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The common fixed point of single-valued generalized φ_f-weakly contractive mappings

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ABSTRACT

Fixed point and coincidence results are presented for single-valued generalized φ_f-weakly contractive mappings on complete metric spaces (X, d), where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $f : E \to X$ is a function such that $E \subseteq X$ is nonempty and closed. Our results extend previous results given by Rhoades (2001) [1] and by Zhang and Song (2009) [2].

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1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a φ-weak contraction if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$.

Also two mappings $T, S : X \to X$ are called generalized φ-weak contractions if there exists a map $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Sy) \le N(x, y) - \varphi(N(x, y)) \tag{1.2}$$

for all $x, y \in X$, where

$$N(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(x, Sy) + d(y, Tx)]\right\}. \tag{1.3}$$

The concept of the φ-weak contraction was defined by Alber and Guerre-Delabriere [1] in 1997, and the generalized φ-weak contraction was defined by Zhang and Song [2] in 2009. Rhoades [3, Theorem 2] proved the following fixed point theorem for φ-weak contraction single-valued mappings, giving another generalization of the Banach contraction principle [4].

Theorem 1.1. *Let (X, d) be a complete metric space, and let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)), \tag{1.4}$$

for every $x, y \in X$ (i.e. it is φ-weakly contractive), where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

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On choosing $\psi(t) = t - \varphi(t)$, φ-weak contractions become mappings of Boyd and Wong type [5], and on defining $k(t) = \frac{1-\varphi(t)}{t}$ for $t > 0$ and $k(0) = 0$, then φ-weak contractions become mappings of Reich type [6].

In 2009 Zhang and Song [2] proved the following theorem on the existence of a common fixed point for two single-valued generalized φ-weak contraction mappings.

Theorem 1.2. *Let (X, d) be a complete metric space, and let $T, S : X \longrightarrow X$ be two mappings such that for all $x, y \in X$*

$$d(Tx, Sy) \le N(x, y) - \varphi(N(x, y)), \tag{1.5}$$

(i.e. they are generalized φ-weak contractions), where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a l.s.c. function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then there exists a unique point $x \in X$ such that $x = Tx = Sx$.

Recently Rouhani and Moradi [7] extended Theorems 1.1 and 1.2 to multivalued mappings.

In Section 3, we extend Theorem 1.1 by assuming φ to be only l.s.c., and extend Theorem 1.2.

2. Preliminaries

In this work, (X, d) denotes a complete metric space and E denotes a nonempty closed subset of X.

Definition 2.1 (*See [8]*)**.** Let f, g be two self-mappings of a metric space (X, d). f and g are said to be weakly compatible if for all $t \in X$ the equality $ft = gt$ implies $fgt = gft$.

Definition 2.2. Two mappings $T, S : E \to E$ are called generalized φ_f-weakly contractive if there exist two maps $\varphi : [0, \infty) \to [0, \infty)$ and $f : E \to X$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Sy) \le M(x, y) - \varphi(M(x, y)) \tag{2.1}$$

for all $x, y \in X$, where

$$M(x, y) := \max\left\{d(fx, fy), d(fx, Tx), d(fy, Sy), \frac{1}{2}[d(fx, Sy) + d(fy, Tx)]\right\}. \tag{2.2}$$

3. Main results

The following theorem extends the Zhang–Song theorem.

Theorem 3.1. *Let (X, d) be a complete metric space, and let E be a nonempty closed subset of X. Let $T, S : E \to E$ be two generalized φ_f-weakly contractive mappings, where φ is a lower semicontinuous (l.s.c.) function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $f : E \to X$ verifying the conditions:*

(A) *f and T and f and S are weakly compatible.*
(B) *$T(E) \subset f(E)$ and $S(E) \subset f(E)$.*

Assume that $f(E)$ is a closed subset of X. Then f, T and S have a unique common fixed point.

Proof. Let $x_0 \in X$. Using (B) there exist two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = Tx_0 = fx_1, y_1 = Sx_1 = fx_2, y_2 = Tx_2 = fx_3, \ldots, y_{2n} = Tx_{2n} = fx_{2n+1}, y_{2n+1} = Sx_{2n+1} = fx_{2n+2}, \ldots$.

We break the argument into four steps.

Step 1. $\lim_{n\to\infty} d(y_{n+1}, y_n) = 0$.

Proof. Using (2.1),

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(Sx_{2n+1}, Tx_{2n}) = d(Tx_{2n}, Sx_{2n+1}) \\ &\le M(x_{2n}, x_{2n+1}) - \varphi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} d(y_{2n}, y_{2n-1}) &\le M(x_{2n}, x_{2n+1}) \\ &= \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})]\right\} \\ &\le \max\left\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\right\} \\ &= \max\left\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\right\} = d(y_{2n}, y_{2n-1}) \quad \text{(by (3.1))}. \end{aligned} \tag{3.2}$$

So $M(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n-1})$. Hence by (3.1),

$$d(y_{2n+1}, y_{2n}) \le d(y_{2n}, y_{2n-1}). \tag{3.3}$$

Similarly, $M(x_{2n+2}, x_{2n+1}) = d(y_{2n+1}, y_{2n})$ and

$$d(y_{2n+2}, y_{2n+1}) \le d(y_{2n+1}, y_{2n}). \tag{3.4}$$

Therefore by (3.3) and (3.4), we conclude that $M(x_{k+1}, x_k) = d(y_k, y_{k-1})$ and

$$d(y_{k+1}, y_k) \le d(y_k, y_{k-1}), \tag{3.5}$$

for all $k \in \mathbb{N}$.

Therefore, the sequence $\{d(y_{k+1}, y_k)\}$ is monotone nonincreasing and bounded below. So there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = \lim_{n\to\infty} M(x_n, x_{n-1}) = r. \tag{3.6}$$

Since φ is l.s.c.,

$$\varphi(r) \le \lim_{n\to\infty} \inf \varphi(M(x_n, x_{n-1})) \le \lim_{n\to\infty} \inf \varphi(M(x_{2n-1}, x_{2n})). \tag{3.7}$$

From (3.1),

$$r \le r - \varphi(r), \tag{3.8}$$

and so $\varphi(0) = 0$. Hence $r = 0$. □

Step 2. $\{y_n\}$ is a bounded sequence.

Proof. If $\{y_n\}$ were unbounded, then by Step 1, $\{y_{2n}\}$ and $\{y_{2n-1}\}$ are unbounded. We choose the sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$, $n(2) > n(1)$ is even and minimal in the sense that $d(y_{n(2)}, y_{n(1)}) > 1$, and similarly $n(3) > n(2)$ is odd and minimal in the sense that $d(y_{n(3)}, y_{n(2)}) > 1, \ldots, n(2k) > n(2k-1)$ is even and minimal in the sense that $d(y_{n(2k)}, y_{n(2k-1)}) > 1$, and $n(2k+1) > n(2k)$ is odd and minimal in the sense that $d(y_{n(2k+1)}, y_{n(2k)}) > 1$.

Obviously $n(k) \ge k$ for every $k \in \mathbb{N}$. By Step 1, there exists $N_0 \in \mathbb{N}$ such that for all $k \ge N_0$ we have $d(y_{k+1}, y_k) < \frac{1}{4}$. So for every $k \ge N_0$, $n(k+1) - n(k) \ge 3$ and

$$\begin{aligned} 1 &< d(y_{n(k+1)}, y_{n(k)}) \\ &\le d(y_{n(k+1)}, y_{n(k+1)-1}) + d(y_{n(k+1)-1}, y_{n(k+1)-2}) + d(y_{n(k+1)-2}, y_{n(k)}) \\ &\le d(y_{n(k+1)}, y_{n(k+1)-1}) + d(y_{n(k+1)-1}, y_{n(k+1)-2}) + 1. \end{aligned} \tag{3.9}$$

Hence $\lim_{k\to\infty} d(y_{n(k+1)}, y_{n(k)}) = 1$. Also

$$\begin{aligned} 1 &< d(y_{n(k+1)}, y_{n(k)}) \\ &\le d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)+1}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) \\ &\le d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)+1}, y_{n(k+1)}) + d(y_{n(k+1)}, y_{n(k)}) + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k)}) \\ &= 2d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)}, y_{n(k)}) + 2d(y_{n(k)+1}, y_{n(k)}), \end{aligned} \tag{3.10}$$

and this shows that $\lim_{k\to\infty} d(y_{n(k+1)+1}, y_{n(k)+1}) = 1$.

So if $n(k+1)$ is odd, then

$$d(y_{n(k+1)+1}, y_{n(k)+1}) \le M(x_{n(k+1)+1}, x_{n(k)+1}) - \varphi(M(x_{n(k+1)+1}, x_{n(k)+1})), \tag{3.11}$$

where

$$\begin{aligned} 1 &< d(y_{n(k+1)}, y_{n(k)}) \le M(x_{n(k+1)+1}, x_{n(k)+1}) \\ &= \max\left\{d(y_{n(k+1)}, y_{n(k)}), d(y_{n(k+1)}, y_{n(k+1)+1}), d(y_{n(k)}, y_{n(k)+1}), \frac{1}{2}[d(y_{n(k+1)}, y_{n(k)+1}) + d(y_{n(k)}, y_{n(k+1)+1})]\right\} \\ &\le \max\left\{d(y_{n(k+1)}, y_{n(k)}), d(y_{n(k+1)}, y_{n(k+1)+1}), d(y_{n(k)}, y_{n(k)+1}), \frac{1}{2}[d(y_{n(k+1)}, y_{n(k+1)+1}) + d(y_{n(k+1)+1}, y_{n(k)+1}) \right. \\ &\quad \left. + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{n(k+1)+1})]\right\}, \end{aligned} \tag{3.12}$$

and this shows that $\lim_{k\to\infty} M(x_{n(k+1)+1}, x_{n(k)+1}) = 1$. Since φ is l.s.c. and (3.11) holds, we have $1 \le 1 - \varphi(1)$. So $\varphi(1) = 0$ and this is a contradiction. □

Step 3. $\{y_n\}$ is Cauchy.

Proof. Let $C_n = \sup\{d(y_i, y_j) : i, j \geq n\}$. Since $\{y_n\}$ is bounded, $C_n < +\infty$ for all $n \in \mathbb{N}$. Obviously $\{C_n\}$ is decreasing. So there exists $C \geq 0$ such that $\lim_{n\to\infty} C_n = C$. We need to show that $C = 0$.

For every $k \in \mathbb{N}$, there exist $n(k), m(k) \in \mathbb{N}$ such that $m(k) > n(k) \geq k$ and

$$C_k - \frac{1}{k} \leq d(y_{m(k)}, y_{n(k)}) \leq C_k. \tag{3.13}$$

By (3.13), we conclude that

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = C. \tag{3.14}$$

From Step 1 and (3.14),

$$\begin{aligned}\lim_{k\to\infty} d(y_{m(k)+1}, y_{n(k)+1}) &= \lim_{k\to\infty} d(y_{m(k)+1}, y_{n(k)}) \\ &= \lim_{k\to\infty} d(y_{m(k)}, y_{n(k)+1}) = \lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = C.\end{aligned} \tag{3.15}$$

So we can assume that for every $k \in \mathbb{N}$, $m(k)$ is odd and $n(k)$ is even.

Hence,

$$d(y_{m(k)+1}, y_{n(k)+1}) \leq M(x_{m(k)+1}, x_{n(k)+1}) - \varphi(M(x_{m(k)+1}, x_{n(k)+1})), \tag{3.16}$$

where

$$\begin{aligned}d(y_{m(k)}, y_{n(k)}) &\leq M(x_{m(k)+1}, x_{n(k)+1}) \\ &= \max\left\{d(y_{m(k)}, y_{n(k)}), d(y_{m(k)}, y_{m(k)+1}), d(y_{n(k)}, y_{n(k)+1}), \frac{1}{2}[d(y_{m(k)}, y_{n(k)+1}) + d(y_{n(k)}, y_{m(k)+1})]\right\}.\end{aligned} \tag{3.17}$$

This inequality shows that $\lim_{k\to\infty} M(x_{m(k)+1}, x_{n(k)+1}) = C$. Since φ is l.s.c. and (3.16) holds, letting $n \to \infty$ in (3.16) we get $C \leq C - \varphi(C)$. Hence $\varphi(C) = 0$ and so $C = 0$. Therefore, $\{y_n\}$ is a Cauchy sequence. □

Step 4. T, S and f have a common fixed point.

Proof. Since (X, d) is complete and $\{y_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Since E is closed and $\{y_n\} \subseteq E$, we have $z \in E$. By assumption $f(E)$ is closed, so there exists $u \in E$ such that $z = fu$.

For all $n \in \mathbb{N}$

$$d(Tu, Sx_{2n+1}) \leq M(u, x_{2n+1}) - \varphi(M(u, x_{2n+1})), \tag{3.18}$$

where

$$\begin{aligned}M(u, x_{2n+1}) &= \max\left\{d(fu, fx_{2n+1}), d(fu, Tu), d(fx_{2n+1}, Sx_{2n+1}), \frac{1}{2}[d(fu, Sx_{2n+1}) + d(fx_{2n+1}, Tu)]\right\} \\ &= \max\left\{d(z, y_{2n}), d(z, Tu), d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(z, y_{2n+1}) + d(y_{2n}, Tu)]\right\},\end{aligned} \tag{3.19}$$

and this shows that $\lim_{n\to\infty} M(u, x_{2n+1}) = d(z, Tu)$. Since φ is l.s.c. and (3.18) holds, letting $n \to \infty$ in (3.18) we get

$$d(Tu, z) \leq d(Tu, z) - \varphi(d(Tu, z)). \tag{3.20}$$

So $\varphi(d(Tu, z)) = 0$ and hence $d(Tu, z) = 0$. Therefore $Tu = z$.

Similarly, $Su = z$. So $Tu = Su = fu = z$. Since f and T, and f and S are weakly compatible, then $Tz = Sz = fz$.

Now we show that z is a common fixed point.

For all $n \in \mathbb{N}$

$$d(Tz, y_{2n+1}) = d(Tz, Sx_{2n+1}) \leq M(z, x_{2n+1}) - \varphi(M(z, x_{2n+1})), \tag{3.21}$$

where

$$\begin{aligned}M(z, x_{2n+1}) &= \max\left\{d(fz, fx_{2n+1}), d(fz, Tz), d(fx_{2n+1}, Sx_{2n+1})\frac{1}{2}[d(fz, Sx_{2n+1}) + d(fx_{2n+1}, Tz)]\right\} \\ &= \max\left\{d(Tz, y_{2n}), 0, d(y_{2n}, y_{2n+1}), \frac{1}{2}[d(Tz, y_{2n+1}) + d(y_{2n}, Tz)]\right\}.\end{aligned} \tag{3.22}$$

This shows that $\lim_{n\to\infty} M(z, x_{2n+1}) = d(Tz, z)$. Since φ is l.s.c. and (3.21) holds, letting $n \to \infty$ in (3.21) we get

$$d(Tz, z) \le d(z, Tz) - \varphi(d(z, Tz)). \tag{3.23}$$

So $\varphi(d(z, Tz)) = 0$ and hence $d(z, Tz) = 0$. Therefore $Tz = z$ and from $Tz = Sz = fz$ we conclude that $Tz = Sz = fz = z$. □

Uniqueness of the common fixed point follows from (2.1), and this completes the proof. □

With a method similar to that in [9], we have another extension of the Zhang–Song theorem.

Theorem 3.2. *Let (X, d) be a complete metric space, and let E be a nonempty closed subset of X. Let $T, S : X \longrightarrow X$ be two generalized φ_f-weakly contractive mappings, where φ is a l.s.c. function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $f : E \to X$ verifying the conditions* (A) *and* (B) *of* Theorem 3.1. *Assume that f is a continuous function on E. If for all $x \in X$,*

$$d(fTx, Tfx) \le d(fx, Tx) \quad \text{and} \quad d(fSx, Sfx) \le d(fx, Sx), \tag{3.24}$$

then f, T and S have a unique common fixed point.

Proof. Uniqueness of the common fixed point follows from (2.1).

Following the proof of Theorem 3.1 we may conclude that $\{y_n\}$ is a Cauchy sequence converging to some z in X and

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} fx_n.$$

Since f is continuous, fy_n converges to fz. Using (3.24) and the triangle inequality, for all $n \in \mathbb{N}$,

$$\begin{aligned} d(Ty_{2n+1}, fz) &\le d(Ty_{2n+1}, fy_{2n+2}) + d(fy_{2n+2}, fz) \\ &= d(Tfx_{2n+2}, fTx_{2n+2}) + d(fy_{2n+2}, fz) \\ &\le d(Tx_{2n+2}, fx_{2n+2}) + d(fy_{2n+2}, fz) \\ &= d(y_{2n+2}, y_{2n+1}) + d(fy_{2n+2}, fz). \end{aligned} \tag{3.25}$$

This shows that $\lim_{n\to\infty} d(Ty_{2n+1}, fz) = 0$.

From (2.1),

$$d(Ty_{2n+1}, Sz) \le M(y_{2n+1}, z) - \varphi(M(y_{2n+1}, z)), \tag{3.26}$$

where

$$M(y_{2n+1}, z) = \max\left\{ d(fy_{2n+1}, fz), d(fy_{2n+1}, Ty_{2n+1}), d(fz, Sz), \frac{1}{2}[d(fy_{2n+1}, Sz) + d(fz, Ty_{2n+1})] \right\},$$

and this shows that $\lim_{n\to\infty} M(y_{2n+1}, z) = d(fz, Sz)$. Hence from (3.26), $d(fz, Sz) \le d(fz, Sz) - \varphi(d(fz, Sz))$. Therefore $d(fz, Sz) = 0$. So $fz = Sz$. Similarly, $fz = Tz$. Therefore $fz = Tz = Sz = t$. Using (3.24), we have $ft = Tt = St$. So from (2.1),

$$d(Tt, t) = d(Tt, Sz) \le M(t, z) - \varphi(M(t, z)), \tag{3.27}$$

where

$$M(t, z) = \max\left\{ d(ft, fz), d(ft, Tt), d(fz, Sz), \frac{1}{2}[d(ft, Sz) + d(fz, Tt)] \right\} = d(Tt, t).$$

Hence from (3.27), $d(Tt, t) \le d(Tt, t) - \varphi(d(Tt, t))$ and so $d(Tt, t) = 0$. This shows that $Tt = t$. Therefore $ft = Tt = St = t$. □

Example 3.3. Let $X = \mathbb{R}$ be endowed with the Euclidean metric and let $E = \left\{0, \frac{1}{2}, 1\right\}$. Let $T, S : E \to E$ be defined by $T0 = T1 = 0$, $T\frac{1}{2} = \frac{1}{2}$ and $Sx = 0$. Obviously

$$d\left(T\frac{1}{2}, S0\right) = \frac{1}{2} = N\left(\frac{1}{2}, 0\right). \tag{3.28}$$

So for every φ verifying the conditions of the Zhang–Song theorem (Theorem 1.2) the inequality (1.2) does not hold. For two functions $f : E \to X$ and $\varphi : [0, +\infty) \to [0, +\infty)$ defined by $f0 = 0$, $f\frac{1}{2} = 2$, $f1 = \frac{1}{2}$ and $\varphi(t) = \frac{t}{3}$ we have

$$d(Tx, Sy) \le M(x, y) - \varphi(M(x, y)). \tag{3.29}$$

So all conditions of Theorem 3.1 hold. Hence T, S and f have a common fixed point ($x = 0$) by Theorem 3.1.

Remark 3.4. In the proof of Theorem 1.2 in [2], the boundedness of the sequence $\{C_n\}$ is used, but not proved. Also, for the proof that $\{x_n\}$ is a Cauchy sequence, the monotonicity of φ is used, without being explicitly mentioned.

In our proof of Theorem 3.1, which is different from that of [2, Theorem 2.1] φ is not assumed to be nondecreasing.

The following theorem extends the Rhoades theorem by assuming φ to be only l.s.c.

Theorem 3.5. *Let (X, d) be a complete metric space, and E be a nonempty closed subset of X. Let $T : E \to E$ be a mapping such that*

$$d(Tx, Ty) \leq d(fx, fy) - \varphi(d(fx, fy)) \tag{3.30}$$

for every $x, y \in E$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is an l.s.c. function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ and $f : E \to X$ verifying the conditions:

(A): *f and T are weakly compatible.*
(B): *$T(E) \subset f(E)$.*

Assume that $f(E)$ is a closed subset of X. Then f and T have a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 3.1, but taking $S = T$, and replacing $M(x, y)$ with $d(x, y)$. □

Remark 3.6. By a method similar to that for Theorem 3.2 one can check without great difficulty that Theorem 3.5 is still true if we have "f is a continuous function on E and (3.24) holds" instead of "$f(E)$ is a closed subset of X".

4. Conclusion and future directions

We have extended the Rhoades theorem to a coincidence theorem without assuming continuity and the condition of being nondecreasing for the map φ. We have also extended the Zhang–Song theorem.

Future directions to be pursued in the context of this research include the investigation of the cases where one or both of the mappings in Theorem 3.1 are multivalued.

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