complex characteristic equation be

$$F_n(\lambda)-iF_{n-1}(\lambda)=0 \quad \ldots\ldots \quad (26)$$

where F_n is a polynomial of the nth degree and F_{n-1} a polynomial of the $(n-1)$th degree in λ, both with real coefficients. Then for stability the 1st, 3rd, 5th, etc., principal diagonal minors of the determinental, discriminant Δ_{2n-1} must be positive.

Thus

$$\Delta_1=b_1, \quad \Delta_3=\begin{vmatrix} b_1 & b_2 & b_3 \\ 1 & a_1 & a_2 \\ 0 & b_1 & b_2 \end{vmatrix}$$

$$\Delta_5=\begin{vmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 & b_3 \end{vmatrix}$$

and so on, must be positive.

These are really the necessary and sufficient conditions for the zeros of F_{n-1}, F_n to be real and to separate one another. (See Appendix.)

Simple Numerical Example

As an example let us consider the complex cubic

$$\Phi_3(\lambda)=F_3(\lambda)-iF_2(\lambda) \quad \ldots\ldots \quad (27)$$

where

$$F_3(\lambda)=\lambda^3-2\lambda^2-4\lambda+4 \quad \ldots\ldots \quad (28)$$

and

$$F_2(\lambda)=3\lambda^2-4\lambda-2 \quad \ldots\ldots \quad (29)$$

It is easy to verify that

$$\Phi_3(\lambda)=(\lambda-1-i)(\lambda+1-i)(\lambda-2-i) \quad \ldots\ldots \quad (30)$$

and thus corresponds to a stable motion. Applying the foregoing Routh-Hurwitz criteria we have

$$a_1=-2, \quad a_2=-4, \quad a_3=4$$
$$b_1=3, \quad b_2=-4, \quad b_3=-2$$

and the discriminant is

$$\Delta_3=\begin{vmatrix} 3, & -4, & -2, & 0, & 0 \\ 1, & -2, & -4, & 4, & 0 \\ 0, & 3, & -4, & -2, & 0 \\ 0, & 1, & -2, & -4, & 4 \\ 0, & 0, & 3, & -4, & 2 \end{vmatrix}$$

We find on evaluation that

$$\Delta_1=+3, \quad \Delta_3=+38, \quad \Delta_5=+520$$

all of which are positive as they should be.

APPENDIX

Conditions for the Roots of Two Real Polynomials of Successive Degrees to be Real and to Separate One Another

Let us first consider the particular case of the polynomials

$$F_3=\lambda^3+b_1\lambda^2+b_2\lambda+b_3 \quad \ldots\ldots \quad (31)$$
$$F_4=\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4 \quad \ldots\ldots \quad (32)$$

where the a's and b's are all real (the coefficient of the highest powers being unity and thus positive).

We find that

(i) $F_4=(\lambda+a_1-b_1)F_3-\Delta_3F_2$ $\quad \ldots\ldots \quad (33)$

where

$$\Delta_3=[b_1(a_1-b_1)-(a_2-b_2)] \quad \ldots\ldots \quad (34)$$
$$F_2=\lambda^2+c_1\lambda+c_2 \quad \ldots\ldots \quad (35)$$
$$c_1=[b_2(a_1-b_1)-(a_3-b_3)]/\Delta_3 \quad \ldots\ldots \quad (36)$$
$$c_2=[b_3(a_1-b_1)-a_4]/\Delta_3 \quad \ldots\ldots \quad (37)$$

(ii) $F_3=(\lambda+b_1-c_1)F_2-\Delta_3{}^{\prime}F_1$ $\quad \ldots\ldots \quad (38)$

where

$$\Delta_3{}^{I}=c_1(b_1-c_1)-(b_2-c_2) \quad \ldots\ldots \quad (39)$$
$$F_1=\lambda+d_1 \quad \ldots\ldots \quad (40)$$

$$d_1=[c_2(b_1-c_1)-b_3]/\Delta_3{}^{I} \quad \ldots\ldots \quad (41)$$

(iii) $F_2=(\lambda+c_1-d_1)F_1-\Delta_3{}^{II}$ ...... (42)

where

$$\Delta_3{}^{II}=d_1(c_1-d_1)-c_2 \quad \ldots\ldots \quad (43)$$

Now it follows from Eq. (42) that the root of $F_1=0$ (which must of course be real) will separate the two roots of $F_2=0$ provided that $\Delta_3{}^{II}$ is positive. The roots of $F_2=0$ must thus be real. Next from Eq. (38), since the two roots of $F_2=0$ are real and straddle that of $F_1=0$ they will, provided $\Delta_3{}^{I}$ is positive, separate the roots of $F_3=0$ which roots must thus be real. Lastly, since the roots of $F_3=0$ are real and straddle those of $F_2=0$ they will, provided Δ_3 is positive, separate the roots of $F_4=0$ which must thus also be real.

Hence the necessary and sufficient conditions for the roots of $F_3=0$, $F_4=0$ to be real and to separate one another are that Δ_3, $\Delta_3{}^{I}$, $\Delta_3{}^{II}$ must all be positive.

Now we find that

$$\Delta_3=\begin{vmatrix} 1 & b_1 & b_2 \\ 1 & a_1 & a_2 \\ 0 & 1 & b_1 \end{vmatrix} \quad \ldots\ldots \quad (44)$$

$$\Delta_3{}^{I}=\Delta_1\Delta_5/\Delta_3{}^2 \quad \ldots\ldots \quad (45)$$

where

$$\Delta_5=\begin{vmatrix} 1 & b_1 & b_2 & b_3 & 0 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & b_1 & b_2 \end{vmatrix} \quad \ldots\ldots \quad (46)$$

and

$$\Delta_3{}^{II}=\Delta_7/\Delta_3{}^3\Delta_3{}^{I2}=\Delta_3\Delta_7/\Delta_5{}^2 \quad \ldots\ldots \quad (47)$$

where

$$\Delta_7=\begin{vmatrix} 1 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 1 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & 1 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 1 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 1 & b_1 & b_2 & b_3 \end{vmatrix} \quad \ldots\ldots \quad (48)$$

It follows that the necessary and sufficient conditions for the roots of $F_3=0$, $F_4=0$ to be real and to separate one another will be fulfilled provided Δ_1, Δ_3, Δ_5, Δ_7 are all positive.

It is fairly clear that the foregoing theorem is quite general for polynomials of any degree. Thus for the polynomial equations $F_{n-1}=0$, $F_n=0$ the necessary and sufficient conditions for their roots to be real and to separate one another will be Δ_1, Δ_3, Δ_5, Δ_7, ... Δ_{2n-1} must all be positive where

$$\Delta_{2n-1}=\begin{vmatrix} 1 & b_1 & b_2 & b_3 \ldots\ldots b_{n-1} & 0 \ldots\ldots\ldots & \\ 1 & a_1 & a_2 & a_3 \ldots\ldots a_{n-1} & a_n \ldots\ldots\ldots & \\ 0 & 1 & b_1 & b_2 \ldots\ldots b_{n-2} & b_{n-1} \ldots\ldots\ldots & \\ 0 & 1 & a_1 & a_2 \ldots\ldots a_{n-2} & a_{n-1} & a_n \ldots \\ 0 & 0 & 1 & b_1 \ldots\ldots b_{n-3} & b_{n-2} & b_{n-1} \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} \quad \ldots \quad (49)$$

The primed Δ's will be as follows, etc.

$$\Delta_3{}^{I}=\Delta_1\Delta_5/\Delta_3{}^2, \quad \Delta_3{}^{II}=\Delta_3\Delta_7/\Delta_5{}^2, \ldots.$$

or generally

$$\Delta_3{}^{N}=\Delta_{2n-1}\Delta_{2n+3}/\Delta_{2n+1}{}^2 \quad \ldots\ldots \quad (50)$$

Additional Conditions for Roots to be Negative

For additional conditions to ensure that the roots of $F_3=0$ and $F_4=0$ are not only real and separate one another but are also all negative, we proceed as follows

(i) $F_4=\lambda F_3+\Delta_2F_3'$ ...... (51)

where

$$\Delta_2=(a_1-b_1)$$

$$F_3'=\lambda^3+\alpha_1\lambda^2+\alpha_2\lambda+\alpha_3$$

$$\alpha_1=\frac{a_2-b_2}{a_1-b_1}, \quad \alpha_2=\frac{a_3-b_3}{a_1-b_1}, \quad \alpha_3=\frac{a_4}{a_1-b_1}$$

(ii) $F_3=F_3'+\Delta_2{}^{I}F_2$ ...... (52)

where

$$\Delta_2{}^{I}=(b_1-\alpha_1)$$

$$F_2=\lambda^2+\beta_1\lambda+\beta_2$$

$$\beta_1=\frac{b_2-\alpha_2}{b_1-\alpha_1}, \quad \beta_2=\frac{b_3-\alpha_3}{b_1-\alpha_1}$$

(iii) $F_3'=\lambda F_2+\Delta_2{}^{II}F_2'$ ...... (53)

where

$$\Delta_2{}^{II}=(\alpha_1-\beta_1)$$

$$F_2'=\lambda^2+\gamma_1\lambda+\gamma_2$$

$$\gamma_1=\frac{\alpha_2-\beta_2}{\alpha_1-\beta_1}, \quad \gamma_2=\frac{\alpha_3}{\alpha_1-\beta_1}$$

(iv) $F_2=F_2'+\Delta_2{}^{III}F_1$ ...... (54)

where

$$\Delta_2{}^{III}=(\beta_1-\gamma_1)$$

$$F_1=\lambda+\delta_1$$

$$\delta_1=\frac{\beta_2-\gamma_2}{\beta_1-\gamma_1}$$

(v) $F_2'=\lambda F_1+\Delta_2{}^{IV}F_1'$ ...... (55)

where

$$\Delta_2{}^{IV}=(\gamma_1-\delta_1)$$

$$F_1'=\lambda+\epsilon_1$$

$$\epsilon_1=\frac{\gamma_2}{\gamma_1-\delta_1}$$

(vi) $F_1=F_1'+\Delta_2{}^{V}$ ...... (56)

where

$$\Delta_2{}^{V}=(\delta_1-\epsilon_1)$$

Proceeding as in the previous case we find that

$$\Delta_2{}^{I}=\Delta_3/\Delta_2, \quad \Delta_2{}^{II}=\Delta_1\Delta_4/\Delta_2\Delta_3, \quad \Delta_2{}^{III}=\Delta_2\Delta_5/\Delta_3\Delta_4$$

$$\Delta_2{}^{IV}=\Delta_3\Delta_6/\Delta_4\Delta_5, \quad \Delta_2{}^{V}=\Delta_4\Delta_7/\Delta_5\Delta_6$$

or generally

$$\Delta_2{}^{N}=\Delta_{n-1}\Delta_{n+2}/\Delta_n\Delta_{n+1}$$

Let us now suppose that $a_r(r=1, 2, 3, 4)$, $b_r(r=1, 2, 3)$ and $\Delta_r(r=1, 2, 3, 4, 5, 6, 7)$ are all positive. From Eq. (51) we find that $\Delta_2\alpha_3=a_4$, so that α_3 is positive. From Eq. (53) $\Delta_2{}^{II}\gamma_2=\alpha_3$ so that γ_2 is positive. From Eq. (55) $\Delta_2{}^{IV}\epsilon_1=\gamma_2$ so that ϵ_1 is positive. Also $\Delta_2{}^{V}=(\delta_1-\epsilon_1)$ so that δ_1 is positive. It follows from these relations and Eq. (54) that β_2 is also positive. Hence from Eq. (55) the roots of $F_2'=0$ must be real and negative; from Eq. (54) the roots of $F_2=0$ must likewise be real and negative; from Eq. (53) the roots of $F_3'=0$ must be real and negative; from Eq. (52) the roots of $F_3=0$ must likewise be real and negative; lastly from Eq. (51) it follows that $F_4=0$ must have real negative roots.

It is evident that the foregoing is quite general and applicable to equations of any degree just as in the previous case for real roots and their separation.

A.R.B. NOTICES

The Council of the Air Registration Board has announced the issue of the undermentioned *Civil Aircraft Inspection Procedures*: all dated October 16, 1961:

Protective Treatments
BL/7-3, Issue 2, Chromate Treatment of Magnesium Alloys.
BL/7-4, Issue 1, Phosphating of Steels.

Testing of Materials and Chemical Solutions
BL/10-3, Issue 1, Tensile Testing of Metallic Materials.

Flying Controls
AL/3-4, Issue 3, Duplicate Inspection.
AL/3-6, Issue 3, Cables-Swaging.

Landing Gear
AL/8-2, Issue 2, Wheels.
AL/8-3, Issue 2, Wheel Brakes—Pressure Operated.

Fuel System
PPL/2-1, Issue 2, Installation and Maintenance.

Control Systems
Leaflet BL/6-4, Issue 2, Oxy-Acetylene Welding.
Leaflet BL/7-2, Issue 4, Cadmium Plating.