Math 117: Monotone and Cauchy Sequences

Some general properties of sequences that we can define include convergent, bounded, and monotone.

Definitions.

- A sequence (s_n) is **convergent** iff

- A sequence (s_n) is **bounded** iff

- A sequence (s_n) is **increasing** iff

- A sequence (s_n) is **decreasing** iff

- A sequence (s_n) is **monotone** iff (s_n) is increasing or decreasing.

We know some theorems relating these properties:

If a sequence is ____________________, then it is ____________________.

(And therefore, if a sequence is ____________________, then it is ____________________.)

Is it true that if a sequence is bounded, then it is convergent?

However, it *is* true that if a sequence is bounded and ____________________, then it is convergent.

Theorem: Monotone Convergence Theorem

If a sequence is ____________________, then it is convergent if and only if it is bounded.

Proof. Let (s_n) be a ____________________ sequence. (If (s_n) is ____________________, the proof is similar – homework!) We already know that if (s_n) is ____________________, it is ____________________. To prove the other direction, we assume that (s_n) is ____________________.

Consider the set

$$S = \left\{ \qquad\qquad\qquad\qquad \right\}.$$

By assumption, S is ____________________. (Also, notice that S is ____________________ since, for example, __________.) By the ________________________________, S has a _____.

Let $s =$ _____________. We claim that $\lim s_n = s$.

Given _____________,

Hence, (s_n) converges to s. $\square$

Example. Define the sequence (s_n) by $s_1 = 1$ and $s_{n+1} = \sqrt{1 + s_n}$ for $n \geq 1$. Prove (s_n) converges and find its limit.

Idea: To show (s_n) converges, show that (s_n) is ____________________ and that (s_n) is bounded by ______.

Then, to find the limit, use the fact that if (s_n) is a convergent sequence, then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n+1}$. (This is easy to prove using the definition of convergence.)

Cauchy Sequences

We now introduce a property of sequences (the Cauchy property) that certainly holds for all convergent sequences. It turns out that this property actually implies convergence as well! In other words, a sequence converges if and only if it is Cauchy. For a sequence to be Cauchy, we don't require that the terms of the sequence are eventually all close to a certain limit, just that the terms of the sequence are eventually all close to one another.

Definition. A sequence (s_n) is said to be a **Cauchy sequence** iff

for every ______, there exists ______ such that for all __________, $\left| \_\_\_\_\_\_\_\_\_\_\_\_ \right| <$ ____.

Lemma 1. Every convergent sequence is a Cauchy sequence.

Proof. Let (s_n) be a convergent sequence, and let $\lim s_n = s$. By the ________________,

Let __________________. Since ____________, there exists ______ such that ____________ __________________. Using this and our computation above, we find that if ____________,

Therefore, (s_n) is a Cauchy sequence. □

Lemma 2. Every Cauchy sequence is bounded.

Proof. (Homework!) Let (s_n) be a Cauchy sequence. By definition, there exists __________ such that __________________________________.

Therefore, for every __________, $|s_n| \leq$ ____________. Let

$$M = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

Then, for every $n \in \mathbb{N}$, $|s_n| \leq M$. Hence, the sequence (s_n) is bounded. □.

Theorem: Cauchy Convergence Criterion

A sequence of real numbers is convergent iff it is a Cauchy sequence.

Proof. (Outline of the proof)

The above lemma proved that a convergent sequence must be a Cauchy sequence. Therefore, we only need to prove the other direction. Let (s_n) be a Cauchy sequence. Consider the set $S = \{s_n : n \in \mathbb{N}\}$. In the case that S has only finitely many elements, the proof that (s_n) converges is left as an exercise.

Now consider the case that S is infinite. Notice that S is bounded since (s_n) is a convergent sequence. By the ____________________________________ theorem, there exists a point $s \in \mathbb{R}$ such that s is ____________________________________. We claim that $\lim s_n = s$.

Given _________, since (s_n) is ____________________,

Since s is ____________________________________,

If _____________, then

$$\left| s_n - s \right|$$

Thus $\lim s_n = s$. □

Note: Cauchy sequences having limits turns out to be related to completeness. For example, it's easy to see that in the ordered field $\mathbb{Q}$, we can have Cauchy sequences that have no limit. E.g., the sequence (q_n) given by $q_n = 1 + 1/1! + 1/2! + ... + 1/n!$ is a sequence of rational numbers that is Cauchy but has no limit in $\mathbb{Q}$.

Application: If a sequence of real numbers is *not* Cauchy, then it is __________________.