Modern index theory on closed manifolds
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Abstract

In this course, we introduce classical and modern approaches to index theory on closed manifolds. We will develop the necessary bacis theory, and then concentrate on particular applications for questions arising from geometry. The course is complemented by a course of Paolo Piazza; he will present the counterpart for manifolds with boundary and the use of cyclic cohomology to derive numerical invariants.

In more detail, in the present course I will cover

(1) First lecture: classical constructions
  - Ellipticity of (pseudo)differential operators and index
  - classical Atiyah-Singer index theorem; cohomological formula and K-theoretic approach
  - Family index theorem
  - Obstructions to positive scalar curvature via index theory

(2) Second lecture: KK-theoretic index
  - C^*-algebras and their K-theory
  - Kasparov's KK-theory
  - index in KK-theory
  - K-homology; the Baum-Connes conjecture
  - The Mishchenko-Fomenko index
  - The Gromov-Lawson-Rosenberg conjecture about positive scalar curvature

(3) Third lecture: Index theory and L^2-invariants
  - Atiyah's L^2-index theorem
  - Vanishing results for L^2-rho invariants (for the concept of rho- and eta-invariants, compare Paolo Piazza's talks)
  - L^2-Betti numbers and how they behave (a survey); integrality of L^2-Betti numbers and approximation results for L^2-Betti numbers.

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(4) Fourth lecture: Miscellaneous and left over subjects (depending on time used up)
  - The index obstruction to positive scalar curvature coming from enlargeability (following Gromov-Lawson), and its relation to the Mishchenko-Fomenko index
  - The Novikov conjecture for low dimensional cohomology classes; using a canonically associated twisting C^*-algebra
  - Codimension 2 index obstructions to positive scalar curvature (a reinterpretation of another result of Gromov-Lawson in the context of higher index theory)
  - The signature operator and homotopy invariance of (higher) signatures

Every elliptic (pseudo)-differential operator D on a closed manifold gives rise to a Fredholm operator acting on L^2-sections of the bundle in question. It therefore has an index $\mathrm{ind}(D) = \dim(\ker(D)) - \dim(\mathrm{coker}(D))$. This index depends only on the *symbol* of D. The Atiyah-Singer index theorem expresses this index by means of a *topological expression* in terms of this symbol.

Using a Chern character and applied to special operators coming from geometry, there is a very explicit cohomological formula for this index.

It turns out that, in more general contexts, the suitable definition of index is not given in terms of the difference of kernel and cokernel, but more precisely as an element of a K-theory group (where this group depends on the geometric situation in question). One instance where this can be observed is the index theorem for families, where the index is an element in the (topological) K-theory of the parameter space. Again, the index theorem gives a topological expression for this index.

In the version we are studying, the index takes values in the K-theory of a C^*-algebra associated to the index problem. In this context, one can make use of the fact that *positivity* implies invertibility to conclude that the index of a positive operator vanishes.

Geometric conditions sometimes imply positivity; the most prominent example is the Dirac operator on a spin manifold, which by the Lichnerowicz formula is positive if the manifold has a metric with positive scalar curvature. Consequently, the index of the Dirac operator is an obstruction to the existence of a metric with positive scalar curvature (as long as the index is independent of the chosen metric; which follows e.g. from the index theorems).

A convenient modern way to describe the K-theory of C^*-algebras and the index element of a pseudo-differential operator is given by Kasparov's bivariant KK-theory. This theory also allows connections between the index of these operators, and topological invariants constructed from this operators and living in the K-homology of the underlying spaces. In this context, we will also introduce the Mishchenko-Fomenko index. This index can be used to define the Baum-Connes assembly map.

The Baum-Connes conjecture expresses the K-theory of C^*-algebras in terms of the K-homology of topological spaces; in those cases where it holds (and so far, no counterexamples are known) this provides new views on the index obstructions to positive scalar curvature; it allows for instance to prove the stable Gromov-Lawson-Rosenberg conjecture which gives a precise description of those spin manifolds which admit a metric with positive scalar curvature -upto a certain stabilization procedure.

A special invariant which can be read off from the Mishchenko-Fomenko index is the L^2-index of Atiyah; which also has an interpretation as a difference of (regularized) dimensions of kernel and cokernel, but now of an operator acting on a (typically non-compact) covering space of the manifold one started with. Atiyah proves that this index coincides with the classical finite dimensional index of the underlying operator. This allows the use of secondary L^2-invariants to classify e.g. different metrics of positive scalar curvature on a given manifold. For this, one uses related L^2-invariants, namely L^2-rho invariants (eta- and rho-invariants are introduced in the series of lectures of Paolo Piazza).

Further related L^2-invariants are the L^2-Betti numbers, which measure the dimension of the kernel of the Laplacian (on forms) of the universal covering. These have interesting properties; one of the most remarkable is that these numbers (a priori arbitrary positive reals) are always integers for large classes of torsion-free fundamental groups; this is related to the existence of zero divisors in the complex group ring. The Atiyah conjecture predicts that this integrality result always holds.

In the last talk, if time permits, we will talk about some (or perhaps more likely none) of the miscellaneous subjects listed above.