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Morrey-Sobolev Spaces on Metric Measure Spaces

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Abstract In this article, the authors introduce the Newton-Morrey-Sobolev space on a metric measure space $(\mathscr{X}, d, \mu)$. The embedding of the Newton-Morrey-Sobolev space into the Hölder space is obtained if $\mathscr{X}$ supports a weak Poincaré inequality and the measure μ is doubling and satisfies a lower bounded condition. Moreover, in the Ahlfors Q-regular case, a Rellich-Kondrachov type embedding theorem is also obtained. Using the Hajłasz gradient, the authors also introduce the Hajłasz-Morrey-Sobolev spaces, and prove that the Newton-Morrey-Sobolev space is a subspace of the Hajłasz-Morrey-Sobolev space when μ is doubling and $\mathscr{X}$ supports a weak Poincaré inequality. In particular, on the Euclidean space $\mathbb{R}^n$, the authors obtain the coincidence among the Newton-Morrey-Sobolev space, the Hajłasz-Morrey-Sobolev space and the classical Morrey-Sobolev space, and also the coincidence between the Hajłasz-Morrey-Sobolev space and the Hardy-Morrey-Sobolev space. In the case when $\mathscr{X}$ is a metric space of homogeneous type, a characterization of the Hajłasz-Morrey-Sobolev space via the grand maximal function is also established. Finally, when μ is doubling and satisfies some measure decay property, the authors obtain the boundedness of some fractional maximal operators on Morrey spaces, Newton-Morrey-Sobolev spaces and Hajłasz-Morrey-Sobolev spaces.

1 Introduction

In 1996, via introducing the notion of Hajłasz gradients, Hajłasz [18] obtained an equivalent characterization of the classical Sobolev space on $\mathbb{R}^n$, which becomes an effective way to define Sobolev spaces on metric spaces. From then on, several different approaches to introduce Sobolev spaces on metric measure spaces were developed; see, for example, [32, 15, 47, 21, 19, 28, 52, 33].

Throughout the paper, $(\mathscr{X}, d, \mu)$ denotes a metric measure space with a nontrivial Borel regular measure μ, which is finite on bounded sets and positive on open sets. Let f be a measurable function on $\mathscr{X}$. Recall that a non-negative function g on $\mathscr{X}$ is called a *Hajłasz*

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gradient of f if there exists a set $E\subset X$ such that $\mu(E)=0$ and, for all $x,\ y\in\mathscr{X}\setminus E$,

$$|f(x)-f(y)|\le d(x,y)[g(x)+g(y)].$$

The *Hajłasz-Sobolev space* $M^{1,p}(\mathscr{X})$ with $p\in[1,\infty]$ is then defined to be the space of all measurable functions $f\in L^p(\mathscr{X})$ which have Hajłasz gradients $g\in L^p(\mathscr{X})$. The *norm* of this space is defined by $\|f\|_{M^{1,p}(\mathscr{X})}:=\|f\|_{L^p(\mathscr{X})}+\inf_g\|g\|_{L^p(\mathscr{X})}$, where the infimum is taken over all Hajłasz gradients g of f. It was proved in [18] that when $\mathscr{X}=\mathbb{R}^n$ and $p\in(1,\infty]$, $M^{1,p}(\mathbb{R}^n)$ coincides with the classical Sobolev space $W^{1,p}(\mathbb{R}^n)$.

Over a decade ago, based on the notions of upper gradients and weak upper gradients, Shanmugalingam [47, 48] introduced another type of Sobolev spaces on metric measure spaces, which are called Newtonian spaces or Newton-Sobolev spaces. These spaces were also proved to coincide with the Hajłasz-Sobolev spaces if $\mathscr{X}$ supports some Poincaré inequality and the measure is doubling. Now we recall their definitions.

Recall that we call γ a *curve* if it is a continuous mapping from an interval into $\mathscr{X}$. A curve γ is *rectifiable* if its length is finite. All rectifiable curve can be arc-length parameterized. *Without loss of generality, we may assume that all curves appearing in this article are always treated as parameterized.*

Let $p\in[1,\infty)$ and Γ be a family of non-constant rectifiable curves on $\mathscr{X}$. The *p-modulus* of Γ is defined as $\mathrm{Mod}_p(\Gamma):=\inf_\rho\|\rho\|^p_{L^p(\mathscr{X})}$, where the infimum is taken over all functions ρ in the admissible class $F(\Gamma)$ for Γ and

$$F(\Gamma):=\left\{\rho\in[0,\infty]:\ \rho\text{ is measurable and }\int_\gamma\rho(s)\,ds\ge1\text{ for all }\gamma\in\Gamma\right\}. \tag{1.1}$$

If Γ contains a constant curve, then $F(\Gamma)=\emptyset$. We let the infimum over the empty set always be infinity.

Let f be a measurable function on $\mathscr{X}$. A non-negative function g is called an *upper gradient* of f if, for any curve $\gamma\in\Gamma_{\rm rect}$,

$$|f\circ\gamma(0)-f\circ\gamma(l(\gamma))|\le\int_\gamma g(s)\,ds, \tag{1.2}$$

where $\Gamma_{\rm rect}$ is the *class of all non-constant rectifiable curves in* $\mathscr{X}$. Moreover, if the inequality (1.2) holds for all the curves except for a family of curves of p-modulus zero, then we call g a *p-weak upper gradient* of f. The notion of p-weak upper gradient was introduced by Heinonen and Koskela in [20]; see also [25] and [47, 48].

For all $p\in[1,\infty)$, denote by the *symbol* $\widetilde{N}^{1,p}(\mathscr{X})$ the space of all measurable functions $f\in L^p(\mathscr{X})$ which have p-weak upper gradients $g\in L^p(\mathscr{X})$ and, for all $f\in\widetilde{N}^{1,p}(\mathscr{X})$, let

$$\|f\|_{\widetilde{N}^{1,p}(\mathscr{X})}:=\|f\|_{L^p(\mathscr{X})}+\inf_g\|g\|_{L^p(\mathscr{X})},$$

where the infimum is taken over all p-weak upper gradients g of f. The *Newton-Sobolev space* $N^{1,p}(\mathscr{X})$ is then defined to be the quotient space $N^{1,p}(\mathscr{X}):=\widetilde{N}^{1,p}(\mathscr{X})/\sim$ with the norm $\|\cdot\|_{N^{1,p}(\mathscr{X})}:=\|\cdot\|_{\widetilde{N}^{1,p}(\mathscr{X})}$, where $\sim$ is an *equivalence relation* defined by setting, for all $f_1, f_2 \in \widetilde{N}^{1,p}(\mathscr{X})$, $f_1 \sim f_2$ if $\|f_1 - f_2\|_{\widetilde{N}^{1,p}(\mathscr{X})} = 0$. It was proved in [47, Theorem 4.9] that the Newton-Sobolev space coincides with the Hajłasz-Sobolev space if (X, μ) supports some Poincaré inequality and the measure μ is doubling. We refer to [47, 25, 7, 17, 6] for more properties about these spaces.

Recently, there are some attempts to study Newtonian type spaces in more general settings. Durand-Cartagena in [14] introduced and studied the Newtonian space $N^{1,\infty}(\mathscr{X})$ in the limit case that $p = \infty$. Tuominen [51] considered Newtonian type spaces associated with Orlicz spaces by replacing the Lebesgue norm in the definition of $N^{1,p}(\mathscr{X})$ with Orlicz norms. Using Lorentz spaces instead of Lebesgue spaces, Costea and Miranda [12] introduced Newtonian type spaces related to Lorentz spaces. Malý [37, 38] studied the Newtonian type spaces associated with a general *quasi-Banach function lattice* X, namely, a quasi-Banach function space X satisfying that, if $f \in X$ and $|g| \le |f|$ almost everywhere, then $g \in X$ and $\|g\|_X \le \|f\|_X$.

Let $0 < p \le q \le \infty$. Recall that the *Morrey space* $\mathcal{M}_p^q(\mathscr{X})$ (see [42]) is defined to be the space of all measurable functions f on $\mathscr{X}$ such that

$$\|f\|_{\mathcal{M}_p^q(\mathscr{X})} := \sup_{B \subset \mathscr{X}} [\mu(B)]^{1/q-1/p} \left[\int_B |f(x)|^p \, d\mu(x)\right]^{1/p} < \infty, \tag{1.3}$$

where the supremum is taken over all balls in $\mathscr{X}$. In recent years, Morrey spaces and the Morrey version of many classical function spaces such as Hardy spaces and Besov spaces, namely, the spaces defined via replacing Lebesgue norms by Morrey norms in their norms, attract more and more attentions and have been proved to be useful in the study of partial differential equations and harmonic analysis; see, for example, [1, 2, 3, 4, 43, 41, 35, 39, 56] and their references.

The main purpose of this article is to develop a theory of Newtonian type spaces based on Morrey spaces, namely, Newton-Sobolev-Morrey spaces, as well as the Hajłasz-Sobolev-Morrey spaces on metric measure spaces.

We begin with the following generalized modulus based on Morrey spaces.

Definition 1.1. Let $1 \le p \le q < \infty$ and Γ be a collection of rectifiable curves. The *Morrey-modulus* of Γ is defined by

$$\mathrm{Mod}_p^q(\Gamma) := \inf_{\rho \in F(\Gamma)} \|\rho\|^p_{\mathcal{M}_p^q(\mathscr{X})},$$

where $F(\Gamma)$ is defined as in (1.1).

Definition 1.2. Let f be a measurable function and g a non-negative Borel measurable function. If the inequality (1.2) holds for all non-constant rectifiable curves in $\mathscr{X}$ except a family of curves of Morrey-modulus zero, then g is called a Mod_p^q-*weak upper gradient* of f.

Via these Mod_p^q-weak upper gradients, the Newton-Morrey-Sobolev space is introduced as follows.

Definition 1.3. Let $1 \le p \le q < \infty$. The *space* $\widetilde{NM_p^q}(\mathscr{X})$ is defined to be the set of all μ-measurable functions f such that $\|f\|_{\widetilde{NM_p^q}(\mathscr{X})} < \infty$, where

$$\|f\|_{\widetilde{NM_p^q}(\mathscr{X})} := \|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \inf_g \|g\|_{\mathcal{M}_p^q(\mathscr{X})}$$

with the infimum being taken over all Mod_p^q-weak upper gradients g of f. The *Newton-Morrey-Sobolev space* $NM_p^q(\mathscr{X})$ is then defined as the quotient space

$$\widetilde{NM_p^q}(\mathscr{X}) / \left\{ f \in \widetilde{NM_p^q}(\mathscr{X}) : \ \|f\|_{\widetilde{NM_p^q}(\mathscr{X})} = 0 \right\}$$

with

$$\|f\|_{NM_p^q(\mathscr{X})} := \|f\|_{\widetilde{NM_p^q}(\mathscr{X})}.$$

It is easy to see that $\|\cdot\|_{NM_p^q(\mathscr{X})}$ is a norm. Moreover, when $p = q$, the space $NM_p^q(\mathscr{X})$ is just the Newton-Sobolev space $N^{1,p}(\mathscr{X})$ introduced by Shanmugalingam [47]. We also remark that, since Morrey spaces are Banach function lattices, these Newton-Sobolev-Morrey spaces are special cases of the Newtonian type spaces associated with quasi-Banach function lattices considered by Malý [37, 38].

This article is organized as follows. In Section 2, we show that the Newton-Morrey-Sobolev space is non-trivial by proving that the set of Lipschitz functions with bounded support is contained in the Newton-Morrey-Sobolev space (see Theorem 2.4 below), but not dense (see Remark 2.5 below), which is different from the Newton-Sobolev space.

In Section 3, the embedding of the Newton-Morrey-Sobolev space into the Hölder space is obtained when $\mathscr{X}$ supports a weak Poincaré inequality, the measure μ is doubling and satisfies a lower bounded condition (see Theorem 3.1 below). Moreover, if the space $\mathscr{X}$ is Ahlfors Q-regular and supports a weak Poincaré inequality, via proving the boundedness of some fractional integrals on Morrey spaces, we also obtain a Rellich-Kondrachov type embedding theorem of the Newton-Morrey-Sobolev space (see Theorem 3.6 below). Both embedding properties on Newton-Morrey-Sobolev spaces generalize the corresponding results for Newton-Sobolev spaces obtained by Shanmugalingam in [47, Theorems 5.1 and 5.2].

In Section 4, using the Hajłasz gradient, we introduce the Hajłasz-Morrey-Sobolev space on metric measure spaces and show that, when $\mathscr{X}$ supports a weak Poincaré inequality and the measure μ is doubling, the Newton-Morrey-Sobolev space is a subspace of the Hajłasz-Morrey-Sobolev space (see Theorem 4.3 below). We also prove that the intersection of the Hajłasz-Morrey-Sobolev space and $\mathcal{C}(\mathscr{X})$, the *set of all functions which are almost everywhere continuous with respect to the Morrey modulus*, is continuously embedded into the Newton-Morrey-Sobolev space (see Theorem 4.6 below). This generalizes the result on the relation between Newton-Sobolev spaces and Hajłasz-Sobolev spaces obtained by Shanmugalingam in [47, Theorem 4.9]. In particular, when $\mathscr{X} = \mathbb{R}^n$, both the Newton-Morrey-Sobolev space $NM_p^q(\mathbb{R}^n)$ when $1 < p \le q < \infty$ and $n < q$ and the Hajłasz-Morrey-Sobolev space $HM_p^q(\mathbb{R}^n)$ when $1 < p \le q < \infty$ are proved to coincide with the classical Morrey-Sobolev space on $\mathbb{R}^n$ (see Theorem 4.8 and Corollary 4.11 below). We point out that, in the proof of Theorem 4.8, we use two key tools: the predual space of the Morrey space, which was found by Adams and Xiao in [1, Theorem 2.3] and [4, Theorem 7], and the density of $C_c^\infty(\mathbb{R}^n)$ in the predual space (see Lemma 4.10 below).

In Section 5, invoking the maximal function characterization of Hardy-Morrey-Sobolev spaces from Jia and Wang [30], and the coincidence between Hardy-Morrey spaces and some special cases of Triebel-Lizorkin-type spaces in [54, 46, 56], together with the lifting and the mapping properties of Fourier multipliers and pseudo-differential operators on Triebel-Lizorkin-type spaces, we first prove that, on the Euclidean space $\mathbb{R}^n$, when $\frac{n}{n+1} < p \le q \le \infty$, the Hajłasz-Morrey-Sobolev space $HM_p^q(\mathbb{R}^n)$ (see Definition 5.3 below) coincides with the Hardy-Morrey-Sobolev space $\mathscr{H}M_{1,p}^q(\mathbb{R}^n)$ (see Theorem 5.5 below), which generalizes the known result for Hardy-Sobolev spaces obtained by Koskela and Saksman [33, Theorem 1]. In a more general case when $\mathscr{X}$ is a metric measure space of homogeneous type in the sense of Coifman and Weiss [10, 11], namely, a *metric measure space satisfying the doubling measure condition*, we give a characterization of the Hajłasz-Morrey-Sobolev space via grand maximal functions (see Theorem 5.11 below), which generalizes the case when $s = 1$ of [34, Theorem 5.2]. It should be pointed out that, differently from the proof of [34, Theorem 5.2], the proof of Theorem 5.11 involves several localized arguments via a partition of unity on homogeneous type spaces.

Finally, Section 6 is devoted to the the boundedness of some fractional maximal operators on Morrey and Morrey-Sobolev spaces. We first show, in Subsection 6.1, the boundedness of some certain modified maximal operators on modified Morrey spaces over geometrically doubling metric spaces (see Theorem 6.5 below). As an application, the boundedness of related fractional maximal operators on modified Morrey spaces is obtained (see Proposition 6.6 below). As further applications, in Subsection 6.2, we show the boundedness of fractional maximal operators on Hajłasz-Morrey-Sobolev spaces when $\mathscr{X}$ is a doubling metric measure space satisfying the relative 1-annular decay property and the measure lower bound condition (see Theorem 6.9 below). If $\mathscr{X}$ supports a weak Poincaré-inequality, and the measure is doubling and satisfies the measure lower bound condition, then the boundedness of discrete fractional maximal operators on Newton-Morrey-Sobolev spaces is also obtained (see Theorem 6.10 below). All these conclusions generalize the corresponding known results on Newton-Sobolev spaces and Hajłasz-Sobolev spaces by Heikkinen et al. in [23, 24].

At the end of this section, we make some conventions on notation. Throughout the paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means $A \le CB$, where C is a positive constant independent of A and B. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*.

2 Some basic properties

In this section, we consider some basic properties of Newton-Morrey-Sobolev spaces including their completeness and non-triviality. Throughout this section, we *only assume that* μ *is a nontrivial Borel regular measure*.

Recall that the Newton-Morrey-Sobolev space is a special case of the Newtonian spaces based on quasi-Banach function lattice X introduced in [37]. The following result is a special case of [37, Theorem 7.1].

Theorem 2.1. *For all $1 \le p \le q < \infty$, the space $NM_p^q(\mathscr{X})$ is a Banach space.*

The next lemma is usually called the *truncation lemma*, which shows how a Mod_p^q-weak upper gradient behaves when multiplying a characteristic function. Its proof is similar to those of [12, Lemmas 4.6 and 4.7] and we omit the details.

Lemma 2.2. *Let $f \in NM_p^q(\mathscr{X})$ and $g_1, g_2 \in \mathcal{M}_p^q(\mathscr{X})$ be two Mod_p^q-weak upper gradients of f.*

(i) *If f is a constant on a closed set E, then $g := g_1\chi_{\mathscr{X}\setminus E}$ is also a Mod_p^q-weak upper gradient of f.*

(ii) *If E is closed in $\mathscr{X}$, then $h := g_1\chi_E + g_2\chi_{\mathscr{X}\setminus E}$ is also a Mod_p^q-weak upper gradient of f.*

We also need the following conclusion.

Proposition 2.3. *Let $1 \le p \le q < \infty$. For any set $E \subset \mathscr{X}$ with finite measure, $\|E\|_{\mathcal{M}_p^q(\mathscr{X})}$ is bounded by a multiple of $\mu(E)^{1/q}$.*

Proof. Notice that

$$\|\chi_E\|_{\mathcal{M}_p^q(\mathscr{X})} = \sup_{B\subset\mathscr{X}} [\mu(B)]^{1/q} \left[\frac{\mu(B\cap E)}{\mu(B)}\right]^{1/p}.$$

If $\mu(B) \ge \mu(E)/2$, then by $p \le q$, we have

$$[\mu(B)]^{1/q} \left[\frac{\mu(B\cap E)}{\mu(B)}\right]^{1/p} \lesssim [\mu(E)]^{1/q-1/p}[\mu(B\cap E)]^{1/p} \lesssim [\mu(E)]^{1/q}.$$

If $\mu(B) \le \mu(E)/2$, then

$$[\mu(B)]^{1/q} \left[\frac{\mu(B\cap E)}{\mu(B)}\right]^{1/p} \lesssim [\mu(E)]^{1/q}.$$

This finishes the proof of Proposition 2.3. □

Recall that $NM_p^p(\mathscr{X}) = N^{1,p}(\mathscr{X})$, which is a non-trivial space. The following conclusion shows that even when $q > p \ge 1$, $NM_p^q(\mathscr{X})$ is also not a trivial space. In what follows, $\mathrm{Lip}_b(\mathscr{X})$ denotes the *set of all Lipschitz functions on $\mathscr{X}$ with bounded support.*

Theorem 2.4. *Let $1 \le p \le q < \infty$. Then $\mathrm{Lip}_b(\mathscr{X}) \subset NM_p^q(\mathscr{X})$.*

Proof. Let B be a ball in $\mathscr{X}$. By Proposition 2.3, we know that $\chi_B \in \mathcal{M}_p^q(\mathscr{X})$ and

$$\|\chi_B\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim [\mu(B)]^{1/q} < \infty.$$

Now let $f\in \mathrm{Lip}_b(\mathscr{X})$ with $\mathrm{supp}\,(f)\subset B$ and L be the Lipschitz constant of f. Since $f\in \mathrm{Lip}_b(\mathscr{X})$, we know that there exists a positive constant M_0 such that $|f|\le M_0\chi_B$. Hence, by (1.3), we see that $f\in \mathcal{M}_p^q(\mathscr{X})$ and $\|f\|_{\mathcal{M}_p^q(\mathscr{X})}\lesssim M_0[\mu(B)]^{1/q}$.

On the other hand, notice that, for all recitable curves γ, it holds that

$$|f\circ\gamma(\ell(0))-f\circ\gamma(\ell(\gamma))|\le L\,d(\gamma(\ell(0)),\gamma(\ell(\gamma)))\le\int_\gamma L\,ds.$$

Hence L is an upper gradient of f. Then by Lemma 2.2, $L\chi_{2B}$ is a Mod_p^q-weak upper gradient of f, which further implies that $f\in NM_p^q(\mathscr{X})$ and

$$\|f\|_{NM_p^q(\mathscr{X})}\lesssim (M_0+L)[\mu(B)]^{1/q}.$$

Thus, $\mathrm{Lip}_b(\mathscr{X})\subset NM_p^q(\mathscr{X})$, which completes the proof of Theorem 2.4. □

Remark 2.5. We point out that the *set* $\mathrm{Lip}_b(\mathscr{X})$ *might not be dense in* $NM_p^q(\mathscr{X})$ *when* $1\le p<q<\infty$, which is different from the Newton-Sobolev space $N^{1,p}(\mathscr{X})$ (see [47, Theorem 4.1]). We give a counterexample in the case $\mathscr{X}=\mathbb{R}$ as follows. For $N\in\mathbb{N}$ and $j\in\mathbb{Z}$, let $I_j^N:=[2^N+jN,2^N+jN+1]$,

$$\psi_j^N(x):=\begin{cases}4x-4c_j^N+2, & x\in[c_j^N-\frac12,\,c_j^N-\frac14),\\ 1, & x\in[c_j^N-\frac14,\,c_j^N+\frac14),\\ -4x+4c_j^N+2, & x\in[c_j^N+\frac14,\,c_j^N+\frac12],\\ 0, & x\in(-\infty,\,c_j^N-\frac12)\cup(c_j^N+\frac12,\,\infty),\end{cases}$$

and

$$\phi_j^N(x):=\begin{cases}4, & x\in[c_j^N-\frac12,\,c_j^N+\frac12],\\ 0, & x\in(-\infty,\,c_j^N-\frac12)\cup(c_j^N+\frac12,\,\infty),\end{cases}$$

where c_j^N is the middle point of the interval I_j^N. Let $f:=\sum_{N>N_0}\sum_{j=0}^{N-1}\psi_j^N$ and $g:=\sum_{N>N_0}\sum_{j=0}^{N-1}\phi_j^N$, where $N_0\in\mathbb{N}$ such that, for every $N\ge N_0$, it holds that $2^{N+1}>2^N+N^2-N+1$. Then, $\{I_j^N:\ N>N_0,\ j\in\{0,\dots,N-1\}\}$ are disjoint with each other. By this and (1.3) with $\mathscr{X}=\mathbb{R}$, we see that, when $p=q$,

$$\|f\|_{L^p(\mathbb{R})}\ge\sum_{N>N_0}\sum_{j=0}^{N-1}\left|\left[c_j^N-\frac14,\,c_j^N+\frac14\right)\right|=\sum_{N>N_0}\frac{N}{2}=\infty,$$

which implies that $f\notin L^p(\mathbb{R})$. When $p<q$, by this and (1.3) with $\mathscr{X}=\mathbb{R}$, together with some computations same as in [8, Lemma 2], we know that $f\in\mathcal{M}_p^q(\mathbb{R})$ and $g\in\mathcal{M}_p^q(\mathbb{R})$. Furthermore, by noticing that curves in $\mathbb{R}$ are intervals and $\{I_j^N:\ N>N_0,\ j\in\{0,\dots,N-1\}\}$ are disjoint with each other, we see that ϕ_j^N is an upper gradient of ψ_j^N and hence g is an upper gradient of f. Now for any $g\in\mathrm{Lip}_b(\mathbb{R})$, let $\mathrm{supp}\,g\subset[A,B]$, $N_1\in\mathbb{N}$ and $j_1\in\{0,\dots,N_1-1\}$ such that $N_1>N_0$ and $2^{N_1}+j_1N_1>B$. Then, we have

$$\|f-g\|_{\mathcal{M}_p^q(\mathbb{R})}\ge\left\|\chi_{[c_{j_1}^{N_1}-\frac14,\,c_{j_1}^{N_1}+\frac14]}\right\|_{\mathcal{M}_p^q(\mathbb{R})}\ge\left(\frac12\right)^{1/q}.$$

Thus, f can not be approximating by any sequence of functions in $\mathrm{Lip}_b(\mathbb{R})$ in the norm of $NM_p^q(\mathbb{R})$ and hence the above claim holds true.

3 Sobolev embeddings

Let $\alpha \in (0,1]$ and $C^{0,\alpha}(\mathscr{X})$ denote the *α-Hölder space* on $\mathscr{X}$, namely, the space of all functions f satisfying that, for all $x, y \in \mathscr{X}$, $|f(x)-f(y)| \le C[d(x,y)]^\alpha$, where C is a positive constant independent of x and y.

It is well known that, when $\mathscr{X} = \mathbb{R}^n$, the following Sobolev embeddings hold true:

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{np/(n-p)}(\mathbb{R}^n) \quad \text{if } p < n, \tag{3.1}$$

and

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,1-n/p}(\mathbb{R}^n) \quad \text{if } p > n, \tag{3.2}$$

where the *symbol* $\hookrightarrow$ means continuous embedding. The generalizations of (3.1) and (3.2) to the Newton-Sobolev space and the Hajłasz-Sobolev space on metric measure spaces were obtained in [47] and [20, 21]. This section is devoted to the corresponding Sobolev embedding theorems for Newton-Morrey-Sobolev spaces.

Recall that a space $\mathscr{X}$ is said to *support a weak* $(1,p)$*-Poincaré inequality* if there exist positive constants C and $\tau \ge 1$ such that, for all open balls B in $\mathscr{X}$ and all pairs of functions f and ρ defined on τB, whenever ρ is an upper gradient of f in τB and f is integrable on B, then

$$\frac{1}{\mu(B)}\int_B |f(x)-f_B|\,d\mu(x) \le C\,\mathrm{diam}(B)\left\{\frac{1}{\mu(B)}\int_{\tau B}[\rho(x)]^p\,d\mu(x)\right\}^{1/p}, \tag{3.3}$$

where above and in what follows, f_B denotes the *integral mean of f on B*, namely,

$$f_B = \frac{1}{\mu(B)}\int_B f(y)\,d\mu(y), \tag{3.4}$$

$\mathrm{diam}(B)$ the *diameter* of B and τB the *ball with the same center as B but τ times the radius of B*. In particular, if $\tau = 1$, then we say that *$\mathscr{X}$ supports a $(1,p)$-Poincaré inequality*.

It is well known that the Euclidean space supports a $(1,p)$-Poincaré inequality. For more information on Poincaré inequalities, we refer to [26, 27, 21] and their references.

A measure μ on $\mathscr{X}$ is said to be *doubling* if there exists a positive constant C such that, for all balls B in $\mathscr{X}$, it holds that $\mu(2B) \le C\mu(B)$. As a generalization of (3.2) to Newton-Morrey-Sobolev spaces, we have the following conclusion.

Theorem 3.1. *Let $1 \le p \le q < \infty$ and $Q \in (0,q)$. Assume that $(\mathscr{X}, d, \mu)$ is a metric measure space with doubling measure μ, supports a weak $(1,p)$-Poincaré inequality and satisfies that there exists a positive constant C such that, for all $x \in \mathscr{X}$ and $0 < r < 2\mathrm{diam}(\mathscr{X})$, it holds that $\mu(B(x,r)) \ge Cr^Q$. Then $NM_p^q(\mathscr{X}) \hookrightarrow C^{0,1-Q/q}(\mathscr{X})$.*

Proof. By the same reason as that stated in the proof of [47, Theorem 5.1], we only need to show that if $f \in NM_p^q(\mathscr{X})$ and x, y are Lebesgue points of f, then

$$|f(x) - f(y)| \lesssim [d(x,y)]^{1-Q/q} \|f\|_{NM_p^q(\mathscr{X})}.$$

To this end, let $B_1 := B(x, d(x,y))$, $B_{-1} := B(y, d(x,y))$ and, for all $i > 1$,

$$B_i = \frac{1}{2} B_{i-1} \text{ and } B_{-i} = \frac{1}{2} B_{-i+1}.$$

Let B_0 be a ball contained in $B_1 \cap B_{-1}$ with radius $r_{B_0} := d(x,y)/4$. Since x, y are Lebesgue points, it follows that

$$|f(x) - f(y)| \le \sum_{i \in \mathbb{Z}} |f_{B_i} - f_{B_{i+1}}|.$$

Let ρ be an upper gradient of f such that

$$\|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \|\rho\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \|f\|_{NM_p^q(\mathscr{X})}.$$

Then, by this, (1.3), the doubling condition of μ and the weak $(1,p)$-Poincaré inequality, together with $\mu(\tau B_i) \gtrsim r_i^Q$, we see that, when $i \in \mathbb{N}$,

$$\begin{aligned}
|f_{B_i} - f_{B_{i+1}}| &\lesssim \frac{1}{\mu(B_i)} \int_{B_i} |f_{B_i} - f(x)| \, d\mu(x) \\
&\lesssim \operatorname{diam}(B_i) \left\{ \frac{1}{\mu(\tau B_i)} \int_{\tau B_i} [\rho(x)]^p \, d\mu(x) \right\}^{1/p} \\
&\lesssim r_i [\mu(\tau B_i)]^{-1/q} \|\rho\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim r_i^{1-Q/q} \|\rho\|_{\mathcal{M}_p^q(\mathscr{X})} \\
&\lesssim 2^{-i(1-Q/q)} [d(x,y)]^{1-Q/q} \|f\|_{NM_p^q(\mathscr{X})},
\end{aligned} \tag{3.5}$$

where we denote by r_i the radius of B_i. Similarly, for all $i \le -2$, we also have

$$|f_{B_i} - f_{B_{i+1}}| \lesssim 2^{i(1-Q/q)} [d(x,y)]^{1-Q/q} \|f\|_{NM_p^q(\mathscr{X})}.$$

On the other hand, by the Hölder inequality and the doubling condition of μ, we see that

$$\begin{aligned}
|f_{B_{-1}} - f_{B_0}| &\le \frac{1}{\mu(B_0)} \int_{B_0} \left| f_{B_{-1}} - f(x) \right| \, d\mu(x) \lesssim \frac{1}{\mu(6B_0)} \int_{B_{-1}} |f_{B_{-1}} - f(x)| \, d\mu(x) \\
&\lesssim \frac{1}{\mu(B_{-1})} \int_{B_{-1}} |f_{B_{-1}} - f(x)| \, d\mu(x)
\end{aligned}$$

and then, similar to (3.5), we further conclude that

$$|f_{B_{-1}} - f_{B_0}| \lesssim [d(x,y)]^{1-Q/q} \|f\|_{NM_p^q(\mathscr{X})}.$$

Meanwhile, by the same method as this, we also find that

$$|f_{B_0} - f_{B_1}| \lesssim [d(x,y)]^{1-Q/q} \|f\|_{NM_p^q(\mathscr{X})}.$$

Thus, combining the above estimates, by $Q \in (0, q)$, we see that

$$\begin{aligned} |f(x) - f(y)| &\lesssim [d(x,y)]^{1-Q/q} \left[\sum_{i\in\mathbb{Z}} 2^{-|i|(1-Q/q)} \right] \|f\|_{NM_p^q(\mathscr{X})} \\ &\lesssim [d(x,y)]^{1-Q/q} \|f\|_{NM_p^q(\mathscr{X})}, \end{aligned}$$

which completes the proof of Theorem 3.1. □

Remark 3.2. Theorem 3.1 generalizes [47, Theorem 5.1] by taking $p = q$.

Next we give a Rellich-Kondrachov type embedding theorem for $NM_p^q(\mathscr{X})$ when p is small, which can be seen as a generalization of (3.1). We begin with the following notion of the Ahlfors Q-regular; see, for example, [47, Definition 5.1].

Definition 3.3. Let $Q \in (0, \infty)$. A metric measure space $\mathscr{X}$ is said to be *Ahlfors Q-regular* (or *Q-regular*), if there exists a constant $C \geq 1$ such that, for any $x \in \mathscr{X}$ and any $r \in (0, 2\mathrm{diam}(\mathscr{X}))$,

$$\frac{1}{C} r^Q \leq \mu(B(x,r)) \leq C r^Q.$$

Let M be the *Hardy-Littlewood maximal operator* defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathscr{X})$, the *set of all functions which are integrable on all balls*, and $x \in \mathscr{X}$,

$$Mf(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y), \tag{3.6}$$

where the supremum is taken over all balls B in $\mathscr{X}$ containing x. The following statement shows that the operator M is bounded on Morrey spaces. For its proof, we refer to [5] for example.

Lemma 3.4. *Let $(\mathscr{X}, d, \mu)$ be a metric space with doubling measure and $1 < p \leq q \leq \infty$. Then there exists a positive constant C such that, for all $f \in \mathcal{M}_p^q(\mathscr{X})$,*

$$\|Mf\|_{\mathcal{M}_p^q(\mathscr{X})} \leq C \|f\|_{\mathcal{M}_p^q(\mathscr{X})}.$$

We also need the following boundedness of fractional integral operators on Morrey spaces.

Proposition 3.5. *Let $\mathscr{X}$ be Ahlfors Q-regular with $Q \in (0, \infty)$, $\alpha \in (0, Q)$ and $1 < p \leq q < Q/\alpha$. Then the following fractional integral I_α is bounded from $\mathcal{M}_p^q(\mathscr{X})$ to $\mathcal{M}_{p^*}^{q^*}(\mathscr{X})$, where $p^* := \frac{Qp}{Q-q\alpha}$, $q^* := \frac{Qq}{Q-q\alpha}$ and I_α is defined by setting, for all $f \in \mathcal{M}_p^q(\mathscr{X})$ and $x \in \mathscr{X}$,*

$$I_\alpha(f)(x) := \int_{\mathscr{X}} \frac{f(y)}{[d(x,y)]^{Q-\alpha}} \, d\mu(y).$$

Proof. Without loss of generality, we may assume that $f \in \mathcal{M}_p^q(\mathscr{X})$ is non-negative. For any $x \in \mathscr{X}$, fix $\delta > 0$ and write

$$\begin{aligned} I_\alpha(f)(x) &\le \int_{B(x,\delta)} \frac{f(y)}{[d(x,y)]^{Q-\alpha}}\,d\mu(y) + \int_{\mathscr{X}\setminus B(x,\delta)} \frac{f(y)}{[d(x,y)]^{Q-\alpha}}\,d\mu(y) \\ &=: b_\delta^{(\alpha)}(x) + g_\delta^{(\alpha)}(x). \end{aligned}$$

By the Hölder inequality, (1.3) and the Ahlfors Q-regular property of $\mathscr{X}$, together with $q < Q/\alpha$, we see that

$$\begin{aligned} g_\delta^{(\alpha)}(x) &= \sum_{j=0}^\infty \int_{B(x,2^{j+1}\delta)\setminus B(x,2^j\delta)} \frac{f(y)}{[d(x,y)]^{Q-\alpha}}\,d\mu(y) \\ &\lesssim \sum_{j=0}^\infty (2^j\delta)^{\alpha-Q}[\mu(B(x,2^{j+1}\delta))]^{1-1/p}\left[\int_{B(x,2^{j+1}\delta)}[f(y)]^p\,d\mu(y)\right]^{1/p} \\ &\lesssim \sum_{j=0}^\infty (2^j\delta)^{\alpha-Q}[\mu(B(x,2^{j+1}\delta))]^{1-1/q}\|f\|_{\mathcal{M}_p^q(\mathscr{X})} \\ &\approx \sum_{j=0}^\infty (2^j\delta)^{\alpha-Q}(2^j\delta)^{(1-1/q)Q}\|f\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \delta^{\alpha-Q/q}\|f\|_{\mathcal{M}_p^q(\mathscr{X})}. \end{aligned} \tag{3.7}$$

For b_δ, let $A_j := B_j \setminus B_{j+1} := B(x, 2^{-j}\delta) \setminus B(x, 2^{-j-1}\delta)$ for all $j \in \mathbb{N} \cup \{0\} =: \mathbb{Z}_+$. Then, by the Ahlfors Q-regular property of $\mathscr{X}$, together with $\alpha > 0$, we see that, for all $x \in \mathscr{X}$,

$$\begin{aligned} b_\delta^{(\alpha)}(x) &= \sum_{j\in\mathbb{Z}_+} \int_{A_j} \frac{f(y)}{[d(x,y)]^{Q-\alpha}}\,d\mu(y) \approx \sum_{j\in\mathbb{Z}_+} (2^{-j}\delta)^{\alpha-Q}\int_{B_j} f(y)\,d\mu(y) \\ &\lesssim \delta^\alpha \sum_{j\in\mathbb{Z}_+} 2^{-j\alpha}\frac{1}{\mu(B_j)}\int_{B_j} f(y)\,\mu(y) \lesssim \delta^\alpha M(f)(x). \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$I_\alpha(f)(x) \lesssim \delta^\alpha M(f)(x) + \delta^{\alpha-Q/q}\|f\|_{\mathcal{M}_p^u(\mathscr{X})}.$$

Now let $\delta := \|f\|_{\mathcal{M}_p^q(\mathscr{X})}^{q/Q}[M(f)(x)]^{-q/Q}$. Then for any $x \in \mathscr{X}$,

$$I_\alpha(f)(x) \lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})}^{\alpha q/Q}[M(f)(x)]^{1-\alpha q/Q},$$

which, together with Lemma 3.4, further implies that

$$\begin{aligned} \|I_\alpha(f)\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})} &\lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})}^{q\alpha/Q}\|[M(f)]^{1-q\alpha/Q}\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})} \\ &\approx \|f\|_{\mathcal{M}_p^q(\mathscr{X})}^{q\alpha/Q}\|M(f)\|_{\mathcal{M}_p^q(\mathscr{X})}^{1-q\alpha/Q} \lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})}. \end{aligned}$$

This finishes the proof of Proposition 3.5. $\square$

Now we have the following Rellich-Kondrachov type embedding result, which generalizes [47, Theorem 5.2] by taking $p=q$, $\alpha=1$ and $\mathscr{X}$ being bounded.

Theorem 3.6. *Let $1\le r<p\le q<\infty$. Let $1<p/r\le q/r<Q/\alpha<\infty$, $\alpha\in(0,r)\cap(0,Q)$ and $\mathscr{X}$ be an Ahlfors Q-regular metric measure space supporting a weak $(1,r)$-Poincaré inequality. Then there exists a positive constant C such that, for all functions $f\in NM_p^q(\mathscr{X})$, upper gradients ρ of f and $R\in(0,\infty)$,*

$$\|f-f_{B(\cdot,R)}\|_{\mathcal{M}^{\frac{Qrq}{Qr-q\alpha}}_{\frac{Qrp}{Qr-q\alpha}}(\mathscr{X})}\le CR^{1-\alpha/r}\|\rho\|_{\mathcal{M}_p^q(\mathscr{X})}.$$

Proof. Let $f\in NM_p^q(\mathscr{X})$ and ρ be an upper gradient of f. For any Lebesgue point x for f, we write $B_0:=B(x,R)$ and $B_i:=B(x,2^{-i}R)$ for all $i\in\mathbb{N}$. Since an Ahlfors Q-regular space is doubling, by the weak $(1,r)$-Poincaré inequality and $r>\alpha$, we see that

$$\begin{aligned}|f(x)-f_{B(x,R)}|&\le\sum_{i=0}^{\infty}|f_{B_i}-f_{B_{i+1}}|\lesssim\sum_{i=0}^{\infty}\frac{1}{\mu(B_i)}\int_{B_i}|f(z)-f_{B_i}|\,d\mu(z)\\&\lesssim\sum_{i=0}^{\infty}\frac{\mathrm{diam}(B_i)}{[\mu(\tau B_i)]^{1/r}}\left\{\int_{\tau B_i}[\rho(z)]^r\,d\mu(z)\right\}^{1/r}\\&\approx\sum_{i=0}^{\infty}\frac{\mathrm{diam}(B_i)}{(2^{-i}\tau R)^{Q/r}}\left\{\int_{\tau B_i}[\rho(z)]^r\,d\mu(z)\right\}^{1/r}\\&\lesssim\sum_{i=0}^{\infty}\frac{2^{-i}R}{(2^{-i}\tau R)^{\alpha/r}}\left\{\int_{\tau B_i}\frac{[\rho(z)]^r}{[d(x,z)]^{Q-\alpha}}\,d\mu(z)\right\}^{1/r}\\&\lesssim R^{1-\alpha/r}\left\{\int_{\mathscr{X}}\frac{[\rho(z)]^r}{[d(x,z)]^{Q-\alpha}}\,d\mu(z)\right\}^{1/r}\approx R^{1-\alpha/r}[I_\alpha(\rho^r)(x)]^{1/r},\end{aligned}$$

where τ is the same as in (3.3). Applying Proposition 3.5, together with $1<p/r\le q/r<Q/\alpha$, we conclude that

$$\begin{aligned}\|f-f_{B(\cdot,R)}\|_{\mathcal{M}^{\frac{Qqr}{Qr-q\alpha}}_{\frac{Qpr}{Qr-q\alpha}}(\mathscr{X})}&\lesssim R^{1-\alpha/r}\|[I_\alpha(\rho^r)]^{1/r}\|_{\mathcal{M}^{\frac{Qrq}{Qr-q\alpha}}_{\frac{Qpr}{Qr-q\alpha}}(\mathscr{X})}\approx R^{1-\alpha/r}\|I_\alpha(\rho^r)\|^{1/r}_{\mathcal{M}^{\frac{Qq}{Qr-q\alpha}}_{\frac{Qp}{Qr-q\alpha}}(\mathscr{X})}\\&\lesssim R^{1-\alpha/r}\|\rho^r\|^{1/r}_{\mathcal{M}^{q/r}_{p/r}(\mathscr{X})}\approx R^{1-\alpha/r}\|\rho\|_{\mathcal{M}_p^q(\mathscr{X})},\end{aligned}$$

which completes the proof of Theorem 3.6. □

Remark 3.7. (i) Let $1<r<p<\infty$, $1<p/r<Q<\infty$, and $\mathscr{X}$ be an Ahlfors Q-regular metric measure space supporting a weak $(1,r)$-Poincaré inequality. Then, by Theorem 3.6, we see that there exists a positive constant C such that, for all functions $f\in N^{1,p}(\mathscr{X})$, upper gradients ρ of f and $R\in(0,\infty)$,

$$\|f-f_{B(\cdot,R)}\|_{L^{\frac{Qpr}{Qr-p}}(\mathscr{X})}\le CR^{1-1/r}\|\rho\|_{L^p(\mathscr{X})},$$

which has its own interest.

(ii) We also remark that Theorem 3.6 generalizes the classical result for Newton spaces in [47, Theorem 5.2]. Indeed, if we further assume that $\mathscr{X}$ is bounded, then we know that $f_{\mathscr{X}} = f_{B(x,\mathrm{diam}(\mathscr{X}))}$ for almost all $x \in \mathscr{X}$. Thus, it follows from (i) that, under the same assumptions on Q, r, p as in (i), there exists a positive constant C such that, for all functions $f \in N^{1,p}(\mathscr{X})$ and upper gradients ρ of f,

$$\|f - f_{\mathscr{X}}\|_{L^{\frac{Qpr}{Qr-p}}(\mathscr{X})} \le C[\mathrm{diam}(\mathscr{X})]^{1-1/r}\|\rho\|_{L^p(\mathscr{X})},$$

which is just [47, Theorem 5.2].

4 Hajłasz-Morrey-Sobolev spaces

In this section, we introduce Morrey-Sobolev spaces associated with Hajłasz gradients and consider the relation between the Hajłasz-Morrey-Sobolev space and the Newton-Morrey-Sobolev space.

Definition 4.1. Let $0 < p \le q \le \infty$. The *inhomogeneous Hajłasz-Morrey-Sobolev space* $HM_p^q(\mathscr{X})$ is defined to be the space of all measurable functions f having Hajłasz gradients $h \in \mathcal{M}_p^q(\mathscr{X})$. The norm of $f \in HM_p^q(\mathscr{X})$ is defined as

$$\|f\|_{HM_p^q(\mathscr{X})} := \|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \inf \|h\|_{\mathcal{M}_p^q(\mathscr{X})},$$

where the infimum is taken over all Hajłasz gradients h of f.

We remark that $HM_p^q(\mathscr{X})$ when $p = q$ is just the Hajłasz-Sobolev space $M^{1,p}(\mathscr{X})$ in [18]. Moreover, $\|f\|_{HM_p^q(\mathscr{X})} = 0$ if and only if $f = 0$ almost everywhere.

Next we consider the relationship between the Hajłasz-Morrey-Sobolev space and the Newton-Morrey-Sobolev space. We first need the following lemma, which is a special case of [37, Lemma 5.6].

Lemma 4.2. *Let $1 \le p \le q < \infty$ and g be a* Mod_p^q*-weak upper gradient of f. Then for any $\varepsilon \in (0,\infty)$, there exists a function g_ε, which is an upper gradient of f, such that $\|g_\varepsilon - g\|_{\mathcal{M}_p^q(\mathscr{X})} \le \varepsilon$ and $g_\varepsilon \ge g$ everywhere on $\mathscr{X}$.*

Applying Lemma 4.2, we obtain the following result.

Theorem 4.3. *Let $1 \le p \le q < \infty$. If $\mathscr{X}$ supports a weak $(1,p)$-Poincaré inequality and the measure μ is doubling, then $NM_p^q(\mathscr{X}) \hookrightarrow HM_p^q(\mathscr{X})$.*

Proof. Let $f \in NM_p^q(\mathscr{X})$. By [31, Theorem 1.0.1], we see that $\mathscr{X}$ supports a weak $(1,r)$-Poincaré inequality for some $r \in (0,p)$. By Lemma 4.2, there exists an upper gradient g of u such that

$$\|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \|g\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \|f\|_{NM_p^q(\mathscr{X})}.$$

Since $\mathscr{X}$ supports a weak $(1,r)$-Poincaré inequality for some $r \in (0,p)$, by [21, Theorem 3.2], we know that there exists a set $E \subset \mathscr{X}$ with $\mu(E) = 0$ such that, for all x, $y \in \mathscr{X} \setminus E$,

$$|f(x) - f(y)| \lesssim d(x,y)\left\{[M(g^r)(x)]^{1/r} + [M(g^r)(y)]^{1/r}\right\}.$$

Hence a constant multiple of $h := [M(g^r)]^{1/r}$ is a Hajłasz gradient of f. Then, by Lemma 3.4, we know that

$$\|f\|_{HM_p^q(\mathscr{X})} \lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \|h\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \|g\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \|f\|_{NM_p^q(\mathscr{X})},$$

which completes the proof of Theorem 4.3. □

Remark 4.4. When $p = q$, under the same assumptions as in Theorem 4.3, it was proved in [47, Theorem 4.9] that $NM_p^p(\mathscr{X}) = HM_p^p(\mathscr{X})$.

Next we turn to consider the inverse embedding of Theorem 4.3. The following property of the Morrey modulus is needed, which is a special case of [37, Proposition 4.7].

Lemma 4.5. *Let $1 \le p \le q < \infty$ and Γ be a collection of rectifiable curves. Then the following two statements are equivalent:*

(i) $\mathrm{Mod}_p^q(\Gamma) = 0$*;*

(ii) *there exists a non-negative Borel measurable function $\rho \in \mathcal{M}_p^q(\mathscr{X})$ such that, for any $\gamma \in \Gamma$, $\int_\gamma \rho(s)\,ds = \infty$.*

Recall that a function f on $\mathscr{X}$ is said to be *almost everywhere continuous with respect to* Mod_p^q if there exists a set $E \subset \mathscr{X}$, with $\mathrm{Mod}_p^q(\Gamma_E) = 0$, such that f is continuous on $\mathscr{X} \setminus E$. Let $\mathcal{C}(\mathscr{X})$ be the *set of all such functions f on $\mathscr{X}$*. Then we have the following conclusion.

Theorem 4.6. *Let $1 \le p \le q < \infty$. Then $\mathcal{C}(\mathscr{X}) \cap HM_p^q(\mathscr{X}) \hookrightarrow NM_p^q(\mathscr{X})$.*

Proof. Let $f \in \mathcal{C}(\mathscr{X}) \cap HM_p^q(\mathscr{X})$. Then, by $f \in \mathcal{C}(\mathscr{X})$, we know that there exists a set $E_1 \subset \mathscr{X}$, with $\mathrm{Mod}_p^q(\Gamma_{E_1}) = 0$, such that f is continuous on $\mathscr{X} \setminus E_1$. Also, by $f \in HM_p^q(\mathscr{X})$, we see that there exists a Hajłasz gradient h of f such that $h \in \mathcal{M}_p^q(\mathscr{X})$, which further implies that there exists a set $E_2 \subset \mathscr{X}$, with $\mu(E_2) = 0$, such that, for all $x,\ y \in \mathscr{X} \setminus E_2$,

$$|f(x) - f(y)| \le d(x,y)[h(x) + h(y)]. \tag{4.1}$$

Since $\mu(E_2) = 0$, similar to [47, Lemma 4.6], we may redefine the values of f and h on a set containing E_2 of measure 0 such that (4.1) holds for all $x, y \in \mathscr{X}$ and with the same $\mathcal{M}_p^q(\mathscr{X})$ norm. Let Γ be the collection of non-constant rectifiable curves such that $\int_\gamma h(s)\,ds = \infty$. By Lemma 4.5, we have $\mathrm{Mod}_p^q(\Gamma) = 0$ and hence $\mathrm{Mod}_p^q(\Gamma \cup \Gamma_{E_1}) = 0$.

Let $\gamma \notin \Gamma \cup \Gamma_{E_1}$. Fix $n \in \mathbb{N}$ and divide γ into n subcurves γ_i, $i \in \{1, \dots, n\}$, with equal length. Since γ is continuous, we know that $\mathrm{I}_i := \gamma^{-1}(\gamma_i)$ is an interval and $[0, \ell(\gamma)] = \cup_{i=1}^n \mathrm{I}_i$. By $\int_\gamma h(s)\,ds < \infty$, we know that $\int_{\gamma_i} h(s)\,ds < \infty$ for any $i \in \{1, \dots, n\}$. Then, by the argument as in [47, Lemma 4.7], we see that, for any $i \in \{1, \dots, n\}$, there exists $t_i \in \mathrm{I}_i$ such that $h(\gamma(t_i)) \le \frac{1}{l(\gamma_i)} \int_{\gamma_i} h(s)\,ds$, where $l(\gamma_i)$ denotes the length of γ_i. Write $\gamma(t_j) =: z_j$. Then, by this and (4.1), together with $l(\gamma_i) = \frac{1}{n} l(\gamma)$ and $\cup_{i=1}^n \gamma_i = \gamma$, we conclude that

$$|f(z_1) - f(z_n)| \le \sum_{i=1}^{n-1} |f(z_i) - f(z_{i+1})| \le \sum_{i=1}^{n-1} d(z_i, z_{i+1})[h(z_i) + h(z_{i+1})]$$

$$\leq \sum_{i=1}^{n-1}[l(\gamma_i)+l(\gamma_{i+1})]\left[\frac{1}{l(\gamma_i)}\int_{\gamma_i}h(s)\,ds+\frac{1}{l(\gamma_{i+1})}\int_{\gamma_{i+1}}h(s)\,ds\right]$$
$$\leq 2\sum_{i=1}^{n-1}\left[\int_{\gamma_i}h(s)\,ds+\int_{\gamma_{i+1}}h(s)\,ds\right]\leq 4\int_{\gamma}h(s)\,ds.$$

Since $\gamma\subset\mathscr{X}\setminus E_1$ and f is continuous on $\mathscr{X}\setminus E_1$, by the fact that $z_1\to\gamma(0)$ and $z_n\to\gamma(\ell(\gamma))$ as $n\to\infty$, we further find that

$$|f(\gamma(0))-f(\gamma(\ell(\gamma)))|\leq\int_{\gamma}4h(s)\,ds,$$

that is, $4h$ is a Mod_p^q-weak upper gradient of f. Thus, $f\in NM_p^q(\mathscr{X})$ and $\|f\|_{NM_p^q(\mathscr{X})}\lesssim\|f\|_{HM_p^q(\mathscr{X})}$. This finishes the proof of Theorem 4.6. □

Remark 4.7. (i) We point out that, in the proof of Theorem 4.6, the condition $f\in\mathcal{C}(\mathscr{X})$ is used to exclude those "bad curves" on which the function f may not be continuous.

(ii) From Theorem 4.6, we deduce that the closure of $\mathcal{C}(\mathscr{X})\cap HM_p^q(\mathscr{X})$ in $HM_p^q(\mathscr{X})$ continuously embeds into $NM_p^q(\mathscr{X})$.

(iii) When $p=q$, then $HM_p^q(\mathscr{X})=M^{1,p}(\mathscr{X})$ and $NM_p^q(\mathscr{X})=N^{1,p}(\mathscr{X})$. Since the set of all Lipschitz functions is dense in $M^{1,p}(\mathscr{X})$ (see [18, Theorem 5]), Theorems 4.3 and 4.6 go back to the known result: under the assumptions of Theorem 4.3, $N^{1,p}(\mathscr{X})=M^{1,p}(\mathscr{X})$ for $p\in[1,\infty)$, which is just [47, Theorem 4.9].

(iv) Let $\mathscr{X}$ be as in Theorem 3.1 with the weak $(1,p)$-Poincaré inequality replaced by the weak $(1,t)$-Poincaré inequality for some $t\in(1,p)$. Observe that if $\mathscr{X}$ supports a weak $(1,t)$-Poincaré inequality for some $t\in(1,p)$, then it also supports a weak $(1,p)$-Poincaré inequality, via the Hölder inequality. Then, by Theorem 3.1, we know that, when $Q\in(0,q)$ and $1\leq p\leq q<\infty$, $NM_p^q(\mathscr{X})\subset\mathcal{C}(\mathscr{X})$, which, together with Theorems 4.3 and 4.6, further implies that

$$\mathcal{C}(\mathscr{X})\cap HM_p^q(\mathscr{X})=NM_p^q(\mathscr{X}). \tag{4.2}$$

(v) It is still unknown whether $HM_p^q(\mathscr{X})\hookrightarrow NM_p^q(\mathscr{X})$ for $1\leq p<q<\infty$ also holds true or not since we do not know whether $HM_p^q(\mathscr{X})$ is contained in $\mathcal{C}(\mathscr{X})$ or not. Observe that, from (4.2) and [47, Theorem 4.9], it follows that $HM_p^p(\mathscr{X})\subset\mathcal{C}(\mathscr{X})$.

We now consider the relation between the Hajłasz-Morrey-Sobolev space and the classical Morrey-Sobolev space on $\mathbb{R}^n$. Let $1\leq p\leq q<\infty$. Recall that the classical *Morrey-Sobolev space* $WM_p^q(\mathbb{R}^n)$ is defined as

$$WM_p^q(\mathbb{R}^n):=\left\{f\in\mathcal{M}_p^q(\mathbb{R}^n):\ \nabla f\in\mathcal{M}_p^q(\mathbb{R}^n)\right\},$$

where ∇f denotes the *weak derivative* of f. The *norm* of $f\in WM_p^q(\mathbb{R}^n)$ is given by

$$\|f\|_{WM_p^q(\mathbb{R}^n)}:=\|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}+\|\nabla f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}.$$

Observe that $WM_p^p(\mathbb{R}^n)$ is just the Sobolev space $W^{1,p}(\mathbb{R}^n)$.

We have the following conclusion.

Theorem 4.8. *Let $1 < p \le q < \infty$. Then $WM_p^q(\mathbb{R}^n) = HM_p^q(\mathbb{R}^n)$ with equivalent norms.*

Before we prove Theorem 4.8, we need some key facts on the predual space of $\mathcal{M}_p^q(\mathbb{R}^n)$ when $1 < p < q < \infty$ from Adams and Xiao [1, 4]. To state them, we begin with recalling some notions.

Let $p \in (1, \infty)$. A non-negative function w on $\mathbb{R}^n$ is called an $A_p(\mathbb{R}^n)$ *weight*, denote by $w \in A_p(\mathbb{R}^n)$, if there exists a positive constant C, depending only on p and n, such that, for all balls B,

$$\frac{1}{|B|}\int_B w(y)\,dy\left\{\frac{1}{|B|}\int_B [w(y)]^{1/(1-p)}\,dy\right\}^{p-1} \le C. \tag{4.3}$$

A non-negative function w is called an $A_1(\mathbb{R}^n)$ *weight*, denoted by $w \in A_1(\mathbb{R}^n)$, if there exists a positive constant C such that, for almost every $x \in \mathbb{R}^n$, $Mw(x) \le Cw(x)$.

For $\alpha \in (0, n)$, the α-th order *Hausdorff capacity* at the level $\varepsilon \in (0, \infty]$ of $E \subset \mathbb{R}^n$ is defined by

$$\Lambda_\alpha^{(\varepsilon)}(E) := \inf\left\{\sum_{j=1}^\infty r_j^\alpha : E \subset \bigcup_{j=1}^\infty B(x_j, r_j) \text{ and } r_j \le \varepsilon,\ j \in \mathbb{N}\right\}.$$

Recall that $\Lambda_\alpha^{(\infty)}$ is finite on all bounded sets and is only a capacity in the sense of Meyers (see also [4]).

Definition 4.9. Let $1 \le p \le q < \infty$ and $n \in \mathbb{N}$. The *space* $H_p^q(\mathbb{R}^n)$ is defined to be the space of all as $f \in L^p_{\rm loc}(\mathbb{R}^n)$ such that

$$\|f\|_{H_p^q(\mathbb{R}^n)} := \inf_w \left\{\int_{\mathbb{R}^n} |f(y)|^p [w(y)]^{1-p}\,dy\right\}^{1/p} < \infty,$$

where $L^p_{\rm loc}(\mathbb{R}^n)$ denotes the *class of all p-locally integrable functions on* $\mathbb{R}^n$ and the infirm is taken over all non-negative weights $w \in A_1(\mathbb{R}^n)$ satisfying that

$$\int_{\mathbb{R}^n} w\,d\Lambda^{(\infty)}_{n-\frac{pn}{q}} := \int_0^\infty \Lambda^{(\infty)}_{n-\frac{pn}{q}}(\{x \in \mathbb{R}^n :\ w(x) > t\})\,dt \le 1.$$

Let $1 < p < q < \infty$ and $\lambda := pn/q$. Observe that the space $H_p^q(\mathbb{R}^n)$ is just the space $H^{p,\lambda}$ defined by Adams and Xiao in [4], where they also proved that $H^{p',\lambda}$, which is $H_{p'}^{q/(p-1)}(\mathbb{R}^n)$ by our notation, is the predual space of $L^{p,\lambda}$, which is the same space as $\mathcal{M}_p^q(\mathbb{R}^n)$ in this paper; see [1, Theorem 2.3] and [4, Theorems 6 and 7]. Using this, we have the following useful and key lemma.

Lemma 4.10. (i) *If $1 < p < q < \infty$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $H_p^q(\mathbb{R}^n)$.*

(ii) *If $1 < p < q < \infty$, then for any $f \in \mathcal{M}_p^q(\mathbb{R}^n)$,*

$$\|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| :\quad g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_{H_{p'}^{q/(p-1)}(\mathbb{R}^n)} \le 1\right\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Notice that (ii) is an immediate corollary of (i) and the fact that $(H_{p'}^{q/(p-1)}(\mathbb{R}^n))^* = \mathcal{M}_p^q(\mathbb{R}^n)$ when $1 < p < q < \infty$ (see [1, Theorem 2.3] and [4, Theorems 6 and 7]), together with a density argument. Thus, we only need to prove (i). To this end, let $f \in H_p^q(\mathbb{R}^n)$. By Definition 4.9, we see that, for any $\varepsilon \in (0, \infty)$, there exists a weight $w \in A_1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} w \, d\Lambda_{n-np/q}^{(\infty)} \le 1$ and

$$\|f\|_{H_p^q(\mathbb{R}^n)} \le \left\{ \int_{\mathbb{R}^n} |f(x)|^p [w(x)]^{1-p} \, dx \right\}^{1/p} < \|f\|_{H_p^q(\mathbb{R}^n)} + \varepsilon.$$

Thus, $f \in L_{w^{1-p}}^p(\mathbb{R}^n)$, where

$$L_{w^{1-p}}^p(\mathbb{R}^n) := \left\{ h \text{ is measurable}: \ \|h\|_{L_{w^{1-p}}^p(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |h(x)|^p [w(x)]^{1-p} \, dx < \infty \right\}.$$

We now claim that if $w \in A_1(\mathbb{R}^n)$, then $C_c^\infty(\mathbb{R}^n)$ is dense in $L_{w^{1-p}}^p(\mathbb{R}^n)$. Let $v := w^{1-p}$ and $h \in L_v^p(\mathbb{R}^n)$. By the weighted boundedness of the Hardy-Littlewood maximal function M on $L_v^p(\mathbb{R}^n)$ (see, for example, [49, p. 201]), and $v \in A_p(\mathbb{R}^n)$, which is deduced from $w \in A_1(\mathbb{R}^n)$ (see [13, Proposition 7.2]), we see that

$$\int_{\mathbb{R}^n} [Mh(x)]^p \, v(x) \, dx \lesssim \int_{\mathbb{R}^n} |h(x)|^p \, v(x) \, dx \approx \|h\|_{L_v^p(\mathbb{R}^n)}^p. \tag{4.4}$$

Let $\phi \in C_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$. For all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $\phi_t(x) := \frac{1}{t^n} \phi(\frac{x}{t})$. From $h \in L_v^p(\mathbb{R}^n)$, we deduce that

$$\lim_{R \to \infty} \int_{\mathbb{R}^n} |h(x) - (h\chi_{B(0,R)})(x)|^p \, v(x) \, dx = 0. \tag{4.5}$$

Moreover, by the Hölder inequality, $h \in L_v^p(\mathbb{R}^n)$ and $v \in A_p(\mathbb{R}^n)$, we conclude that, for any ball $B \subset \mathbb{R}^n$,

$$\begin{aligned} \int_B |h(x)| \, dx &= \int_B |h(x)| [v(x)]^{1/p} [v(x)]^{-1/p} \, dx \\ &\le \left[\int_B |h(x)|^p v(x) \, dx \right]^{1/p} \left[\int_B [v(x)]^{-p'/p} \, dx \right]^{1/p'} < \infty. \end{aligned}$$

Thus, $h \in L_{\rm loc}^1(\mathbb{R}^n)$, which implies that, for any $R \in (0, \infty)$, $h\chi_{B(0,R)} \in L^1(\mathbb{R}^n)$. By this and [50, Theorem 1.25], we see that, for any $R \in (0, \infty)$ and almost every $x \in \mathbb{R}^n$,

$$\lim_{t \to 0} (h\chi_{B(0,R)} * \phi_t)(x) = h\chi_{B(0,R)}(x). \tag{4.6}$$

Observe that both h and $(h\chi_{B(0,R)}) * \phi_t$ for any $R \in (0, \infty)$ and $t > 0$ are controlled by Mh. By this, (4.4), the Lebesgue dominated convergence theorem, (4.5) and (4.6), we conclude that

$$\lim_{R \to \infty} \lim_{t \to 0} \int_{\mathbb{R}^n} |h(x) - (h\chi_{B(0,R)}) * \phi_t(x)|^p v(x) \, dx$$

$$\begin{aligned}&\lesssim \lim_{R\to\infty}\int_{\mathbb{R}^n} |h(x)-(h\chi_{B(0,R)})(x)|^p v(x)\,dx\\&\quad+\lim_{R\to\infty}\int_{\mathbb{R}^n}\lim_{t\to 0}|(h\chi_{B(0,R)}(x)-(h\chi_{B(0,R)})*\phi_t(x)|^p v(x)\,dx=0\end{aligned}$$

which, together with the fact that $(h\chi_{B(0,R)})*\phi_t\in C_c^\infty(\mathbb{R}^n)$ for all t, $R\in(0,\infty)$, further implies that h is approximated by functions in $C_c^\infty(\mathbb{R}^n)$. This finishes the proof of the claim.

Moreover, by this claim and $f\in L^p_{w^{1-p}}(\mathbb{R}^n)$, we see that there exists $g\in C_c^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}|f(x)-g(x)|^p[w(x)]^{1-p}\,dx<\varepsilon,$$

which, by Definition 4.9, further implies that

$$\|f-g\|_{H_p^q(\mathbb{R}^n)}\le\int_{\mathbb{R}^n}|f(x)-g(x)|^p[w(x)]^{1-p}\,dx<\varepsilon.$$

This finishes the proof of (i) and hence Lemma 4.10. □

Now we are ready to present the proof of Theorem 4.8.

Proof of Theorem 4.8. Observe that the conclusion of Theorem 4.8 when $1<p=q<\infty$ is just [18, Theorem 1]. Thus, *in what follows of this proof, we always assume that* $1<p<q<\infty$.

We first show that $WM_p^q(\mathbb{R}^n)\subset HM_p^q(\mathbb{R}^n)$. Let $f\in WM_p^q(\mathbb{R}^n)$. By the definition of $WM_p^q(\mathbb{R}^n)$, we see that $\nabla f\in L^p(Q)$ for all cubes Q in $\mathbb{R}^n$ and then, following the argument as in [18, p. 404], we know that, for all Lebesgue points x, $y\in\mathbb{R}^n$ of f,

$$|f(x)-f(y)|\lesssim|x-y|\,[M(|\nabla f|)(x)+M(|\nabla f|)(y)]\,.$$

Hence a constant multiple of $M(|\nabla f|)$ is a Hajłasz gradient of f. Moreover, by Definition 4.1 and Lemma 3.4, we further see that

$$\|f\|_{HM_p^q(\mathbb{R}^n)}\le\|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}+\|M(|\nabla f|)\|_{\mathcal{M}_p^q(\mathbb{R}^n)}\lesssim\|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}+\|\nabla f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}\approx\|f\|_{WM_p^q(\mathbb{R}^n)}.$$

This shows that $WM_p^q(\mathbb{R}^n)\subset HM_p^q(\mathbb{R}^n)$.

Next we prove $HM_p^q(\mathbb{R}^n)\subset WM_p^q(\mathbb{R}^n)$. Let $f\in HM_p^q(\mathbb{R}^n)$. Then, by Definition 4.1, we see that there exists a nonnegative function $h\in\mathcal{M}_p^q(\mathbb{R}^n)$ such that, for almost every $x,y\in\mathbb{R}^n$,

$$|f(x)-f(y)|\le|x-y|[h(x)+h(y)]\tag{4.7}$$

and

$$\|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}+\|h\|_{\mathcal{M}_p^q(\mathbb{R}^n)}\lesssim\|f\|_{HM_p^q(\mathbb{R}^n)}.$$

To complete the proof, it suffices to show that there exists a sequence of functions $\{\eta_\varepsilon\}_{\varepsilon\in(0,\infty)}\subset\mathcal{M}_p^q(\mathbb{R}^n)$ such that, for all $\varphi\in C_c^\infty(\mathbb{R}^n)$ and $i\in\{1,\ldots,n\}$,

$$\left|\left\langle\frac{\partial f}{\partial x_i},\varphi\right\rangle\right|\le\sup_{\varepsilon\in(0,\infty)}\left|\int_{\mathbb{R}^n}\varphi(x)\eta_\varepsilon(x)\,dx\right|\tag{4.8}$$

and

$$\sup_{\varepsilon\in(0,\infty)} \|\eta_\varepsilon\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim \|h\|_{\mathcal{M}_p^q(\mathbb{R}^n)}. \tag{4.9}$$

Assume that the sequence $\{\eta_\varepsilon\}_{\varepsilon\in(0,\infty)}$ satisfying (4.8) and (4.9) exists for a moment. To complete the proof, by Lemma 4.10(ii), (4.8) and (4.9) for any $i \in \{1, \dots, n\}$, we conclude that

$$\begin{aligned}\left\|\frac{\partial f}{\partial x_i}\right\|_{\mathcal{M}_p^q(\mathbb{R}^n)} &= \sup_{\substack{\varphi\in C_c^\infty(\mathbb{R}^n)\\ \|\varphi\|_{H_{p'}^{q/(p-1)}(\mathbb{R}^n)}\le 1}} \left|\left\langle \frac{\partial f}{\partial x_i}, \varphi\right\rangle\right| \le \sup_{\substack{\varphi\in C_c^\infty(\mathbb{R}^n)\\ \|\varphi\|_{H_{p'}^{q/(p-1)}(\mathbb{R}^n)}\le 1}} \sup_{\varepsilon\in(0,\infty)} \left|\int_{\mathbb{R}^n} \varphi(x)\eta_\varepsilon(x)\,dx\right| \\ &= \sup_{\varepsilon\in(0,\infty)} \sup_{\substack{\varphi\in C_c^\infty(\mathbb{R}^n)\\ \|\varphi\|_{H_{p'}^{q/(p-1)}(\mathbb{R}^n)}\le 1}} \left|\int_{\mathbb{R}^n} \varphi(x)\eta_\varepsilon(x)\,dx\right| \\ &= \sup_{\varepsilon\in(0,\infty)} \|\eta_\varepsilon\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim \|h\|_{\mathcal{M}_p^q(\mathbb{R}^n)}.\end{aligned}$$

From this, the definition of $WM_p^q(\mathbb{R}^n)$ and the choice of h, we deduce that

$$\|f\|_{WM_p^q(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}_p^q(\mathbb{R}^n)} + \|h\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim \|f\|_{HM_p^q(\mathbb{R}^n)},$$

which implies that $f \in WM_p^q(\mathbb{R}^n)$ and $\|f\|_{WM_p^q(\mathbb{R}^n)} \lesssim \|f\|_{HM_p^q(\mathbb{R}^n)}$. Thus, $HM_p^q(\mathbb{R}^n) \subset WM_p^q(\mathbb{R}^n)$, which is desired.

We now turn to the proof of the existence of $\{\eta_\varepsilon\}_{\varepsilon\in(0,\infty)}$. Fix $z \in \mathbb{R}^n$ and let $\varepsilon \in (0,\infty)$. Integrating (4.7) twice on $B(z,\varepsilon)$, we see that

$$\begin{aligned}\int_{B(z,\varepsilon)}\int_{B(z,\varepsilon)} |f(x)-f(y)|\,dxdy &\le 2\varepsilon \int_{B(z,\varepsilon)}\int_{B(z,\varepsilon)} [h(x)+h(y)]\,dxdy \\ &= 2\varepsilon|B(z,\varepsilon)|^2 \frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)} h(x)\,dx,\end{aligned}$$

which further implies that

$$\begin{aligned}\frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)} |f(x)-f_{B(z,\varepsilon)}|\,dx &\le \frac{1}{|B(z,\varepsilon)|^2}\int_{B(z,\varepsilon)}\int_{B(z,\varepsilon)} |f(x)-f(y)|\,dxdy \\ &\le 2\varepsilon \frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)} h(x)\,dx.\end{aligned} \tag{4.10}$$

Choose $\psi \in C_c^\infty(B(0,1))$ such that $\int_{\mathbb{R}^n}\psi(x)\,dx = 1$ and let $\psi_\lambda(\cdot) := \lambda^{-n}\psi(\cdot/\lambda)$ for all $\lambda \in (0,\infty)$. Notice that $\int_{\mathbb{R}^n} \frac{\partial\psi_\lambda}{\partial x_i}(x)\,dx = 0$. Then

$$f * \frac{\partial\psi_\lambda}{\partial x_i} = (f - f_{B(z,\varepsilon)}) * \frac{\partial\psi_\lambda}{\partial x_i}$$

and hence, for all $x \in \mathbb{R}^n$,

$$\left|f * \frac{\partial\psi_\lambda}{\partial x_i}(x)\right| = \left|(f - f_{B(z,\varepsilon)}) * \frac{\partial\psi_\lambda}{\partial x_i}(x)\right|$$

$$\begin{aligned}&\le \lambda^{-n-1}\int_{B(x,\lambda)}|f(t)-f_{B(z,\varepsilon)}|\left|\frac{\partial\psi}{\partial x_i}\left(\frac{x-t}{\lambda}\right)\right|\,dt\\&\lesssim \lambda^{-1}\|\nabla\psi\|_{L^\infty(\mathbb{R}^n)}\frac{1}{|B(x,\lambda)|}\int_{B(x,\lambda)}|f(t)-f_{B(z,\varepsilon)}|\,dt.\end{aligned}$$

Let $x=z$ and $\lambda=\varepsilon$. Then by (4.10), we conclude that

$$\left|f*\frac{\partial\psi_\varepsilon}{\partial x_i}(z)\right|\lesssim\frac{1}{|B(z,\varepsilon)|}\int_{B(z,\varepsilon)}h(x)\,dx\lesssim M(h)(z). \tag{4.11}$$

On the other hand, since $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\varphi\in C_c^\infty(\mathbb{R}^n)$, it follows that

$$\int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial x_i}(x)f(x)\,dx=\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial x_i}(x)(\psi_\varepsilon*f)(x)\,dx=-\lim_{\varepsilon\to0}\int_{\mathbb{R}^n}\varphi(x)\left(\frac{\partial\psi_\varepsilon}{\partial x_i}*f\right)(x)\,dx.$$

For any $\varepsilon\in(0,\infty)$, let $\eta_\varepsilon:=\frac{\partial\psi_\varepsilon}{\partial x_i}*f$. Then we see that, for any $\varepsilon\in(0,\infty)$,

$$\left|\int_{\mathbb{R}^n}f(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx\right|\le\lim_{\varepsilon\to0}\left|\int_{\mathbb{R}^n}\varphi(x)\eta_\varepsilon(x)(x)\,dx\right|\le\sup_{\varepsilon\in(0,\infty)}\left|\int_{\mathbb{R}^n}\varphi(x)\eta_\varepsilon(x)\,dx\right|.$$

Moreover, by (4.11), we have $|\eta_\varepsilon|\lesssim M(h)$, which, together with Lemma 3.4 again, implies that, for all $\varepsilon\in(0,\infty)$,

$$\|\eta_\varepsilon\|_{\mathcal{M}_p^q(\mathbb{R}^n)}\lesssim\|M(h)\|_{\mathcal{M}_p^q(\mathbb{R}^n)}\lesssim\|h\|_{\mathcal{M}_p^q(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 4.8. □

We remark that Theorem 4.8 when $p=q$ goes back to the equivalence between Sobolev spaces and Hajłasz Sobolev spaces on $\mathbb{R}^n$ obtained in [18].

From Theorems 4.3, 4.6 and 4.8, we deduce the following corollary.

Corollary 4.11. *Let $1<p\le q<\infty$ and $n<q$. Then $WM_p^q(\mathbb{R}^n)=NM_p^q(\mathbb{R}^n)$ with equivalent norms.*

Proof. We first observe that, when $p=q$, the conclusion is proved by [47, Theorem 4.5]. Now, we assume that $1<p<q<\infty$. By Theorems 4.3, 4.6 and 4.8, it suffices to show that $WM_p^q(\mathbb{R}^n)\subset\mathcal{C}(\mathbb{R}^n)$. Notice that, if $f\in WM_p^q(\mathbb{R}^n)$, then $\nabla f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Then, following the argument in [18, p. 404], by [16, Lemma 7.16], we know that, for almost every $x\in\mathbb{R}^n$ and any ball B,

$$|f(x)-f_B|\lesssim\int_B\frac{|\nabla f(y)|}{|x-y|^{n-1}}\,dy. \tag{4.12}$$

Therefore, from (4.12), the Hölder inequality and (1.3), together with $q>n$, it follows that, for almost every $x\in\mathbb{R}^n$ and all $R\in(0,\infty)$,

$$|f(x)-f_{B(x,R)}|\lesssim\int_{B(x,R)}\frac{|\nabla f(y)|}{|x-y|^{n-1}}\,dy \tag{4.13}$$

$$
\begin{aligned}
&\approx \sum_{k=0}^{\infty} \int_{B(x,\frac{R}{2^k})\setminus B(x,\frac{R}{2^{k+1}})} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\,dy \\
&\lesssim R \sum_{k=0}^{\infty} 2^{-k} \left[\frac{1}{|B(x,\frac{R}{2^k})|}\int_{B(x,\frac{R}{2^k})} |\nabla f(y)|^p\,dy\right]^{1/p} \\
&\lesssim R^{1-n/q} \sum_{k=0}^{\infty} 2^{-k(1-n/q)} \|\nabla f\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim R^{1-n/q}\|\nabla f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}.
\end{aligned}
$$

Thus, for almost every $x, y \in \mathbb{R}^n$, letting $R := 2|x-y|$, by (4.13) and $B(x,R) \subset B(y, \frac{3R}{2})$, we then conclude that

$$
\begin{aligned}
|f(x)-f(y)| &\le |f(x)-f_{B(x,R)}| + |f(y)-f_{B(x,R)}| \\
&\lesssim \int_{B(x,R)} \frac{|\nabla f(z)|}{|x-z|^{n-1}}\,dz + \int_{B(x,R)} \frac{|\nabla f(z)|}{|y-z|^{n-1}}\,dz \\
&\lesssim |x-y|^{1-n/q}\|\nabla f\|_{\mathcal{M}_p^q(\mathbb{R}^n)} + \int_{B(y,\frac{3R}{2})} \frac{|\nabla f(x)|}{|y-z|^{n-1}}\,dz \\
&\lesssim |x-y|^{1-n/q}\|\nabla f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}.
\end{aligned}
$$

Then, by [40, Corollary 1], there exists a $(1-n/q)$-Hölder continuous function $\widetilde{f}$ such that $f = \widetilde{f}$ almost everywhere, and hence in $WM_p^q(\mathbb{R}^n)$. Further, we could treat f and $\widetilde{f}$ as the same function in $WM_p^q(\mathbb{R}^n)$. In this sense, $f \in \mathcal{C}(\mathbb{R}^n)$, which completes the proof of Corollary 4.11. □

Thus, by Theorem 4.8 and Corollary 4.11, we know that both Newton-Morrey-Sobolev spaces and Hajłasz-Morrey-Sobolev spaces coincide with classical Morrey-Sobolev spaces on $\mathbb{R}^n$.

5 Pointwise characterization of Hardy-Morrey-Sobolev spaces

In this section, following some ideas used in [33], we consider the coincidence between the Hajłasz-Morrey-Sobolev space and the Hardy-Morrey-Sobolev space on $\mathbb{R}^n$ (see Definition 5.3 below). When $\mathscr{X}$ is a space of homogeneous type in the sense of Coifman and Weiss, a characterization of the Hajłasz-Morrey-Sobolev space via the grand maximal function is also established.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the *set of all Schwartz functions on* $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual*, namely, the *class of all tempered distributions*. Let $\mathcal{S}_\infty(\mathbb{R}^n)$ be the *set of all* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} \phi(x)x^\gamma\,dx = 0$ *for all multi-indices* $\gamma \in \mathbb{Z}_+^n$, and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its *topological dual*. We first recall that the definition of Hardy-Morrey space, which was originally introduced by Jia and Wang [30].

Definition 5.1. Let $0 < p \le q \le \infty$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$ and supp $\psi \subset \{x \in \mathbb{R}^n : |x| \le 1\}$. The *Hardy-Morrey space* $\mathscr{H}M_p^q(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$, which denotes the , such that $\mathcal{M}_\psi f \in \mathcal{M}_p^q(\mathbb{R}^n)$, where $\mathcal{M}_\psi f(x) := \sup_{t>0} |f * \psi_t(x)|$ for all $x \in \mathbb{R}^n$, and $\psi_t(\cdot) := t^{-n}\psi(\cdot/t)$. Moreover, $\|f\|_{\mathscr{H}M_p^q(\mathbb{R}^n)} := \|\mathcal{M}_\psi f\|_{\mathcal{M}_p^q(\mathbb{R}^n)}$.

Remark 5.2. (i) From [49, p. 57] and the boundedness of the Hardy-Littlewood maximal function on $\mathcal{M}_p^q(\mathbb{R}^n)$ (see Lemma 3.4), it follows that, when $1 < p \le q \le \infty$, $\mathcal{M}_p^q(\mathbb{R}^n) = \mathscr{H}M_p^q(\mathbb{R}^n)$ with equivalent norms.

(ii) By the results in [30, Section 2], we know that $\mathscr{H}M_p^q(\mathbb{R}^n)$ is independent of the choice of ψ as in Definition 5.1.

The Hardy-Morrey-Sobolev space is then defined as follows.

Definition 5.3. Let $0 < p \le q \le \infty$. The *homogeneous Hardy-Morrey-Sobolev space* $\mathscr{H}\dot{M}_{1,p}^q(\mathbb{R}^n)$ is defined to be the set of distributions $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $D_j f \in \mathscr{H}M_p^q(\mathbb{R}^n)$ for all $j \in \{1, \ldots, n\}$, where $D_j f$ denotes the *jth distributional derivative of f*. Moreover,

$$\|f\|_{\mathscr{H}\dot{M}_{1,p}^q(\mathbb{R}^n)} := \sum_{j=1}^n \|D_j f\|_{\mathscr{H}M_p^q(\mathbb{R}^n)}.$$

The homogeneous Hajłasz-Morrey-Sobolev space on $\mathbb{R}^n$ is defined as follows.

Definition 5.4. Let $0 < p \le q \le \infty$. The *homogeneous Hajłasz-Morrey-Sobolev space* $H\dot{M}_p^q(\mathscr{X})$ is defined to be the space of all measurable functions f who has a Hajłasz gradient $h \in \mathcal{M}_p^q(\mathscr{X})$. The *norm* of $f \in H\dot{M}_p^q(\mathscr{X})$ is then defined as

$$\|f\|_{H\dot{M}_p^q(\mathscr{X})} := \inf \|h\|_{\mathcal{M}_p^q(\mathscr{X})},$$

where the infimum is taken over all Hajłasz gradients h of f.

Then we have the following conclusion on the relation between these spaces.

Theorem 5.5. *Let $\frac{n}{n+1} < p \le q \le \infty$. Then $\mathscr{H}\dot{M}_{1,p}^q(\mathbb{R}^n) = H\dot{M}_p^q(\mathbb{R}^n)$ with equivalent norms.*

To prove this theorem, we need some knowledge on the Triebel-Lizorkin-type spaces on $\mathbb{R}^n$ (see [54, 46, 56]). Let $\mathscr{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ be the *collection of all dyadic cubes in* $\mathbb{R}^n$, where $Q_{jk} := 2^{-j}([0,1)^n + k)$. For $Q \in \mathscr{Q}$, we denote by $\ell(Q)$ its *side length* and let $j_Q := -\log_2 \ell(Q)$.

Recall that, for all $s \in \mathbb{R}$, $\tau \in [0,\infty)$, $p \in (0,\infty)$ and $r \in (0,\infty]$, the *Triebel-Lizorkin-type space* $\dot{F}_{p,r}^{s,\tau}(\mathbb{R}^n)$ is defined as the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{F}_{p,r}^{s,\tau}(\mathbb{R}^n)} := \sup_{P \in \mathscr{Q}} |P|^{-\tau} \left\{ \int_P \left[\sum_{j=j_P}^\infty 2^{jsr} |\varphi_j * f(x)|^r \right]^{p/r} dx \right\}^{1/p} < \infty,$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\operatorname{supp} \widehat{\varphi} \subset \left\{ \xi \in \mathbb{R}^n : \ \frac{1}{2} \le |\xi| \le 2 \right\}, \quad |\widehat{\varphi}(\xi)| \ge C > 0 \text{ if } \frac{3}{5} \le |\xi| \le \frac{5}{3}.$$

It was proved in [46, Corollary 3.2] that, for all $0 < p \le q < \infty$, the Hardy-Morrey space $\mathscr{H}M_p^q(\mathbb{R}^n) = \dot{F}_{p,2}^{0,1/p-1/q}(\mathbb{R}^n)$ with equivalent quasi-norms.

Proposition 5.6. *Let $0 < p \le q < \infty$. Then $\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n) = \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Proof. We first show $\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n) \hookrightarrow \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$. Let $f \in \mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)$. Then for all $i \in \{1,\dots,n\}$, $D_i f \in \mathscr{H}M^q_p(\mathbb{R}^n)$ and $\|D_i f\|_{\mathscr{H}M^q_p(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)}$. Recall that, by [46, Corollary 3.2], $\mathscr{H}M^q_p(\mathbb{R}^n) = \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ with equivalent quasi-norms. We then know that $D_i f \in \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and

$$\|D_i f\|_{\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)}. \tag{5.1}$$

For $i \in \{1,\dots,n\}$, let R_i be the Riesz transform defined by setting, for all $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$(R_i g)^\wedge(\xi) := -\imath \frac{\xi_i}{|\xi|}\widehat{g}(\xi), \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\imath$ denotes the imaginary unit, and I_σ with $\sigma \in \mathbb{R}$ be the Riesz potential operator defined by setting, for all $g \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$(I_\sigma g)^\wedge(\xi) := |\xi|^\sigma \widehat{g}(\xi), \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then it is well known that

$$D_i f = -I_1 R_i f \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^n). \tag{5.2}$$

Recall that, by [54, Proposition 3.5], $f \in \dot{F}^{s+\sigma,\tau}_{p,r}(\mathbb{R}^n)$ if and only if $I_\sigma f \in \dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)$, and $\|f\|_{\dot{F}^{s+\sigma,\tau}_{p,r}(\mathbb{R}^n)} \approx \|I_\sigma f\|_{\dot{F}^{s,\tau}_{p,r}(\mathbb{R}^n)}$. Hence, by $D_i f \in \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and (5.2), together with (5.1), we know that $R_i f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and

$$\|R_i f\|_{\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)} \approx \|D_i f\|_{\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)}$$

for all $i \in \{1,\dots,n\}$.

On the other hand, by the mapping properties of Fourier multipliers on Triebel-Lizorkin-type spaces in [55, Theorem 1.5], we know that R_i is a bounded operator on $\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and hence $R_i R_i f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$, which, together with the fact that

$$f = \sum_{i=1}^n R_i R_i f \quad \text{in } \mathcal{S}'_\infty(\mathbb{R}^n),$$

further implies that $f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$ and $\|f\|_{\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)}$. This finishes the proof for $\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n) \hookrightarrow \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$.

Now we prove $\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n) \hookrightarrow \mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)$. Let $f \in \dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)$. Then by the mapping properties of pseudo-differential operators on Triebel-Lizorkin-type spaces in [46, Theorem 1.5], we know that, for all $i \in \{1, \ldots, n\}$, $D_i f \in \dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n) = \mathscr{H}M^q_p(\mathbb{R}^n)$ and

$$\|D_i f\|_{\mathscr{H}M^q_p(\mathbb{R}^n)} \approx \|D_i f\|_{\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n)}.$$

This implies that $\dot{F}^{1,1/p-1/q}_{p,2}(\mathbb{R}^n) \hookrightarrow \mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)$ and hence finishes the proof of Proposition 5.6. □

In particular, when $p = q$, Proposition 5.6 goes back to the known coincidence between Hardy-Sobolev spaces and Triebel-Lizorkin spaces.

By [56, Proposition 8.2], we know that $\dot{F}^{0,1/p-1/q}_{p,2}(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$, which, together with Proposition 5.6, further implies the following conclusion, the details being omitted.

Corollary 5.7. *Let $0 < p \le q < \infty$. Then $\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$.*

Next we turn to the proof of Theorem 5.5.

Proof of Theorem 5.5. The case $q = \infty$ is a special case of Hajłasz [18, Theorem 1], and we only need to consider the case $q < \infty$. Assume first that $f \in \mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)$. Then, by Definition 5.3, we see that, for each $j \in \{1, \ldots, n\}$, $D_j f \in \mathscr{H}M^q_p(\mathbb{R}^n)$. By Corollary 5.7 and [33, Theorem 7], we know that, for all balls $B \subset \mathbb{R}^n$, there exists a set $E \subset \mathbb{R}^n$ of measure 0 such that, for all $x, y \in B \setminus E$,

$$|f(x) - f(y)| \lesssim |x - y|[M_1(Df)(x) + M_1(Df)(y)],$$

where we used the notation in [33] that, for all $x \in \mathbb{R}^n$,

$$M_1(Df)(x) := \max_{j\in\{1,\ldots,n\}} M_1(D_j f)(x) := \max_{j\in\{1,\ldots,n\}} \sup |\langle D_j f, \varphi\rangle|, \tag{5.3}$$

in which the supremum is taken over all compactly supported smooth functions φ satisfying that, for some $r \in (0, \infty)$ and all $j \in \{1, \ldots, n\}$,

$$\operatorname{supp} \varphi \subset B(x, r),\ \|\varphi\|_{L^\infty(\mathbb{R}^n)} \le r^{-n} \text{ and } \|D_j \varphi\|_{L^\infty(\mathbb{R}^n)} \le r^{-n-1}. \tag{5.4}$$

Therefore, by the definition of Hajłasz gradients, we see that $g := M_1(Df)$ is a positive constant multiple of a Hajłasz gradient of f.

We now choose that ψ satisfies (5.4) with φ replaced by ψ, and $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$. Using Remark 5.2(ii) and repeating the proofs of Lemma 2.1 and Lemma 2.4 in [30], we find that, if $\frac{n}{n+1} < 1 \le q < \infty$, then it holds that

$$\|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \approx \sum_{j=1}^n \|\mathcal{M}_\psi(D_j f)\|_{\mathcal{M}^q_p(\mathbb{R}^n)},$$

which, together with Definitions 5.1 and 5.3, further implies that

$$\|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \approx \sum_{j=1}^n \|\mathcal{M}_\psi(D_j f)\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \approx \sum_{j=1}^n \|D_j f\|_{\mathscr{H}M^q_p(\mathbb{R}^n)} \approx \|f\|_{\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)}.$$

Thus, $f \in H\dot{M}_p^q(\mathbb{R}^n)$ and $\|f\|_{H\dot{M}_p^q(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}\dot{M}_{1,p}^q(\mathbb{R}^n)}$. This implies that $\mathscr{H}\dot{M}_{1,p}^q(\mathbb{R}^n) \subset H\dot{M}_p^q(\mathbb{R}^n)$.

Now we suppose $f \in H\dot{M}_p^q(\mathbb{R}^n)$. Then, by Definition 5.4, there exists a Hajłasz gradient $g \in \mathcal{M}_p^q(\mathbb{R}^n)$ of f such that $\|g\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \le 2\|f\|_{H\dot{M}_p^q(\mathbb{R}^n)}$. Notice that, for any ball $B(a,r) \subset \mathbb{R}^n$ with the center $a \in \mathbb{R}^n$ and the radius $r \in (0,\infty)$, by the Hölder inequality, we have $g \in L^p_{\mathrm{loc}}(B(a,2r)) \subset L^{\frac{n}{n+1}}_{\mathrm{loc}}(B(a,2r))$, since $p > \frac{n}{n+1}$. From this and [33, Proposition 5], it follows that

$$\inf_{c\in\mathbb{R}} \frac{1}{|B(a,r)|}\int_{B(a,r)} |f(x)-c|\,dx \lesssim r\left\{\frac{1}{|B(a,2r)|}\int_{B(a,2r)} [g(x)]^{n/(n+1)}\,dx\right\}^{(n+1)/n}. \tag{5.5}$$

This inequality implies that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, from (5.5), we further deduce that

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(x)-f_{B(a,r)}|\,dx \lesssim r\left\{\frac{1}{|B(a,2r)|}\int_{B(a,2r)} [g(x)]^{n/(n+1)}\,dx\right\}^{(n+1)/n}. \tag{5.6}$$

By (5.6), (1.3) and Definition 5.4, we conclude that, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)\phi(x)\,dx\right|\\
&\quad= \left|\int_{\mathbb{R}^n} [f(x)-f_{B(0,1)}]\phi(x)\,dx\right|\\
&\quad\lesssim \int_{B(0,1)} \frac{|f(x)-f_{B(0,1)}|}{(1+|x|)^N}\,dx + \sum_{i=1}^\infty \int_{B(0,2^i)\setminus B(0,2^{i-1})} \frac{|f(x)-f_{B(0,1)}|}{(1+|x|)^N}\,dx\\
&\quad\lesssim \left\{\int_{B(0,2)} [g(x)]^{n/(n+1)}\,dx\right\}^{(n+1)/n} + \sum_{i=1}^\infty 2^{-iN}\int_{B(0,2^i)} |f(x)-f_{B(0,2^i)}|\,dx\\
&\qquad+ \sum_{i=1}^\infty 2^{-iN}\int_{B(0,1)} |f(x)-f_{B(0,1)}|\,dx\\
&\quad\lesssim \left\{\int_{B(0,2)} [g(x)]^{n/(n+1)}\,dx\right\}^{(n+1)/n} + \sum_{i=1}^\infty \frac{2^{-i(N-n)}}{|B(0,2^i)|}\int_{B(0,2^i)} |f(x)-f_{B(0,2^i)}|\,dx\\
&\quad\lesssim \sum_{i=0}^\infty 2^{-i(N-n-1)}\left\{\frac{1}{|B(0,2^{i+1})|}\int_{B(0,2^{i+1})} [g(x)]^p\,dx\right\}^{1/p}\\
&\quad\lesssim \sum_{i=0}^\infty 2^{-i(N-n-1+n/q)}\|g\|_{\mathcal{M}_p^q(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{M}_p^q(\mathbb{R}^n)},
\end{aligned}$$

where we chose $N \in \mathbb{N}$ and $N > n+1-n/q$. Thus, $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$.

Moreover, similar to the proof of [33, Theorem 1], by (5.6), we conclude that, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}_\psi(D_j f)(x) \lesssim \left[M(g^{n/(n+1)})(x)\right]^{(n+1)/n}. \tag{5.7}$$

Applying Lemma 3.4, together with $1 < \frac{n+1}{n}p \le \frac{n+1}{n}q \le \infty$, we know that $M(g^{n/(n+1)}) \in \mathcal{M}^{\frac{n+1}{n}q}_{\frac{n+1}{n}p}(\mathbb{R}^n)$, which, together with (5.7) and (1.3), implies that $\mathcal{M}_\psi(D_j f) \in \mathcal{M}^q_p(\mathbb{R}^n)$ and $\|\mathcal{M}_\psi(D_j f)\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)}$. Therefore, by Definition 5.3 and the choice of g, we know that $f \in \mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)$ and

$$\|f\|_{\mathscr{H}\dot{M}^q_{1,p}(\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{M}^q_p(\mathbb{R}^n)} \lesssim \|f\|_{H\dot{M}^q_p(\mathbb{R}^n)},$$

which completes the proof of Theorem 5.5. □

Remark 5.8. Observe that Theorem 5.5 generalizes [33, Theorem 1] by taking $p = q$.

Next we consider the corresponding result of Theorem 5.5 on spaces of homogeneous type, which were introduced by Coifman and Weiss in [10, 11]. Recall that a metric measure space $(\mathscr{X}, d, \mu)$ is called a *space of homogeneous type*, if there exists a constant $C_0 \in [1, \infty)$ such that, for all $x \in \mathscr{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \le C_0 \mu(B(x, r)) \quad \text{(doubling property)}. \tag{5.8}$$

We now recall the notions of test functions and the approximation of the identity on spaces of homogeneous type (see, for example, [22, 34]). In what follows, for any $x, y \in \mathscr{X}$ and $r \in (0, \infty)$, let $V(x, y) := \mu(B(x, d(x, y)))$ and $V_r(x) := \mu(B(x, r))$. It is easy to see that $V(x, y) \approx V(y, x)$ for all $x, y \in \mathscr{X}$.

Definition 5.9. Let $x_1 \in \mathscr{X}$, $r \in (0, \infty)$, $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. A function φ on $\mathscr{X}$ is said to be in the *space* $\mathcal{G}(x_1, r, \beta, \gamma)$, if there exists a nonnegative constant C such that

(i) $|\varphi(x)| \le C \frac{1}{V_r(x_1) + V(x_1, x)} [\frac{r}{r + d(x_1, x)}]^\gamma$ for all $x \in \mathscr{X}$;

(ii) $|\varphi(x) - \varphi(y)| \le C [\frac{d(x,y)}{r + d(x_1, x)}]^\beta \frac{1}{V_r(x_1) + V(x_1, x)} [\frac{r}{r + d(x_1, x)}]^\gamma$ for all $x, y \in \mathscr{X}$ satisfying that $d(x, y) \le [r + d(x_1, x)]/2$.

Moreover, for any $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$, its *norm* is defined by

$$\|\varphi\|_{\mathcal{G}(x_1, r, \beta, \gamma)} := \inf\{C : \text{ (i) and (ii) hold}\}.$$

Fix $x_1 \in \mathscr{X}$, let $\mathcal{G}(\beta, \gamma) := \mathcal{G}(x_1, 1, \beta, \gamma)$ and $\mathring{\mathcal{G}}(\beta, \gamma) := \{f \in \mathcal{G}(\beta, \gamma) : \int_{\mathscr{X}} f(x)\, d\mu(x) = 0\}$. Denote by $(\mathcal{G}(\beta, \gamma))'$ and $(\mathring{\mathcal{G}}(\beta, \gamma))'$, respectively, the dual spaces of $\mathcal{G}(\beta, \gamma)$ and $\mathring{\mathcal{G}}(\beta, \gamma)$. Obviously, by the Definition of $\mathring{\mathcal{G}}(\beta, \gamma)$, we see that $(\mathring{\mathcal{G}}(\beta, \gamma))' = (\mathcal{G}(\beta, \gamma))'/\mathbb{C}$. Let $\mathcal{A} := \{\mathcal{A}_k(x)\}_{x \in \mathscr{X}, k \in \mathbb{Z}}$ with

$$\mathcal{A}_k(x) := \{\phi \in \mathring{\mathcal{G}}(1, 2) : \|\phi\|_{\mathring{\mathcal{G}}(x, 2^{-k}, 1, 2)} \le 1\} \tag{5.9}$$

for all $x \in \mathscr{X}$ (see [34, Definition 5.2]).

Definition 5.10. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of bounded linear integral operators is called an *approximation of the identity of order 1* (for short, *1*-AOTI) with bounded support, if there exist positive constants C_3, C_4 such that, for all $k \in \mathbb{Z}$ and $x, x', y, y' \in \mathscr{X}$, $S_k(x, y)$, the integral kernel of S_k is a measurable function from $\mathscr{X} \times \mathscr{X}$ into $\mathbb{C}$ satisfying that

(i) $S_k(x,y)=0$ if $d(x,y)>C_4 2^{-k}$ and $|S_k(x,y)|\le C_3\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;

(ii) $|S_k(x,y)-S_k(x',y)|\le C_3 2^k d(x,x')\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$ if $d(x,x')\le \max\{C_4,1\}2^{1-k}$;

(iii) $|S_k(y,x)-S_k(y,x')|\le C_3 2^k d(x,x')\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$ if $d(x,x')\le \max\{C_4,1\}2^{1-k}$;

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|\le C_3 2^{2k}\frac{d(x,x')d(y,y')}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$ for $d(x,x')\le \max\{C_4,1\}2^{1-k}$ and $d(y,y')\le \max\{C_4,1\}2^{1-k}$;

(v) $\int_{\mathscr X} S_k(x,y)\,d\mu(y)=1=\int_{\mathscr X} S_k(x,y)\,d\mu(x)$.

It is known that there always exists an 1-AOTI with bounded support on a space of homogeneous type; see [22, Theorem 2.6].

Let $\mathscr X$ be a space of homogeneous type, $s\in(0,1]$, $p\in(0,\infty)$, $q\in[p,\infty]$ and $r\in(0,\infty]$. The *homogeneous grand Triebel-Lizorkin-Morrey space* $\mathcal A\dot F^s_{p,q,r}(\mathscr X)$ is defined to be the set of all $f\in(\mathcal G(1,2))'$ such that

$$\|f\|_{\mathcal A\dot F^s_{p,q,r}(\mathscr X)}:=\left\|\left\{\sum_{k\in\mathbb Z}2^{ksr}\sup_{\phi\in\mathcal A_k(\cdot)}|\langle f,\phi\rangle|^r\right\}^{1/r}\right\|_{\mathcal M^q_p(\mathscr X)}<\infty$$

with the usual modification made when $r=\infty$.

Theorem 5.11. *Let $\mathscr X$ be a space of homogeneous type. If $p\in(\frac{n}{n+1},\infty)$ and $q\in[p,\infty]$, then $\mathcal A\dot F^1_{p,q,\infty}(\mathscr X)=H\dot M^q_p(\mathscr X)$ with equivalent norms.*

To prove this theorem, we need the following partition of unity for $\mathscr X$, which is an immediate consequence of Christ's dyadic cube decomposition for spaces of homogeneous type in [9, Theorem 11], and the partition of unity for open sets with finite measure constructed by Macías and Segovia in [36, Lemmas (2.9) and (2.16)].

Lemma 5.12. *Let $\mathscr X$ be a space of homogeneous type. Then there exist a sequence $\{B_j\}_j$ of open balls with the finite intersection property and a sequence $\{\phi_j\}_j$ of non-negative functions in $\mathcal G(1,2)$ such that $\mu(\mathscr X\setminus(\cup_j B_j))=0$, $\operatorname{supp}\phi_j\subset B_j$ for all j and $\sum_j\phi_j(x)=1$ for almost every $x\in\mathscr X$.*

Proof. By Christ's dyadic cube decomposition in [9, Theorem 11], there exists a sequence $\{Q_i\}_{i\in\mathbb N}$ of open sets such that $\mu(\mathscr X\setminus(\cup_i Q_i))=0$ and $Q_i\cap Q_k=\emptyset$ if $i\ne k$.

For each Q_i, applying Macías-Segovia's partition of unity for open sets with finite measure (see [36, Lemmas (2.9) and (2.16)]), we know that there exist a sequence $\{B^i_j\}_j$ of open balls with the finite intersection property and a sequence $\{\phi^i_j\}_j$ of functions in $\mathcal G(1,2)$ such that $\cup_j B^i_j=Q_i$ for all i, $\operatorname{supp}\phi^i_j\subset B^i_j$ for all i,j, and $\sum_j\phi^i_j=\chi_{Q_i}$ for all i. Then $\mu(\mathscr X\setminus(\cup_{i,j}B^i_j))=0$, $\operatorname{supp}\phi^i_j\subset B^i_j$ for all i, j, and $\sum_{i,j}\phi^i_j(x)=1$ for almost every $x\in\mathscr X$. This finishes the proof of Lemma 5.12. $\square$

We also need the following two technical lemmas.

Lemma 5.13. *Let $\mathscr{X}$ be a space of homogeneous type, $x, y \in \mathscr{X}$ and $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} < d(x,y) \le 2^{-k_0}$. Assume that $k \le k_0$. Then there exists a positive constant $\widetilde{C}$, independent of k_0, k, x and y, such that*

$$\widetilde{C} 2^{-k}[d(x,y)]^{-1}[S_k(x,\cdot) - S_k(y,\cdot)] \in \mathcal{A}_k(x),$$

where $\mathcal{A}_k(x)$ is as in (5.9).

Proof. For $x,\ y \in \mathscr{X}$ and $k \in \mathbb{Z}$ as in Lemma 5.13, let

$$\phi_k^{(x,y)}(z) := 2^{-k}[d(x,y)]^{-1}[S_k(x,z) - S_k(y,z)] \text{ for all } z \in \mathscr{X}.$$

By Definition 5.10(v), it is easy to see that

$$\int_{\mathscr{X}} \phi_k^{(x,y)}(z)\, d\mu(z) = 0.$$

We now prove that $\phi_k^{(x,y)}$ satisfies Definition 5.9(i) with $\gamma = 2$ and $r = 2^{-k}$, namely, for all $z \in \mathscr{X}$,

$$\left|\phi_k^{(x,y)}(z)\right| \lesssim \frac{1}{V_{2^{-k}}(x) + V(x,z)} \left[\frac{2^{-k}}{2^{-k} + d(x,z)}\right]^2. \tag{5.10}$$

We establish this estimate by considering the following two cases.

Case 1) $d(x,z) > C_4 2^{-k}$ and $d(y,z) > C_4 2^{-k}$. In this case, (5.10) automatically holds true by Definition 5.10(i).

Case 2) $d(x,z) \le C_4 2^{-k}$ or $d(y,z) \le C_4 2^{-k}$. In this case, by $k \le k_0$ and hence

$$d(x,y) \le 2^{-k_0} \le 2^{-k} < \max\{C_4, 1\} 2^{1-k}, \tag{5.11}$$

together with Definition 5.10(ii), we see that

$$|S_k(x,z) - S_k(y,z)| \lesssim 2^k d(x,y) \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(z)}. \tag{5.12}$$

When $d(x,z) \le C_4 2^{-k}$, we see that

$$\frac{2^{-k}}{2^{-k} + d(x,z)} \approx 1, \quad \text{and} \quad V_{2^{-k}}(x) + V_{2^{-k}}(z) \approx V_{2^{-k}}(x) + V(x,z) \text{ (by (5.8))}. \tag{5.13}$$

When $d(y,z) \le C_4 2^{-k}$, by $k \le k_0$, we know that

$$d(x,z) \le d(x,y) + d(y,z) \le 2^{-k_0} + C_4 2^{-k} \le (1 + C_4) 2^{-k}.$$

Thus, in this case, (5.13) also holds true. By (5.13) and (5.12), we then obtain (5.10).

We turn to proving that $\phi_k^{(x,y)}$ satisfies Definition 5.9(ii) with $\beta = 1$, $\gamma = 2$ and $r = 2^{-k}$, that is, for all $z, w \in \mathscr{X}$ with $d(z,w) \le \frac{2^{-k} + d(x,z)}{2}$,

$$|\phi_k^{(x,y)}(z) - \phi_k^{(x,y)}(w)| \lesssim \frac{d(z,w)}{2^{-k} + d(x,z)} \frac{1}{V_{2^{-k}}(x) + V(x,z)} \left[\frac{2^{-k}}{2^{-k} + d(x,z)}\right]^2. \tag{5.14}$$

Write

$$\mathrm{I} := |\phi_k^{(x,y)}(z) - \phi_k^{(x,y)}(w)| = 2^{-k}[d(x,y)]^{-1}|[S_k(x,z) - S_k(x,w)] - [S_k(y,z) - S_k(y,w)]|.$$

We establish the estimate by considering the following three cases.

Case a) $d(x,z) \le \max\{C_4, 1\}2^{1-k}$. In this case, we see that (5.13) holds true and $d(z,w) \le \frac{2^{-k}+d(x,z)}{2} \le \max\{C_4,1\}2^{1-k}$. Thus, by the last estimate, (5.11) and Definition 5.10(iv), we see that

$$\mathrm{I} \lesssim \frac{2^k d(z,w)}{V_{2^{-k}}(x) + V_{2^{-k}}(z)}. \tag{5.15}$$

From this and (5.13), we further deduce (5.14) in this case.

Case b) $d(x,z) > \max\{C_4, 1\}2^{2-k}$. In this case, by Definition 5.10(i), we know that $S_k(x,z) = 0$. Observing that

$$d(y,z) \ge d(x,z) - d(x,y) > \max\{C_4,1\}2^{2-k} - 2^{-k} > C_4 2^{-k},$$

by Definition 5.10(i) again, we also have $S_k(y,z) = 0$.

If $d(z,w) \le \frac{1}{4}d(x,z)$, then

$$d(x,w) \ge d(x,z) - d(z,w) \ge \frac{3}{4}d(x,z) > C_4 2^{-k} \tag{5.16}$$

and

$$d(y,w) \ge d(x,z) - d(x,y) - d(z,w) \ge \frac{3}{4}d(x,z) - d(x,y) > C_4 2^{-k}. \tag{5.17}$$

Thus, $S_k(x,w) = S_k(y,w) = 0$, and hence $\mathrm{I} = 0$ in this case.

Recall that, since $d(x,y) \le 2^{-k_0} \le 2^{-k}$, we have $2^{-k} + d(y,w) \approx 2^{-k} + d(x,w)$ and $V_{2^{-k}}(y) + V(y,w) \approx V_{2^{-k}}(x) + V(x,w)$. By this, together with $S_k(y,z) = S_k(x,z) = 0$ and Definition 5.10(i), we see that, if $d(z,w) > \frac{1}{4}d(x,z)$, then

$$\mathrm{I} = 2^{-k}[d(x,y)]^{-1}|S_k(x,w) - S_k(y,w)| \lesssim \frac{1}{V_{2^{-k}}(x) + V(x,w)}\left[\frac{2^{-k}}{2^{-k} + d(x,w)}\right]^2. \tag{5.18}$$

Observing that, in the present case, it holds true that $\frac{1}{4}d(x,z) < d(z,w) \le \frac{2^{-k}+d(x,z)}{2}$ and $d(x,z) \ge 2^{-k}$, which implies that

$$2^{-k} + d(x,w) \approx 2^{-k} + d(x,z). \tag{5.19}$$

From these estimates, we further deduce that

$$V_{2^{-k}}(x) + V(x,w) \approx V_{2^{-k}}(x) + V(x,z) \tag{5.20}$$

and

$$1 \lesssim \frac{d(z,w)}{d(x,z)} \lesssim \frac{d(z,w)}{2^{-k} + d(x,z)}. \tag{5.21}$$

Combining (5.18), (5.19), (5.20) and (5.21), we then obtain (5.14).

Case c) $\max\{C_4,1\}2^{1-k} < d(x,z) \le \max\{C_4,1\}2^{2-k}$. In this case, we further conclude that $S_k(x,z)=0$ and

$$d(z,w) \le \frac{2^{-k}+d(x,z)}{2} < \max\{C_4,1\}2^{2-k}. \tag{5.22}$$

If $d(z,w) \le \max\{C_4,1\}2^{1-k}$, the proof is same as that in Case a). If $\max\{C_4,1\}2^{1-k} < d(z,w) \le \max\{C_4,1\}2^{2-k}$, then $d(z,w) \approx d(x,z) \approx 2^{-k}$ and $d(y,z) \ge d(x,z)-d(x,y) > \max\{C_4,1\}2^{1-k}-2^{-k_0} > C_4 2^{-k}$. By this and Definition 5.10(i), we see that $S_k(y,z)=0$ and, in this case, $\mathrm{I} = 2^{-k}[d(x,y)]^{-1}|S_k(x,w)-S_k(y,w)|$. Then, repeating the proof of Case b), we also obtain (5.14). This finishes the proof of Lemma 5.13. □

Lemma 5.14. *Let $\mathscr{X}$ be a space of homogeneous type, $x\in\mathscr{X}$ and $k\in\mathbb{Z}$. Then, there exists a positive constant $\widetilde{C}$, independent of k and x, such that $\widetilde{C}[S_{k+1}(x,\cdot)-S_k(x,\cdot)] \in \mathcal{A}_k(x)$.*

Proof. For any $x\in\mathscr{X}$ and $k\in\mathbb{Z}$, let $\phi_k^x(y) := |S_{k+1}(x,y)-S_k(x,y)|$ for all $y\in\mathscr{X}$. Then, by Definition 5.10(v), we know that $\int_{\mathscr{X}}\phi_k^x(y)\,d\mu(y)=0$. Write

$$\mathrm{I}_1 := |S_{k+1}(x,y)-S_k(x,y)|$$

and

$$\mathrm{I}_2 := |[S_{k+1}(x,y)-S_k(x,y)]-[S_{k+1}(x,z)-S_k(x,z)]|.$$

It suffices to show that

$$\mathrm{I}_1\,[V_{2^{-k}}(x)+V(x,y)]\left[\frac{d(x,y)+2^{-k}}{2^{-k}}\right]^2 \lesssim 1 \tag{5.23}$$

and, for any $d(y,z) \le \frac{2^{-k}+d(x,y)}{2}$,

$$\mathrm{I}_2\frac{\left[2^{-k}+d(x,y)\right]^3[V_{2^{-k}}(x)+V(x,y)]}{2^{-2k}d(y,z)} \lesssim 1. \tag{5.24}$$

To prove (5.23), we consider the following two cases.

Case 1) $d(x,y) > C_4 2^{-k}$. In this case, from Definition 5.10(i), we deduce that $\mathrm{I}_1=0$ and hence (5.23) holds true.

Case 2) $d(x,y) \le C_4 2^{-k}$. In this case, by Definition 5.10(i), we know that

$$\begin{aligned}\mathrm{I}_1\,[V_{2^{-k}}(x)+V(x,y)]\left[\frac{d(x,y)+2^{-k}}{2^{-k}}\right]^2 \\ \lesssim \frac{V_{2^{-k}}(x)+V(x,y)}{V_{2^{-k-1}}(x)+V_{2^{-k-1}}(y)}+\frac{V_{2^{-k}}(x)+V(x,y)}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}.\end{aligned} \tag{5.25}$$

From (5.8), it follows that

$$V(x,y)\lesssim V_{2^{-k}}(y),\quad V_{2^{-k}}(x)\lesssim V_{2^{-k-1}}(x) \text{ and } V_{2^{-k}}(y)\lesssim V_{2^{-k-1}}(y). \tag{5.26}$$

Combining (5.25) and (5.26), we then obtain (5.23).

To prove (5.24), we let $d(y,z) \le \frac{2^{-k}+d(x,y)}{2}$ and consider the following three cases.

Case a) $d(y,z) \le \max\{C_4, 1\}2^{-k}$. In this case, by (5.26) and Definition 5.10(iii), we see that

$$\text{I}_2 \le |S_{k+1}(x,y) - S_{k+1}(x,z)| + |S_k(x,y) - S_k(x,z)| \lesssim \frac{2^k d(y,z)}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}. \tag{5.27}$$

If $d(x,y) \le C_4 2^{-k}$, then (5.27) implies (5.24). If $d(x,y) > C_4 2^{-k}$ and $d(x,z) > C_4 2^{-k}$, then (5.24) is trivially true, since, by Definition 5.10(i), $\text{I}_2 = 0$ in this subcase. If $d(x,y) > C_4 2^{-k}$ and $d(x,z) \le C_4 2^{-k}$, then $d(x,y) \lesssim 2^{-k}$ since $d(y,z) \le \frac{2^{-k}+d(x,y)}{2}$ and $d(x,y) \le d(x,z) + d(y,z)$; thus, in this subcase, (5.24) also holds true.

Case b) $\max\{C_4, 1\}2^{-k} < d(y,z) \le \max\{C_4, 1\}2^{-k+1}$. In this case, by (5.8), we see that

$$V_{2^{-k-1}}(x) + V(x,y) \approx V_{2^{-k-1}}(x) + V(x,z) \approx V_{2^{-k}}(x) + V(x,y) \approx V_{2^{-k}}(x) + V(x,z).$$

From this and Definition 5.10(i), we further deduce that

$$\text{I}_2 \lesssim \frac{1}{V_{2^{-k}}(x) + V(x,y)}. \tag{5.28}$$

If $d(x,y) \le C_4 2^{-k}$ or $d(x,z) \le C_4 2^{-k}$, then, by $d(y,z) \lesssim 2^{-k}$, we always have $d(x,y) \lesssim 2^{-k}$, which further implies that $2^{-k} + d(x,y) \approx 1$; by this and (5.28), we conclude that (5.24) holds true. If $d(x,y) > C_4 2^{-k}$ and $d(x,z) > C_4 2^{-k}$, then, in this subcase, $\text{I}_2 = 0$ and (5.24) is trivially true. Thus, (5.24) always holds true in Case b).

Case c) $d(y,z) > \max\{C_4, 1\}2^{-k+1}$. From this assumption and $d(y,z) \le \frac{2^{-k}+d(x,y)}{2}$, We further deduce that

$$d(x,y) \ge 2d(y,z) - 2^{-k} > \max\{C_4, 1\}2^{-k+2} - 2^{-k} > C_4 2^{-k+1}. \tag{5.29}$$

If $d(x,z) > C_4 2^{-k}$, then, in this subcase, $\text{I}_2 = 0$ and (5.24) is trivially true. If $d(x,z) \le C_4 2^{-k}$, then $d(x,y) \le d(x,z) + d(y,z) \le C_4 2^{-k} + \frac{2^{-k}+d(x,y)}{2}$ and hence $d(x,y) \lesssim 2^{-k}$, which, together with (5.29), (5.8) and $d(y,z) \le \frac{2^{-k}+d(x,y)}{2}$, implies that $d(x,y) \approx 2^{-k}$, $V(x,y) \lesssim V_{2^{-k}}(x)$, $d(y,z) \lesssim 2^{-k}$ and hence $d(y,z) \approx 2^{-k}$; from these estimates and Definition 5.10(i), we finally deduce that

$$\text{I}_2 \lesssim \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(z)} \lesssim \frac{1}{V_{2^{-k}}(x) + V(x,y)} \lesssim \frac{2^{-2k} d(y,z)}{[2^{-k} + d(x,y)]^3 \, [V_{2^{-k}}(x) + V(x,y)]}.$$

This finishes the proof of (5.24) and hence Lemma 5.14. □

Proof of Theorem 5.11. The proof of $H\dot{M}^q_p(\mathscr{X}) \subset \mathcal{A}\dot{F}^1_{p,q,\infty}(\mathscr{X})$ is similar to that of [34, Theorem 1.1], the details being omitted.

We now prove the converse inequality, namely, $\mathcal{A}\dot{F}^1_{p,q,\infty}(\mathscr{X}) \subset H\dot{M}^q_p(\mathscr{X})$. Let $f \in \mathcal{A}\dot{F}^1_{p,q,\infty}(\mathscr{X})$ and $\{S_k\}_{k\in\mathbb{Z}}$ be a 1-AOTI with bounded support. We first assume that f is a locally integrable function. In this case, applying [22, Proposition 2.7], we know that, for almost every $x \in \mathscr{X}$,

$$\lim_{k\to\infty} S_k(f)(x) = f(x),$$

from which we further deduce that, for almost every $x, y \in \mathscr{X}$,

$$(5.30)\quad |f(x) - f(y)| \le |S_{k_0}(f)(x) - S_{k_0}(f)(y)| + \sum_{k\ge k_0} \left[|S_{k+1}(f)(x) - S_k(f)(x)| + |S_{k+1}(f)(y) - S_k(f)(y)|\right],$$

where $k_0 \in \mathbb{Z}$ such that $2^{-k_0} < d(x,y) \le 2^{-k_0}$. Let $\phi_{k_0}^{(x,y)}(z) := S_{k_0}(x,z) - S_{k_0}(y,z)$ and $\phi_k^x(y) := S_{k+1}(x,y) - S_k(x,y)$. Then, by Lemmas 5.13 and 5.14, there exists a positive constant $\widetilde{C}$ such that $\widetilde{C}\phi_{k_0}^{(x,y)} \in \mathcal{A}_{k_0}(x)$ and $\widetilde{C}\phi_k^x \in \mathcal{A}_k(x)$. For every $x \in \mathscr{X}$, let

$$(5.31)\qquad g(x) := \sup_{k\in\mathbb{Z}} \sup_{\phi\in\mathcal{A}_k(x)} 2^k |\langle f, \phi\rangle|,$$

where $\mathcal{A}_k(x)$ is as in (5.9). Then, by $f \in \mathcal{A}\dot{F}^1_{p,q,\infty}(\mathscr{X})$, we know that $g \in \mathcal{M}^q_p(\mathscr{X})$ and, by (5.30), (5.31) and the choice of k_0, we conclude that

$$(5.32)\qquad \begin{aligned} |f(x) - f(y)| &\lesssim \sup_{\phi\in\mathcal{A}_{k_0}(x)} |\langle f, \phi\rangle| + \sum_{k\ge k_0} \left[\sup_{\phi\in\mathcal{A}_k(x)} |\langle f, \phi\rangle| + \sup_{\phi\in\mathcal{A}_k(y)} |\langle f, \phi\rangle|\right] \\ &\lesssim \sum_{k\ge k_0} 2^{-k} [g(x) + g(y)] \lesssim d(x,y)[g(x) + g(y)]. \end{aligned}$$

Therefore, from Definition 5.4 and $g \in \mathcal{M}^q_p(\mathscr{X})$, we deduce that $f \in H\dot{M}^q_p(\mathscr{X})$ and

$$\|f\|_{H\dot{M}^q_p(\mathscr{X})} \lesssim \|g\|_{\mathcal{M}^q_p(\mathscr{X})} \approx \|f\|_{\mathcal{A}\dot{F}^1_{p,q,\infty}(\mathscr{X})}.$$

To complete the proof, we only need to show that, for every $f \in \mathcal{A}\dot{F}^1_{p,q,\infty}(\mathscr{X})$, there exists a locally integrable function $\widetilde{f}$ which coincides with f in $(\mathcal{G}(1,2))'$. In the case that $p \in (1,\infty)$, the proof is similar to that of [34, Theorem 1.1], the details being omitted. We now assume that $p \in (n/(n+1), 1]$. Let $x,\ y \in \mathscr{X}$. We pick $k_0 \in \mathbb{Z}$ such that $2^{-k_0-1} < d(x,y) \le 2^{-k_0}$. If $k > k_0$, then, by the same reason as that used in the proof of (5.32), we see that

$$(5.33)\qquad \begin{aligned} &|S_k(f)(x) - S_k(f)(y)| \\ &\quad\lesssim \sum_{j=k_0}^{k-1} \left[|S_{j+1}(f)(x) - S_j(f)(x)| + |S_{j+1}(f)(y) - S_j(f)(y)|\right] \\ &\qquad + |S_{k_0}(f)(x) - S_{k_0}(f)(y)| \\ &\quad\lesssim \sum_{j=k_0}^{\infty} 2^{-j}[g(x) + g(y)] + 2^{-k_0} g(x) \lesssim 2^{-k_0}[g(x) + g(y)] \end{aligned}$$

$$\approx d(x,y)[g(x)+g(y)].$$

If $k\le k_0$, then by Lemma 5.13 again, we know that there exists a positive constant $\widetilde{C}$ such that

$$\widetilde{C}2^{-k}[d(x,y)]^{-1}[S_k(x,\cdot)-S_k(y,\cdot)]\in\mathcal{A}_k(x).$$

From this, we deduce that

$$2^{-k}[d(x,y)]^{-1}\int_{\mathscr{X}}[S_k(x,z)-S_k(y,z)]f(z)\,d\mu(z)\lesssim 2^{-k}g(x)$$

and hence

$$\begin{aligned}(5.34)\qquad &|S_k(f)(x)-S_k(f)(y)|\\ &=2^kd(x,y)\left\{2^{-k}[d(x,y)]^{-1}\left|\int_{\mathscr{X}}[S_k(x,z)-S_k(y,z)]f(z)\,d\mu(z)\right|\right\}\\ &\lesssim d(x,y)g(x)\lesssim d(x,y)[g(x)+g(y)].\end{aligned}$$

Thus, by (5.33) and (5.34), we find that g is a Hajłasz gradient of $S_k(f)$ for all $k\in\mathbb{Z}$.

Fix a bounded set $B\subset\mathscr{X}$. By [34, Lemma 4.1], we know that, for any $k\in\mathbb{Z}$, there exists $C_k\in\mathbb{R}$ such that $S_k(f)-C_k\in L^{p_*}(B)$ with

$$(5.35)\qquad \left[\frac{1}{\mu(B)}\int_B|S_k(f)(x)-C_k|^{p_*}\,d\mu(x)\right]^{1/p_*}\lesssim r_B\left\{\frac{1}{\mu(2B)}\int_{2B}[g(x)]^p\,d\mu(x)\right\}^{1/p},$$

where $p_*:=\frac{np}{n-p}>1$, since $p>\frac{n}{n+1}$. From the weak compactness of $L^{p_*}(B)$, it follows that $\{S_k(f)-C_k\}_{k\in\mathbb{Z}}$ has a subsequence, denoted by $\{S_k(f)-C_k\}_{k\in\mathbb{Z}}$ again without loss of generality, which converges weakly in $L^{p_*}(B)$ and hence almost everywhere in B to a certain function $\widetilde{f}^B\in L^{p_*}(B)$. Moreover, notice that, for all $x\in\mathscr{X}$, $k\in\mathbb{Z}$ and $i\in\mathbb{N}$,

$$(5.36)\qquad |S_k(f)(x)-S_{k+i}(f)(x)|\le\sum_{j=0}^{i-1}|S_{k+j}(f)(x)-S_{k+j+1}(f)(x)|\lesssim 2^{-k}g(x).$$

Thus, by $g\in\mathcal{M}_p^q(\mathscr{X})$, we see that, for all $k,\ k'\in\mathbb{Z}$, $S_k(f)-S_{k'}(f)\in\mathcal{M}_p^q(\mathscr{X})$. Since B is a bounded subset of $\mathscr{X}$, by $p^*>p$ and the Hölder inequality, we further know that $S_k(f)-S_{k'}(f)\in L^p(B)$. On the other hand, by the Hölder inequality, (5.36) and (5.35), we deduce that, for all $k,\ k'\in\mathbb{Z}_+$,

$$\begin{aligned}|C_k-C_{k'}|&=\frac{1}{\mu(B)}\int_B|C_k-C_{k'}|\,d\mu(x)\\ &\le\frac{1}{\mu(B)}\int_B|S_k(f)(x)-C_k-S_{k'}(f)(x)+C_{k'}|\,d\mu(x)\\ &\quad+\frac{1}{\mu(B)}\int_B|S_k(f)(x)-S_{k'}(f)(x)|\,d\mu(x)\\ &\le\frac{1}{[\mu(B)]^{1/p_*}}\left[\|S_k(f)-C_k\|_{L^{p_*}(B)}+\|S_{k'}(f)-C_{k'}\|_{L^{p_*}(B)}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{[\mu(B)]^{1/p}}\|S_k(f)-S_{k'}(f)\|_{L^p(B)}\\
&\le r_B\left\{\frac{1}{\mu(2B)}\int_{2B}[g(x)]^p\,d\mu(x)\right\}^{1/p}+2^{-\min(k,k')}\frac{1}{[\mu(B)]^{1/p}}\|g\|_{L^p(B)}\\
&\lesssim (r_B+1)[\mu(B)]^{1/q}\|g\|_{\mathcal{M}_p^q(\mathscr{X})}.
\end{aligned}
$$

Thus, $|C_k-C_{k'}|$ is dominated by a positive number depending on B but not on $k,\ k'\in\mathbb{Z}_+$. This observation implies that we can choose a subsequence $\{C_{k_j}\}_{j\in\mathbb{N}}$ which converges to a positive constant $\widetilde{C}^B$, which depends on B. Since $\{S_{k_j}(f)-C_{k_j}\}_{j\in\mathbb{N}}$ converges almost everywhere in B to $\widetilde{f}^B$, and $S_k(f)$ converges to f almost everywhere, it follows that f coincides with $\widetilde{f}^B+\widetilde{C}^B$ almost everywhere in B, and hence in $(\mathcal{G}(1,2)\cap\Phi(B))'$, where $\Phi(B)$ is the set of all functions on $\mathscr{X}$ with supports in B. Write $f^B:=\widetilde{f}^B+\widetilde{C}^B$. Then $f=f^B$ in $(\mathcal{G}(1,2)\cap\Phi(B))'$ and, by $\widetilde{f}^B\in L^{p_*}(B)$, f^B is also a locally integrable function.

We still need to show that there exists $\widetilde{f}\in L^1_{\rm loc}(\mathscr{X})$ such that $\langle f,\psi\rangle=\langle\widetilde{f},\psi\rangle$ for all $\psi\in\mathcal{G}(1,2)$. To this end, by Lemma 5.12, choose a partition of unity, $\{\phi_j\}_j\subset\mathcal{G}(1,2)$, on $\mathscr{X}$ and a sequence $\{B_j\}_j$ of open balls, with the finite intersection property, such that $\mu(\mathscr{X}\setminus(\cup_jB_j))=0$, $\operatorname{supp}\phi_j\subset B_j$, ϕ_j is non-negative and $\sum_j\phi_j(x)=1$ for almost every $x\in\mathscr{X}$.

Let f^{B_j} be the locally integrable representation of f in $(\mathcal{G}(1,2)\cap\Phi(B_j))'$ and define $\widetilde{f}:=f^{B_j}$ pointwise on B_j for all j. Notice that, by the construction of f^{B_j}, for almost every $x\in B_i\cap B_j$, $f^{B_i}(x)=f(x)=f^{B_j}(x)$. Thus, $\widetilde{f}$ is well defined. Moreover, it is easy to see that, for all $\psi\in\mathcal{G}(1,2)$, $\sum_j\psi\phi_j$ converges in $\mathcal{G}(1,2)$. Indeed, we may assume that the cardinality of $\{\phi_j\}_j$ is infinite and indicated by $\mathbb{N}$; otherwise, it is trivially true. For any $\varepsilon\in(0,\infty)$, by the construction of $\{B_j\}_{j\in\mathbb{N}}$ and $\phi_j\in\mathcal{G}(1,2)$, especially, the finite intersection property, we know that there exists $L\in\mathbb{N}$ such that $B_j\cap B(x_1,1/\varepsilon)=\emptyset$ for all $j\ge L$. For $x\in B(x_1,1/\varepsilon)$, $\sum_{j\in\mathbb{Z}}|\psi(x)\phi_j(x)|=0$; for $x\notin B(x_1,1/\varepsilon)$, by $\sum_{j\ge L}\phi_j\le1$ and $\psi\in\mathcal{G}(1,2)$, we see that

$$
\begin{aligned}
\sum_{j\ge L}|\psi(x)\phi_j(x)|&\lesssim\frac{1}{[V_1(x_1)+V(x_1,x)]^2}\left(\frac{1}{1+d(x_1,x)}\right)^4\\
&\lesssim\frac{1}{V_1(x_1)}\frac{1}{V_1(x_1)+V(x_1,x)}\left(\frac{1}{1+\frac{1}{\varepsilon}}\right)^2\left[\frac{1}{1+d(x_1,x)}\right]^2\\
&\lesssim\varepsilon^2\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^2.
\end{aligned}
$$

Therefore, for all $x\in\mathscr{X}$, we have

$$
\sum_{j\ge L}|\psi(x)\phi_j(x)|\lesssim\varepsilon^2\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^2.
$$

Moreover, for all $x,y\in\mathscr{X}$ satisfying that $d(x,y)\le(1+d(x_1,x))/2$, it holds true that

$$
\sum_{j\ge L}|\psi(x)\phi_j(x)-\psi(y)\phi_j(y)|\lesssim\varepsilon^2\left[\frac{d(x,y)}{1+d(x_1,x)}\right]\frac{1}{V_1(x_1)+V(x_1,x)}\left[\frac{1}{1+d(x_1,x)}\right]^2.
$$

Indeed, if $x,\ y \in B(x_1, 1/\varepsilon)$, then $\sum_{j\geq L} |\psi(x)\phi_j(x) - \psi(y)\phi_j(y)| = 0$; if $y \in B(x_1, 1/\varepsilon)$ but $x \notin B(x_1, 1/\varepsilon)$, then $\phi_j(y) = 0$ for all $j \geq L$, and hence

$$\begin{aligned}\sum_{j\geq L} |\psi(x)\phi_j(x) - \psi(y)\phi_j(y)| &= \sum_{j\geq L} |\psi(x)\phi_j(x) - \psi(x)\phi_j(y)| = \sum_{j\geq L} |\psi(x)||\phi_j(x) - \phi_j(y)| \\ &\lesssim \frac{d(x,y)}{1+d(x_1,x)} \frac{1}{[V_1(x_1)+V(x_1,x)]^2} \left[\frac{1}{1+d(x_1,x)}\right]^4 \\ &\lesssim \varepsilon^2 \frac{d(x,y)}{1+d(x_1,x)} \frac{1}{V_1(x_1)+V(x_1,x)} \left(\frac{1}{1+d(x_1,x)}\right)^2;\end{aligned}$$

if $x, y \notin B(x_1, 1/\varepsilon)$, by the finite intersection property of $\{B_j\}_{j\in\mathbb{N}}$ and an argument similar to that used in the above, we also have the desired inequality. Thus, $\sum_{j\in\mathbb{N}} \psi\phi_j$ converges in $\mathcal{G}(1,2)$, and hence

$$\langle f, \psi \rangle = \left\langle f, \psi \sum_{j\in\mathbb{N}} \phi_j \right\rangle = \sum_{j\in\mathbb{N}} \langle f, \psi\phi_j \rangle = \sum_{j\in\mathbb{N}} \langle f^{B_j}, \psi\phi_j \rangle = \left\langle \widetilde{f}, \sum_{j=1}^{\infty} \psi\phi_j \right\rangle = \langle \widetilde{f}, \psi \rangle.$$

This finishes the proof of Theorem 5.11. □

We remark that, Theorem 5.11, when $p = q$, goes back to [34, Theorem 5.2] in the case $s = 1$ therein. Moreover, the proof of Theorem 5.11 is different from that of [34, Theorem 5.2]. Indeed, comparing with the proof of [34, Theorem 5.2], the proof of Theorem 5.11 needs several localized arguments, in which a partition of unity on spaces of homogeneous type plays a key role (see Lemma 5.12).

To end this section, we consider the case of inhomogeneous spaces.

Definition 5.15. Let $0 < p \leq q \leq \infty$. The *inhomogeneous Hardy-Morrey-Sobolev space* $\mathscr{H}M^q_{1,p}(\mathbb{R}^n)$ is defined to be the set of functions $f \in \mathcal{M}^q_p(\mathbb{R}^n)$ such that $D_j f \in \mathscr{H}M^q_p(\mathbb{R}^n)$ for all $j \in \{1, \ldots, n\}$, where $D_j f$ denotes the *jth distributional derivative of f*. Moreover,

$$\|f\|_{\mathscr{H}M^q_{1,p}(\mathbb{R}^n)} := \|f\|_{\mathcal{M}^q_p(\mathbb{R}^n)} + \sum_{j=1}^{n} \|D_j f\|_{\mathscr{H}M^q_p(\mathbb{R}^n)}.$$

The inhomogeneous grand Triebel-Lizorkin Morrey space is defined as follows.

Definition 5.16. Let $\mathscr{X}$ be a space of homogeneous type, $s \in (0,1]$, $p \in (0,\infty)$, $q \in [p,\infty]$ and $r \in (0,\infty]$. The *inhomogeneous grand Triebel-Lizorkin-Morrey space* $\mathcal{A}F^s_{p,q,r}(\mathscr{X})$ is defined to be the set of all $f \in (\mathcal{G}(1,2))'$ such that

$$\|f\|_{\mathcal{A}F^s_{p,q,r}(\mathscr{X})} := \left\| \left\{ \sum_{k\in\mathbb{Z}_+} 2^{ksr} \sup_{\phi\in\mathcal{A}_k(\cdot)} |\langle f, \phi \rangle|^r \right\}^{1/r} \right\|_{\mathcal{M}^q_p(\mathscr{X})} < \infty$$

with the usual modification made when $r = \infty$.

Similar to the proofs of Theorems 5.5 and 5.11, respectively, we have the following results, the details being omitted.

Theorem 5.17. *Let* $p \in (\frac{n}{n+1}, \infty]$ *and* $q \in [p, \infty]$. *Then,* $\mathscr{H}M^q_{1,p}(\mathbb{R}^n) = HM^q_p(\mathbb{R}^n)$ *with equivalent norms.*

We remark that, when $p \in (1, \infty]$ and $q \in [p, \infty]$, Theorem 5.17 goes back to Theorem 4.8, since, in this case, $\mathscr{H}M^q_{1,p}(\mathbb{R}^n) = WM^q_p(\mathbb{R}^n)$ (see Remark 5.2). Also, Theorem 5.17 completely covers [33, Corollary 10] by taking $p = q$.

Theorem 5.18. *Let* $\mathscr{X}$ *be a space of homogeneous type,* $p \in (\frac{n}{n+1}, \infty)$ *and* $q \in [p, \infty]$. *Then,* $\mathcal{A}F^1_{p,q,\infty}(\mathscr{X}) = HM^q_p(\mathscr{X})$ *with equivalent norms.*

Observe that, when $p = q$, Theorem 5.18 coincide with [34, Theorem 5.2].

6 Boundedness of (fractional) maximal operators

This section is devoted to the boundedness of (fractional) maximal operators on Morrey type spaces over metric measure spaces.

In Subsection 6.1, for a geometrically doubling metric measure space $(\mathscr{X}, d, \mu)$ in the sense of Hytönen [29], we show, in Theorem 6.5 below, that the modified maximal operator $M^{(\beta)}_0$ (see (6.1) below) is bounded on the modified Morrey space $\mathcal{M}^{q,(k)}_p(\mathscr{X})$, which, when $(\mathscr{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mu)$ with μ being a Radon measure satisfying the polynomial growth condition (also called the non-doubling measure), was introduced by Sawano and Tanaka [45]. As an application, the boundedness of the fractional maximal operator $M^{(\beta)}_\alpha$ on this space is also obtained in Proposition 6.6 below.

In Subsection 6.2, if μ is a doubling measure, as applications of Theorem 6.5 and Proposition 6.6, we show the boundedness of the fractional maximal operator $\widetilde{M}_\alpha$ on Morrey spaces (see Corollary 6.7 below), from which, we further deduce, in Corollary 6.8 below, the boundedness of the fractional maximal operator $\widetilde{M}_\alpha$ on Morrey spaces when μ further satisfies the measure lower bound condition (see (6.3) below). If μ is doubling, satisfies (6.3) and has the relative 1-annular decay property (see (6.6) below), we then obtain the boundedness of $\widetilde{M}_\alpha$ on $HM^q_p(\mathscr{X})$ (see Theorem 6.9 below). Finally, we prove that, if μ is doubling and satisfies (6.3), and $\mathscr{X}$ supports a weak $(1,p)$-Poincaré inequality, then the discrete fractional maximal function M^*_α is bounded on $NM^q_p(\mathscr{X})$ (see Theorem 6.10 below).

6.1 Maximal operators on $\mathcal{M}^{q,(k)}_p(\mathscr{X})$

In 2010, Hytönen [29] introduced the notion of geometrically doubling metric measures which include both spaces of homogeneous type and the Euclidean spaces with non-doubling measures satisfying the polynomial growth condition; see also the monograph [53] for some recent developments of this subject.

A key notion appeared in [29] is the following geometrically doubling.

Definition 6.1. A metric measure space $(\mathscr{X}, d, \mu)$ is said to be *geometrically doubling*, if there exists $N_0 \in \mathbb{N}$ such that any given ball contains no more than N_0 points at distance exceeding half its radius.

From the geometrically doubling property, we deduce the following conclusion, which is used later on.

Proposition 6.2. *Let $(\mathscr{X}, d, \mu)$ be a geometrically doubling metric measure. Then, for any ball $B(x,r) \subset \mathscr{X}$, with $x \in \mathscr{X}$ and $r \in (0,\infty)$, and any $n_1 \geq n_2 > 1$, there exist $r_0 \in (0,\infty)$, a constant $\widetilde{N} \in \mathbb{N}$, depending only on n_1, n_2 and the constant N_0 in Definition 6.1, and $\widetilde{N}$ balls $\{B(x_i, r_0)\}_{i=1}^{\widetilde{N}}$ such that $n_1 B(x_i, r_0) \subset n_2 B(x,r)$ for all $i \in \{1,\ldots,\widetilde{N}\}$ and $B(x,r) \subset \cup_{i=1}^{\widetilde{N}} B(x_i, r_0)$.*

Proof. Let n_1 and n_2 be as in Proposition 6.2, and $k := \lfloor \log_2(\frac{n_1+1}{n_2-1}) \rfloor + 1$, where $\lfloor t \rfloor$ denotes the *maximal integer not more than* $t \in \mathbb{R}$. We claim that, for any $y \in \mathscr{X}$ and ball $B(x,r) \subset \mathscr{X}$ with $x \in \mathscr{X}$ and $r \in (0,\infty)$, if $B(y, \frac{r}{2^k}) \cap B(x,r) \neq \emptyset$, then $n_1 B(y, \frac{r}{2^k}) \subset n_2 B(x,r)$. Indeed, by choosing $z \in B(y, \frac{r}{2^k}) \cap B(x,r)$ and observing that $k > \log_2(\frac{n_1+1}{n_2-1})$, we have

$$d(x,y) \leq d(x,z) + d(z,y) < \left(1 + \frac{1}{2^k}\right) r < \left(n_2 - \frac{n_1}{2^k}\right) r.$$

Thus, for all $w \in n_1 B(y, \frac{r}{2^k})$,

$$d(w,x) \leq d(w,y) + d(y,x) < \frac{n_1 r}{2^k} + \left(n_2 - \frac{n_1}{2^k}\right) r = n_2 r,$$

which shows the above claim. Then, by repeating the proof of (1) ⇒ (2) in [29, Lemma 2.3], we obtain the desired conclusion, which completes the proof of Proposition 6.2. □

Now we recall the definition of the modified Morrey space, which, when $(\mathscr{X}, d, \mu) := (\mathbb{R}^n, |\cdot|, \mu)$ with μ being a Radon measure satisfying the polynomial growth condition, was originally introduced by Sawano and Tanaka [45].

Definition 6.3. Let $k \in (0,\infty)$, $1 \leq p \leq q < \infty$ and $\mathscr{X}$ be a metric measure space. The *modified Morrey space* $\mathcal{M}_p^{q,(k)}(\mathscr{X})$ is defined as

$$\mathcal{M}_p^{q,(k)}(\mathscr{X}) := \left\{ f \in L^p_{\mathrm{loc}}(\mathscr{X}) : \|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})} < \infty \right\},$$

where

$$\|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})} := \sup_{B(x,r) \subset \mathscr{X}} [\mu(B(x,kr))]^{1/q-1/p} \left[\int_{B(x,r)} |f(y)|^p \, d\mu(y) \right]^{1/p},$$

where the supremum is taken over all balls $B(x,r)$, with $x \in \mathscr{X}$ and $r \in (0,\infty)$, of $\mathscr{X}$.

Proposition 6.4. *Let $(\mathscr{X}, d, \mu)$ be a geometrically doubling metric measure space and $1 \leq p \leq q < \infty$. Then, the space $\mathcal{M}_p^{q,(k)}(\mathscr{X})$ is independent of the choice of $k \in (1,\infty)$.*

Proof. Let $k_1,\ k_2 \in (1,\infty)$. We need to show that $\mathcal{M}_p^{q,(k_1)}(\mathscr{X})$ and $\mathcal{M}_p^{q,(k_2)}(\mathscr{X})$ coincide with equivalent norms. To this end, without loss of generality, we may assume that $k_1 < k_2$. By Definition 6.3, we easily find that $\mathcal{M}_p^{q,(k_1)}(\mathscr{X}) \subset \mathcal{M}_p^{q,(k_2)}(\mathscr{X})$. Thus, we still need to show the inverse embedding. Let B be a ball in $\mathscr{X}$. By Proposition 6.2, there exist $\widetilde{N}$ balls $\{B_i\}_{i=1}^{\widetilde{N}}$ with the same radius such that, for all $i \in \{1,\dots,\widetilde{N}\}$, $k_2B_i \subset k_1B$ and $B \subset \cup_{i=1}^{\widetilde{N}} B_i$, where $\widetilde{N}$ depends only on k_1, k_2 and N_0 in Definition 6.1. By these, we see that

$$\begin{aligned}[\mu(k_1B)]^{1/q-1/p}\left[\int_B |f(x)|^p\,d\mu(x)\right]^{1/p} &\le \sum_{i=1}^{\widetilde{N}}[\mu(k_1B)]^{1/q-1/p}\left[\int_{B_i}|f(x)|^p\,d\mu(x)\right]^{1/p}\\ &\le \sum_{i=1}^{\widetilde{N}}[\mu(k_2B_i)]^{1/q-1/p}\left[\int_{B_i}|f(x)|^p\,d\mu(x)\right]^{1/p}\\ &\le \widetilde{N}\|f\|_{\mathcal{M}_p^{q,(k_2)}(\mathscr{X})}.\end{aligned}$$

By the arbitrariness of B and Definition 6.3, we conclude that

$$\|f\|_{\mathcal{M}_p^{q,(k_1)}(\mathscr{X})} \le \widetilde{N}\|f\|_{\mathcal{M}_p^{q,(k_2)}(\mathscr{X})},$$

which further implies that $\mathcal{M}_p^{q,(k_2)}(\mathscr{X}) \subset \mathcal{M}_p^{q,(k_1)}(\mathscr{X})$ and hence completes the proof of Proposition 6.4. □

Recall that, for $\alpha \in [0,1]$ and $\beta \in [1,\infty)$, the *modified fractional maximal operator* $M_\alpha^{(\beta)}$ is defined by setting, for all $f \in L^1_{\rm loc}(\mathscr{X})$ and $x \in \mathscr{X}$,

$$M_\alpha^{(\beta)}f(x) := \sup_{r>0}[\mu(B(x,\beta r))]^{\alpha-1}\int_{B(x,r)}|f(y)|\,d\mu(y). \tag{6.1}$$

In particular, we write $M_\alpha := M_\alpha^{(1)}$.

Then we have the following conclusion, which generalizes [45, Theorem 2.3], wherein the corresponding result on the power bounded measure spaces on $\mathbb{R}^n$ was obtained. The proof of Theorem 6.5 is similar to that of [45, Theorem 2.3], and one key tool used in the proof is the $L^p(\mu)$-boundedness of the central Hardy-Littlewood maximal operator on geometrically doubling metric measure spaces (see [44]). For the convenience of the readers, we give the details.

Theorem 6.5. *Let $\mathscr{X}$ be a geometrically doubling metric measure space, $1 < p \le q < \infty$, $\beta \in (1,\infty)$ and $k \in (1,\infty)$. Then, there exists a positive constant C such that, for all $f \in \mathcal{M}_p^{q,(k)}(\mathscr{X})$,*

$$\|M_0^{(\beta)}f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})} \le C\|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})}. \tag{6.2}$$

Proof. By Proposition 6.4, it suffices to consider the case that $k := \frac{2\beta}{\beta+1} > 1$. Let $f \in \mathcal{M}_p^{q,(k)}(\mathscr{X})$ and $B_0 \subset \mathscr{X}$ be a ball. Define $\widetilde{\beta} := \frac{\beta+7}{\beta-1} > 1$, $f_1 := f\chi_{\widetilde{\beta}B_0}$ and $f_2 := f - f_1$. Then, by Definition 6.3, together with $1 < p \le q < \infty$, we have

$$\begin{aligned}
&\frac{1}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\left\{\int_{B_0}[M_0^{(\beta)}f_1(y)]^p\,d\mu(y)\right\}^{1/p}\\
&\quad\le \frac{1}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\left\{\int_{\mathscr{X}}[M_0^{(\beta)}f_1(y)]^p\,d\mu(y)\right\}^{1/p}\\
&\quad\lesssim \frac{1}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\left\{\int_{\widetilde{\beta}B_0}|f(y)|^p\,d\mu(y)\right\}^{1/p}\lesssim \|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})},
\end{aligned}$$

where we used the fact that $M_0^{(\beta)}$ is bounded on $L^p(\mu)$, which follows from the boundedness on $L^p(\mu)$ with $p \in (1,\infty]$ of the central Hardy-Littlewood maximal operator $M_0^{(1)}$ in the present setting (see [44]) and $M_0^{(\beta)} \le M_0^{(1)}$.

To estimate f_2, observe that, if $B \subset \mathscr{X}$ is a ball satisfying that $B \cap B_0 \ne \emptyset$ and $B \cap (\mathscr{X} \setminus \widetilde{\beta}B_0) \ne \emptyset$, then the radius $r_B > \frac{\widetilde{\beta}-1}{2} r_{B_0} = \frac{4}{\beta-1} r_{B_0}$, where r_B and r_{B_0} denote, respectively, the radii of B and B_0, and hence $B_0 \subset \frac{\beta+1}{2}B$. Therefore, we see that, for any $x \in B_0$,

$$\begin{aligned}
M_0^{(\beta)}f_2(x) &\le \sup_{x\in B}\frac{1}{\mu(\beta B)}\int_B |f_2(y)|\,d\mu(y) \le \sup_{B_0\subset\frac{\beta+1}{2}B}\frac{1}{\mu(\beta B)}\int_B |f(y)|\,d\mu(y)\\
&\le \sup_{B_0\subset B}\frac{1}{\mu(\frac{2\beta}{\beta+1}B)}\int_B |f(y)|\,d\mu(y) = \sup_{B_0\subset B}\frac{1}{\mu(kB)}\int_B |f(y)|\,d\mu(y),
\end{aligned}$$

and hence, by this, the Hölder inequality and Definition 6.3, we further see that

$$\begin{aligned}
&\frac{1}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\left[\int_{B_0}[M_0^{(\beta)}f_2(y)]^p\,d\mu(y)\right]^{1/p}\\
&\quad\le \frac{[\mu(B_0)]^{1/p}}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\sup_{B_0\subset B}\frac{1}{\mu(kB)}\int_B |f(y)|\,d\mu(y)\\
&\quad\le \frac{[\mu(B_0)]^{1/p}}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\sup_{B_0\subset B}\frac{[\mu(B)]^{1-1/p}}{\mu(kB)}\left[\int_B |f(y)|^p\,d\mu(y)\right]^{1/p}\\
&\quad\le \sup_{B_0\subset B}\frac{[\mu(B_0)]^{1/p}}{[\mu(k\widetilde{\beta}B_0)]^{1/p-1/q}}\frac{[\mu(B)]^{1-1/p}}{\mu(kB)}[\mu(kB)]^{1/p-1/q}\|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})}\\
&\quad\le \sup_{B_0\subset B}\frac{[\mu(B)]^{1-1/p+1/q}}{[\mu(kB)]^{1-1/p+1/q}}\|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})} \le \|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})},
\end{aligned}$$

where the last inequality follows from the fact that $B_0 \subset B$ and $k,\ \widetilde{\beta} > 1$. This estimate for f_2, together with the previous estimate for f_1 and Proposition 6.4, further implies that

$$\|M_0^{(\beta)}f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})} \approx \|M_0^{(\beta)}f\|_{\mathcal{M}_p^{q,(k\widetilde{\beta})}(\mathscr{X})} \lesssim \|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})},$$

which completes the proof of Theorem 6.5. □

Using Theorem 6.5, we have the following boundedness of the modified fractional maximal operator $M_\alpha^{(\beta)}$ on the modified Morrey space.

Proposition 6.6. *Let $\mathscr{X}$ be a geometrically doubling metric measure space, $1 < p \le q < \infty$, $\beta \in (1,\infty)$, $\alpha \in (0, 1/q)$ and $k \in (1,\infty)$. Then, there exists a positive constant C such that, for all $f \in \mathcal{M}_p^{q,(k)}(\mathscr{X})$,*

$$\|M_\alpha^{(\beta)} f\|_{\mathcal{M}_{\widetilde{p}}^{\widetilde{q},(k)}(\mathscr{X})} \lesssim \|f\|_{\mathcal{M}_p^{q,(k)}(\mathscr{X})},$$

where $\widetilde{p} := \frac{p}{1-\alpha q}$ and $\widetilde{q} := \frac{q}{1-\alpha q}$.

Proof. By Proposition 6.4, it suffices to consider the case that $k = \beta \in (1,\infty)$.

For any ball $B(x,r) \subset \mathscr{X}$ with $x \in \mathscr{X}$ and $r > 0$ and $f \in \mathcal{M}_p^{q,(\beta)}(\mathscr{X})$, we know, by the Hölder inequality, (1.3) and (3.6), that

$$\begin{aligned}
&[\mu(B(x,\beta r))]^{\alpha-1} \int_{B(x,r)} |f(y)|\, d\mu(y)\\
&\quad= [\mu(B(x,\beta r))]^{\alpha-1} \left[\int_{B(x,r)} |f(y)|\, d\mu(y)\right]^{\alpha q} \left[\int_{B(x,r)} |f(y)|\, d\mu(y)\right]^{1-\alpha q}\\
&\quad\le [\mu(B(x,\beta r))]^{\alpha-1+(1-\frac{1}{p})\alpha q} \left[\int_{B(x,r)} |f(y)|^p\, d\mu(y)\right]^{\frac{\alpha q}{p}} \left[\int_{B(x,r)} |f(y)|\, d\mu(y)\right]^{1-\alpha q}\\
&\quad= \left\{\frac{1}{[\mu(B(x,\beta r))]^{1-\frac{p}{q}}} \int_{B(x,r)} |f(y)|^p\, d\mu(y)\right\}^{\frac{\alpha q}{p}} \left[\frac{1}{\mu(B(x,\beta r))} \int_{B(x,r)} |f(y)|\, d\mu(y)\right]^{1-\alpha q}\\
&\quad\le \|f\|_{\mathcal{M}_p^{q,(\beta)}(\mathscr{X})}^{\alpha q} [M_0^{(\beta)} f(x)]^{1-\alpha q},
\end{aligned}$$

which, together with (6.1), implies that, for all $x \in \mathscr{X}$,

$$M_\alpha^{(\beta)} f(x) \le \|f\|_{\mathcal{M}_p^{q,(\beta)}(\mathscr{X})}^{\alpha q} [M_0^{(\beta)} f(x)]^{1-\alpha q}.$$

Then, by Theorem 6.5, we see that

$$\|M_\alpha^{(\beta)} f\|_{\mathcal{M}_{\widetilde{p}}^{\widetilde{q},(\beta)}(\mathscr{X})} \le \|f\|_{\mathcal{M}_p^{q,(\beta)}(\mathscr{X})}^{\alpha q} \|[M_0^{(\beta)} f]^{1-\alpha q}\|_{\mathcal{M}_{\widetilde{p}}^{\widetilde{q},(\beta)}(\mathscr{X})} \lesssim \|f\|_{\mathcal{M}_p^{q,(\beta)}(\mathscr{X})},$$

which completes the proof of Proposition 6.6. □

6.2 Fractional maximal operators on $HM_p^q(\mathscr{X})$ and $NM_p^q(\mathscr{X})$

Recently, Heikkinen et al. [23, 24] studied the boundedness of some fractional maximal operators on the Newton-Sobolev space and the Hajłasz-Sobolev space over metric measure spaces. In this section, we consider the corresponding problem for Newton-Morrey-Sobolev spaces and Hajłasz-Morrey-Sobolev spaces.

Throughout this section, we *always assume that the measure μ is doubling* (see (5.8)).

Recall that it is well known that any space of homogeneous type is also geometrically doubling (see [10, pp. 66-67]). Moreover, since μ is doubling, we see that, for any $\beta \in (1,\infty)$ and $\alpha \in [0,1]$, there exists a positive constant C, depending on β and α, such that, for all $f \in L^1_{\mathrm{loc}}(\mathscr{X})$ and $x \in \mathscr{X}$, $M_\alpha f(x) \le C M_\alpha^{(\beta)} f(x)$. From these facts, Theorem 6.5 and Proposition 6.6, we immediately deduce the following conclusion.

Corollary 6.7. *Let $1 < p \le q < \infty$ and $\alpha \in [0, 1/q)$. Then, there exists a positive constant C, depending on α, p and q, such that, for all $f \in \mathcal{M}_p^q(\mathscr{X})$,*

$$\|M_\alpha f\|_{\mathcal{M}_{\widetilde{p}}^{\widetilde{q}}(\mathscr{X})} \le C\|f\|_{\mathcal{M}_p^q(\mathscr{X})},$$

where $\widetilde{p} := \frac{p}{1-\alpha q}$ and $\widetilde{q} := \frac{q}{1-\alpha q}$.

As an application of Theorem 6.5, we obtain the following boundedness of fractional maximal operators on modified Morrey spaces.

Recall that a measure μ is said to satisfy the *measure lower bound condition*, if there exists a positive constant C such that, for any $x \in \mathscr{X}$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \ge Cr^Q \tag{6.3}$$

for some $Q \in (0,\infty)$.

Recently, if μ satisfies (6.3), Heikkinen et al. [24] established the boundedness from $L^p(\mathscr{X})$ to $L^s(\mathscr{X})$ for $p \in (1, Q)$ and $s := \frac{Qp}{Q-\alpha p}$ of the following modified fractional maximal function $\widetilde{M}_\alpha$, defined by setting, for any $\alpha \in [0,1]$, $f \in L^1_{\mathrm{loc}}(\mathscr{X})$ and $x \in \mathscr{X}$,

$$\widetilde{M}_\alpha f(x) := \sup_{r>0} \frac{r^\alpha}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y). \tag{6.4}$$

It is easy to see that, in the present setting, there exists a positive constant C, depending on α and Q, such that, for all $f \in L^1_{\mathrm{loc}}(\mathscr{X})$ and $x \in \mathscr{X}$, $\widetilde{M}_\alpha f(x) \le C M_\alpha f(x)$, which, together with Corollary 6.7, implies the following conclusion.

Corollary 6.8. *Let $1 < p \le q < \infty$ and $\alpha \in [0, 1/q)$. Assume that μ satisfies* (6.3)*. Then, there exists a positive constant C, depending on α, p and q, such that, for all $f \in \mathcal{M}_p^q(\mathscr{X})$,*

$$\left\|\widetilde{M}_\alpha f\right\|_{\mathcal{M}_{\widetilde{p}}^{\widetilde{q}}(\mathscr{X})} \le C\|f\|_{\mathcal{M}_p^q(\mathscr{X})}, \tag{6.5}$$

where $\widetilde{p} := \frac{p}{1-\alpha q}$ and $\widetilde{q} := \frac{q}{1-\alpha q}$.

Recall that $\mathscr{X}$ is said to satisfy the *relative* 1*-annular decay property*, if there exists a positive constant C such that, for all $x \in \mathscr{X}$, $R \in (0,\infty)$ and $h \in (0,R)$,

$$\mu\left(B \cap [B(x,R) \setminus B(x,R-h)]\right) \le C\frac{h}{r_B}\mu(B) \tag{6.6}$$

for all balls B with radius $r_B < 3R$; see, for example, [24, (2.5)].

Now we turn to the boundedness of the fractional maximal operator $\widetilde{M}_\alpha$ on Hajłasz-Morrey-Sobolev spaces.

Theorem 6.9. *Assume that μ satisfies (6.3) and $\mathscr{X}$ has the relative 1-annular decay property (6.6). Let $1 < p \le q < \infty$ and $\alpha \in (0, Q/q)$. Then, for any $f \in HM_p^q(\mathscr{X})$, $\widetilde{M}_\alpha f \in HM_{p^*}^{q^*}(\mathscr{X})$, where $p^* := \frac{Qp}{Q-\alpha q}$ and $q^* := \frac{Qq}{Q-\alpha q}$. Moreover, there exists a positive constant C, depending only on the doubling constant, Q, p, q and α, such that, for all $f \in HM_p^q(\mathscr{X})$,*

$$\|\widetilde{M}_\alpha f\|_{HM_{p^*}^{q^*}(\mathscr{X})} \le C\|f\|_{HM_p^q(\mathscr{X})}.$$

Proof. The proof is similar to that of [24, Theorem 4.5]. Let $f \in HM_p^q(\mathscr{X})$ and $g \in \mathcal{M}_p^q(\mathscr{X})$ be a Hajłasz gradient of f such that $\|g\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \|f\|_{HM_p^q(\mathscr{X})}$. It is easy to see that g is also a Hajłasz gradient of $|f|$. Let $r \in (1, p)$ and define

$$\widetilde{g} := \left[\widetilde{M}_{\alpha r}(g^r)\right]^{1/r}.$$

By an argument similar to that used in the proof of [24, Theorem 4.5], we know that $\widetilde{g}$ is a Hajłasz gradient of $\widetilde{M}_\alpha(|f|)$, as well as $\widetilde{M}_\alpha f$, since $\widetilde{M}_\alpha(|f|) = \widetilde{M}_\alpha f$. Moreover, by $p/r > 1$, (1.3) and (6.5), we see that

$$\|\widetilde{g}\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})} = \|\widetilde{M}_{\alpha r}(g^r)\|^{1/r}_{\mathcal{M}_{p^*/r}^{q^*/r}(\mathscr{X})} \lesssim \|g^r\|^{1/r}_{\mathcal{M}_{p/r}^{q/r}(\mathscr{X})} \approx \|g\|_{\mathcal{M}_p^q(\mathscr{X})}. \tag{6.7}$$

Combining (6.7) and Definition 4.1, we obtain the desired conclusion and then complete the proof of Theorem 6.9. □

We point out that, Theorem 6.9 when $p = q$ goes back to [24, Theorem 4.5].

Now we recall the *discrete fractional maximal operator* M_α^* introduced in [23, Section 5]. Let $\{B(x_i, r)\}_{i\in\mathbb{N}}$ be a ball covering of $\mathscr{X}$ such that $\{B(x_i, r)\}_{i\in\mathbb{N}}$ are of finite overlap. Since $\mathscr{X}$ is doubling, the overlap number N depends only on the doubling constant and is independent of r. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a partition of unity related to $\{B(x_i, r)\}_{i\in\mathbb{N}}$ such that $0 \le \varphi_i \le 1$, $\varphi_i = 0$ on $\mathscr{X} \setminus B(x_i, 6r)$, $\varphi_i \ge v$ on $B(x_i, 3r)$ and φ_i is Lipschitz function with Lipschitz constant L/r, where $L \in (0, \infty)$ and $v \in (0, 1]$ are constants depending only on the doubling constant, and $\sum_{i\in\mathbb{N}} \varphi_i \equiv 1$. The discrete convolution of $u \in L^1_{\rm loc}(\mathscr{X})$ at the scale $3r$ is defined by setting, for all $x \in \mathscr{X}$,

$$u_r(x) := \sum_{i\in\mathbb{N}} \varphi_i(x) u_{B(x_i,3r)},$$

where $u_{B(x_i,3r)}$ denotes the integral mean of u on $B(x_i, 3r)$ (see (3.4)). Now, let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence of the positive rational numbers, and $\{B(x_{i,j}, r_j)\}_{i\in\mathbb{N}}$ for each j is a ball covering of $\mathscr{X}$ as above. Then, the *discrete fractional maximal function* $M_\alpha^* u$ of u is defined as by setting, for all $x \in \mathscr{X}$,

$$M_\alpha^* u(x) := \sup_{j\in\mathbb{N}} r_j^\alpha |u|_{r_j}(x).$$

Similar to the proof of [23, Theorem 6.3], we obtain the following result on the boundedness of M_α^* on Newton-Morrey-Sobolev spaces.

Theorem 6.10. *Let μ satisfy* (6.3), $1 < p \le q < \infty$ *and* $\alpha \in (0, Q/q)$. *Assume that* $\mathscr{X}$ *is complete and supports a weak* $(1,p)$*-Poincaré inequality. Then, for any* $f \in NM_p^q(\mathscr{X})$, *it holds that,* $M_\alpha^* f \in NM_{p^*}^{q^*}(\mathscr{X})$ *with* $p^* := Qp/(Q-\alpha q)$ *and* $q^* := Qq/(Q-\alpha q)$. *Moreover, there exists a positive constant* C, *independent of* f, *such that*

$$\|M_\alpha^* f\|_{NM_{p^*}^{q^*}(\mathscr{X})} \le C \|f\|_{NM_p^q(\mathscr{X})}.$$

Proof. Let $f \in NM_p^q(\mathscr{X})$ and $g \in \mathcal{M}_p^q(\mathscr{X})$ be a Mod_p^q-weak upper gradient of f such that

$$\|g\|_{\mathcal{M}_p^q(\mathscr{X})} \le 2\|f\|_{NM_p^q(\mathscr{X})}. \tag{6.8}$$

By [23, Lemma 5.1] and (6.5), we have

$$\|M_\alpha^* f\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})} \lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})}. \tag{6.9}$$

By the same reason as that used in the proof [23, Theorem 6.3], observing that the pointwise Lipschitz constant of a function is also an upper gradient of that function, we see that a positive constant multiple of $(M_{\alpha\theta}^* g^\theta)^{1/\theta}$ is a Mod_p^q-weak upper gradient of $M_\alpha^* f$, where θ lies in $(1,p)$ such that the weak $(1,\theta)$-Poincaré inequality is supported by $\mathscr{X}$. By $g^\theta \in \mathcal{M}_{p/\theta}^{q/\theta}(\mathscr{X})$ and $p/\theta > 1$, together with [23, Lemma 5.1] and (6.5), we know that

$$\|[M_{\alpha\theta}^*(g^\theta)]^{1/\theta}\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})} \lesssim \|g\|_{\mathcal{M}_p^q(\mathscr{X})}. \tag{6.10}$$

Combining (6.8), (6.9) and (6.10), we obtain

$$\begin{aligned}\|M_\alpha^* f\|_{NM_{p^*}^{q^*}(\mathscr{X})} &\lesssim \|M_\alpha^* f\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})} + \|[M_{\alpha\theta}^*(g^\theta)]^{1/\theta}\|_{\mathcal{M}_{p^*}^{q^*}(\mathscr{X})}\\ &\lesssim \|f\|_{\mathcal{M}_p^q(\mathscr{X})} + \|g\|_{\mathcal{M}_p^q(\mathscr{X})} \lesssim \|f\|_{NM_p^q(\mathscr{X})},\end{aligned}$$

which completes the proof of Theorem 6.10. □

We remark that Theorem 6.10 when $p = q$ goes back to [23, Theorem 6.3].

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