A Note on the Escalator Process

Routh's Stability Criteria for Polynomial Characteristic Equations Derived by Algebra

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Summary

If an algebraic polynomial equation has roots which are negative if real and have negative real parts if complex, the coefficients must satisfy certain fundamental conditions originally formulated by Routh. These conditions are here derived by comparatively simple algebra for the sextic equation by a method which can be generalized; its extension to equations of the eighth and tenth degree is indicated.

The case of damped Lagrangian frequency equations is considered as an appropriate epilogue.

Introduction

ARTIFICES inherent in the 'Escalator' process are here used to formulate Routh's classic criteria for a polynomial equation of even degree in m, say, to have all its roots negative if real and with negative real parts if complex. The necessary and sufficient conditions can be expressed as follows:

(*a*) All the coefficients must have the same sign.

(*b*) The two equations in m^2 formed respectively by taking the terms of odd degree (divided by m) and those of even degree must have all their roots in m^2 real (and negative) and the roots of each equation must separate those of the other.

It is hoped that the presentation here used will be of interest to those unskilled in or ill-acquainted with the intricacies of the theory of equations, determinants, etc., involved or implicit in the classical treatment.

The case of damped Lagrangian frequency equations is included because of its particular practical interest; there appears to be a species of root separation theorem for the determinantal characteristic equations of any two consecutive orders.

1. Case of the Sextic from First Principles

We consider for exemplification the algebraic polynomial equation of the sixth degree, viz.

$$m^6+bm^5+cm^4+dm^3+em^2+fm+g=0 \quad \ldots\ldots\ldots \quad (1)$$

where the coefficients b, c, d, e, f, g are real.

Factorizing the polynomial into quadratic factors, the Eq. (1) may be expressed in the form

$$(m^2+p_1m+q_1)(m^2+p_2m+q_2)(m^2+p_3m+q_3)=0 \quad \ldots\ldots\ldots \quad (2)$$

where the p's and q's are real.

If in addition to being real they are positive, then the roots of (1) will be negative if real and if conjugate complex their real parts will be negative.

It is evident therefore that a pre-requisite condition for such to be the case is that the coefficients b, c, d, e, f, g of Eq. (1) must each be positive as well as real.

In what follows we assume that these coefficients are of that nature.

Next we notice that (1) may be written

$$-bm=f_3(m^2)/f_2(m^2) \quad \ldots\ldots\ldots \quad (3)$$

where

$$f_2(m^2)=m^4+dm^2/b+f/b,$$
$$f_3(m^2)=m^6+cm^4+em^2+g,$$

and thus (3) in turn can be expressed by the H.C.F. process as

$$-bm=(bc-d)/b+m^2-D/[(d/b-F/D)+m^2-Q/(F/D+m^2)] \quad \ldots \quad (4)$$

where

$$\left.\begin{aligned} D&=\{(bc-d)/b\}d/b+f/b-e \\ F&=\{(bc-d)/b\}f/b-g \\ Q&=\{d/b-F/D\}F/D-f/b \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (5)$$

It is convenient also to express D and F in terms of the p's and q's in (2): we find that

$$bc-d=(p_2+p_3)(p_3+p_1)(p_1+p_2)+p_1q_1+p_2q_2+p_3q_3 \quad \ldots\ldots\ldots \quad (6)$$

whence we obtain

$$\left.\begin{aligned} b^2D&=p_1p_2p_3(p_2+p_3)(p_3+p_1)(p_1+p_2) \\ &\quad+\Sigma p_1(p_2+p_3)q_1\{p_1(p_2+p_3)+p_2^2+p_3^2\}+\Sigma p_1p_2(q_1-q_2)^2 \\ b^2F&=q_1q_2q_3[(p_2+p_3)(p_3+p_1)(p_1+p_2)(p_1/q_1+p_2/q_2+p_3/q_3) \\ &\quad+\Sigma p_2p_3(q_2/q_3+q_3/q_2-2)] \end{aligned}\right\} \quad (7)$$

Now Q, save for a factor, will be found to be the discriminant of $f_2(m^2)=0$ and $f_3(m^2)=0$.

By such discriminant is meant the expression (of the coefficients) which must be zero for these two equations to have a common root in m^2 and is, in determinantal form

$$\Delta_6=\begin{vmatrix} b & d & f & 0 & 0 \\ 1 & c & e & g & 0 \\ 0 & b & d & f & 0 \\ 0 & 1 & c & e & g \\ 0 & 0 & b & d & f \end{vmatrix} \quad \ldots\ldots\ldots \quad (8)$$

It is to be noticed that this discriminant is the product $\Pi(m_r+m_s)$ where m_r, m_s are two different roots in m of the Eq. (1) or its equivalent (2); and this product, for the sextic equation in question, may be proved to be

$$\begin{aligned} &p_1p_2p_3[(q_1-q_2)^2+(p_1+p_2)(p_1q_2+p_2q_1)] \\ &\quad\times[(q_2-q_3)^2+(p_2+p_3)(p_2q_3+p_3q_2)] \\ &\quad\times[(q_3-q_1)^2+(p_3+p_1)(p_3q_1+p_1q_3)] \quad \ldots\ldots\ldots \quad (9) \end{aligned}$$

Thus if the p's and q's are positive the discriminant Δ_6 will clearly be positive.

Now in the circumstances under consideration Q will be positive as the factor by which it is linked to Δ_6 will be found to be positive.

Hence Q must be positive if the roots of Eq. (1) are to have the required qualities, in which case it follows that F/D and $(d/b-F/D)$ must be positive. Also $(bc-d)$ is seen from the first of (6) to be positive when the p's and q's are of that sign; from which it follows that F will be positive and in consequence D will likewise be positive.

A cardinal implication of these conditions is that the roots in m^2 of $f_2(m^2)=0$ and $f_3(m^2)=0$ must be real and negative and in addition must separate each other.

The corresponding criteria in accordance with Routh's treatment are: b, c, d, e, f, g, $(bc-d)$ and all subsequent minors included in and including the discriminant Δ_6 itself must be positive.

It may be verified that the criteria arrived at in the foregoing are in consonance with their classical determinantal counterparts.

For example

$$b^2D=\begin{vmatrix} b & d & f \\ 1 & c & e \\ 0 & b & d \end{vmatrix}$$

$$b^3D^2Q+gb^3D(d/b-F/D)=f\begin{vmatrix} b & d & f & 0 \\ 1 & c & e & g \\ 0 & b & d & f \\ 0 & 1 & c & e \end{vmatrix}$$

and $\qquad b^3D^2Q=\Delta_6$

Although we have dealt only with a sixth degree polynomial equation it may be noticed that the method outlined is of a general nature, and that criteria for the kind of roots in question are based on the essential requirement that for the algebraic polynomial equation of the $2n$th degree of the form

$$f_n(m^2)+bmf_{n-1}(m^2)=0 \quad \ldots\ldots\ldots \quad (10)$$

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the equations $f_{n-1}(m^2)=0, f_n(m^2)=0$ must have real negative roots in m^2 which roots must separate each other.

In dealing with equations of odd degree as for example the quintic

$$m^5+bm^4+cm^3+dm^2+em+f=0 \quad \ldots\ldots \quad (11)$$

supposed of the form

$$(m^2+p_1m+q_1)(m^2+p_1m+q_1)(m+p_3)=0 \quad \ldots\ldots \quad (12)$$

we may regard it as the sextic with one zero root consequent on the coefficient g being put equal to zero.

In the particular quintic it will be noticed that the applicable criteria are: b, c, d, e, f must be positive; $(bc-d)$ must be positive and subsequent principal minors included in and including the discriminant

$$\Delta_5=\begin{vmatrix} b & d & f & 0 \\ 1 & c & e & 0 \\ 0 & b & d & f \\ 0 & 1 & c & e \end{vmatrix}$$

must be positive.

This discriminant may be deduced from (9) (in terms of the p's and q's) by putting $q_3=0$ and dividing by $p_3q_1q_2(=f)$, i.e. that part of the product $\Pi(m_r+m_s)$ due to the zero root of the curtailed sextic. Thus

$$\Delta_5=p_1p_2[(q_1-q_2)^2+(p_1+p_2)(p_1q_2+p_1q_1)] \times[q_2+(p_2+p_3)p_3][q_1+(p_3+p_1)p_3] \quad \ldots\ldots \quad (14)$$

2. Application of the Escalator Process

As an illustration of this let us consider the octic equation in m, viz.

$$-bm=f_4(m^2)/f_3(m^2) \quad \ldots\ldots \quad (15)$$

where $f_3(m^2)=m^6+dm^4/b+fm^2/b+h/b,$

$f_4(m^2)=m^8+cm^6+em^4+gm^2+k,$

and since the roots of $f_3(m^2)=0$, $f_4(m^2)=0$ in m^2 are supposed real and negative and separate each other (15) may be expressed in the Escalator form

$$-bm=(bc-d)/b+m^2-[P_1^2/(m_1^2+m^2)+P_2^2/(m_2^2+m^2)+P_3^2/(m_3^2+m^2)] \quad \ldots\ldots \quad (16)$$

where $-m_1^2, -m_2^2, -m_3^2$ are the roots in m^2 of $f_3(m^2)=0$.

These latter suppositions require that $(bc-d)/b$, P_1^2, P_2^2, P_3^2 should all be real positive quantities.

We notice that if we put $m=\alpha+i\beta$ in (16) where α and β are real and $i=\sqrt{(-1)}$, we obtain

$$-b(\alpha+i\beta)=(bc-d)/b+\alpha^2-\beta^2+2i\alpha\beta -\Sigma P_1^2\{m_1^2+\alpha^2-\beta^2-2i\alpha\beta\}/\{(m_1^2+\alpha^2-\beta^2)^2+4\alpha^2\beta^2\} \ldots \quad (17)$$

Equating imaginary parts we find that

$$-b=2\alpha[1+\Sigma P_1^2/\{(m_1^2+\alpha^2-\beta^2)^2+4\alpha^2\beta^2\}] \quad \ldots\ldots \quad (18)$$

and thus since b is positive 2α must be negative.

Hence if the roots of $f_3(m^2)=0$ and $f_4(m^2)=0$ in m^2 are real and separate each other then the conjugate complex roots of $f_4(m^2)+bmf_3(m^2)=0$ have their real parts negative.

Similarly it may be seen that pairs of real roots of the form say $-p/2\pm(p^2/4-q)^{1/2}$, arising from a quadratic factor m^2+pm+q, must be negative.

Reverting now to the octic Eq. (15) we have

$$f_4(m^2)/f_3(m^2)=(bc-d)/b+m^2 -[P_1^2/(m_1^2+m^2)+P_2^2/(m_2^2+m^2)+P_3^2/(m_3^2+m^2)] \ldots \quad (19)$$

where $f_3(m^2)=(m_1^2+m^2)(m_2^2+m^2)(m_3^2+m^2)$.

Multiplying both sides of (19) by $f_3(m^2)$ and equating coefficients of corresponding powers of m^2 on both sides we find that

$$D=\{(bc-d)/b\}d/b+f/b-e=P_1^2+P_2^2+P_3^2 \quad \ldots\ldots \quad (20)$$

$$F=\{(bc-d)/b\}f/b+h/b-g=\Sigma P_1^2(m_2^2+m_3^2) \quad \ldots\ldots \quad (21)$$

$$H=\{(bc-d)/b\}h/b-k=\Sigma P_1^2m_2^2m_3^2 \quad \ldots\ldots \quad (22)$$

Thus $Dm^4+Fm^2+H=\Sigma P_1^2(m_1^2+m^2)(m_2^2+m^2) \quad \ldots\ldots \quad (23)$

so that the roots $-m_{12}^2, -m_{23}^2$ of

$$f_2(m^2)=(m^4+Fm^2/D+H/D)=0 \quad \ldots\ldots \quad (24)$$

must separate the roots $-m_1^2, -m_2^2, -m_3^2$ of $f_3(m^2)=0$.

Next

$$f_3(m^2)/f_2(m^2)=(m^6+dm^4/b+fm^2/b+h/b)/(m^4+Fm^2/D+H/D) \quad \ldots \quad (25)$$

or

$$f_3(m^2)/f_2(m^2)=(d/b-F/D)+m^2-\{F^1(m^2+H^1/F^1)\}/\{m^4+Fm^2/D+H/D\} \quad \ldots\ldots \quad (26)$$

where $F^1=(d/b-F/D)F/D+H/D-f/b \quad \ldots\ldots \quad (27)$

$$H^1=(d/b-F/D)H/D-h/b \quad \ldots\ldots \quad (28)$$

and since the roots of $f_2(m^2)=0$ in m^2 must be real and separate the roots of $f_3(m^2)=0$, which must likewise be real, it follows that F^1 must be positive; H^1/F^1 must be positive and in consequence H^1 must also be positive and

$$(H^1)^2/(F^1)^2-(F/D)(H^1/F^1)+H/D \quad \ldots\ldots \quad (29)$$

must be negative; or alternatively

$$(F/D)(H^1/F^1)-(H^1)^2/(F^1)^2-H/D \quad \ldots\ldots \quad (30)$$

must be positive.

These criteria are in conformity with the corresponding classical counterparts, viz. b, c, d, e, f, g, h, k; $(bc-d)$ and subsequent principal minors included in and including the discriminant

$$\Delta_8=\begin{vmatrix} b & d & f & h & 0 & 0 & 0 \\ 1 & c & e & g & k & 0 & 0 \\ 0 & b & d & f & h & 0 & 0 \\ 0 & 1 & c & e & g & k & 0 \\ 0 & 0 & b & d & f & h & 0 \\ 0 & 0 & 1 & c & e & g & k \\ 0 & 0 & 0 & b & d & f & h \end{vmatrix} \quad \ldots\ldots \quad (31)$$

must be positive.

To illustrate how in the quest for the appropriate criteria we may 'escalate' from equations of lower degree to those of higher degree let us consider the following equation of the tenth degree, viz.

$$-bm=\frac{m^{10}+cm^8+em^6+gm^4+km^2+n}{m^8+\frac{d}{b}m^6+\frac{f}{b}m^4+\frac{h}{b}m^2+\frac{l}{b}}$$

$$\text{or } -bm=\frac{bc-d}{b}+m^2-\frac{(Dm^6+Fm^4+Hm^2+L)}{\left(m^8+\frac{d}{b}m^6+\frac{f}{b}m^4+\frac{h}{b}m^2+\frac{l}{b}\right)} \quad \ldots\ldots \quad (32)$$

where $D=\{(bc-d)/b\}d/b+f/b-e \quad \ldots\ldots \quad (33)$

$$F=\{(bc-d)/b\}f/b+h/b-g \quad \ldots\ldots \quad (34)$$

$$H=\{(bc-d)/b\}h/b+l/b-k \quad \ldots\ldots \quad (35)$$

$$L=\{(bc-d)/b\}l/b-n \quad \ldots\ldots \quad (36)$$

all of which must be positive.

For the further conditions we proceed as for the case of the octic with $d/b, f/b, h/b, l/b$ respectively in place of c, e, g, k; and $F/D, H/D, L/D$ respectively in place of $d/b, f/b, h/b$.

3. Case of Damped Lagrangian Frequency Equations

Generally the polynomial characteristic equation will arise from damped Lagrangian frequency equations of the form

$$(a_{11}+b_{11}m+c_{11}m^2)x+(a_{12}+b_{12}m+c_{12}m^2)y+(a_{13}+b_{13}m+c_{13}m^2)z=0 \quad (37)$$

$$(a_{12}+b_{12}m+c_{12}m^2)x+(a_{22}+b_{22}m+c_{22}m^2)y+(a_{23}+b_{23}m+c_{23}m^2)z=0 \quad (38)$$

$$(a_{13}+b_{13}m+c_{13}m^2)x+(a_{23}+b_{23}m+c_{23}m^2)y+(a_{33}+b_{33}m+c_{33}m^2)z=0 \quad (39)$$

Associated with any root m of the characteristic equation let

$$x=r_1\exp(i\alpha_1),\ y=r_2\exp(i\alpha_2),\ z=r_3\exp(i\alpha_3),$$

where r_1, r_2, r_3 are real and positive and i is the imaginary $\sqrt{(-1)}$.

Multiply (26), (27), (28) respectively by $r_1\exp(-i\alpha_1)$, $r_2\exp(-i\alpha_2)$, $r_3\exp(-i\alpha_3)$ and add. We obtain the quadratic equation

$$A+Bm+Cm^2=0 \quad \ldots\ldots \quad (40)$$

where $A=a_{11}r_1^2+2a_{12}r_1r_2\cos(\alpha_1-\alpha_2)+a_{22}r_2^2+\text{etc.} \quad \ldots\ldots \quad (41)$

$$B=b_{11}r_1^2+2b_{12}r_1r_2\cos(\alpha_1-\alpha_2)+b_{22}r_2^2+\text{etc.} \quad \ldots\ldots \quad (42)$$

$$C=c_{11}r_1^2+2c_{12}r_1r_2\cos(\alpha_1-\alpha_2)+c_{22}r_2^2+\text{etc.} \quad \ldots\ldots \quad (43)$$

and thus if the a's, b's and c's respectively are coefficients of positive homogeneous quadratic functions, A, B and C will each be positive; and, in consequence, the cognate root m will be negative if real, and if one of a pair of conjugate complex roots will have its real part negative.

This theorem is due to Routh and is given in his *Advanced Rigid Dynamics.*

Now the characteristic equation corresponding to the three equations (41), (42), (43) will be, in determinantal form

$$\begin{vmatrix} a_{11}+b_{11}m+c_{11}m^2 & a_{12}+b_{12}m+c_{12}m^2 & a_{13}+b_{13}m+c_{13}m^2 \\ a_{12}+b_{12}m+c_{12}m^2 & a_{22}+b_{22}m+c_{22}m^2 & a_{23}+b_{23}m+c_{23}m^2 \\ a_{13}+b_{13}m+c_{13}m^2 & a_{23}+b_{23}m+c_{23}m^2 & a_{33}+b_{33}m+c_{33}m^2 \end{vmatrix}=0 \quad (44)$$

If this Eq. (44) be expanded and expressed as

$$F_3(m^2)+mF_2(m^2)=0 \quad \ldots\ldots \quad (45)$$

in which $F_2(m^2)$, $F_3(m^2)$ are polynomials of the second and third degrees respectively in m^2, then the roots of these equations in m^2 will be real and will separate each other provided the a's and b's and c's respectively are coefficients of positive homogeneous quadratic functions.

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