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Some weighted Bartholdi zeta function of a digraph

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ABSTRACT

Related to the weighted bond scattering matrix of a graph, we define some weighted Bartholdi zeta function of a digraph D, and give a determinant expression for it. We present a decomposition formula for the weighted zeta function of a group covering of D. Furthermore, we define an L-function of D, and give a determinant expression for it. As a corollary, we express the weighted Bartholdi zeta function of a group covering of D by means of its L-functions.

1. Introduction

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [10]. In [10], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [17,18]. Hashimoto [9] generalized Ihara's result on the zeta function of a regular graph to an irregular graph,

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and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [3] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

For two variable zeta function of a graph, Bartholdi [2] defined and gave a determinant expression of the Bartholdi zeta function of a graph. Mizuno and Sato [13] considered a new zeta function of a digraph, and defined a new zeta function of a digraph by using not an infinite product but a determinant.

The spectral determinant of the Laplacian on a quantum graph is closely related to the Ihara zeta function of a graph (see [4,6,8,16]). Smilansky [16] considered spectral zeta functions and trace formulas for (discrete) Laplacians on ordinary graphs, and expressed some determinant on the bond scattering matrix of a graph G by using the characteristic polynomial of its Laplacian. Furthermore, Smilansky [16] considered a weighted version of the determinant on the bond scattering matrix of a graph G, and expressed it by using the characteristic polynomial of the weighted Laplacian of G. Oren, Godel and Smilansky [14] defined a new Bartholdi zeta function of a graph related to its weighted scattering matrix.

In this paper, we generalize the above zeta function of a digraph and the above Bartholdi zeta function of a graph. We define a new weighted Bartholdi zeta function of a digraph with respect to a vertex weight and an arc weight, and present its determinant expression. Furthermore, we present a decomposition formula for the new weighted Bartholdi zeta function of a group covering of a digraph D, and a determinant expression for the new weighted Bartholdi L-function of D.

2. Preliminaries

Graphs and digraphs treated here are finite. Let $G=(V(G),E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. Furthermore, let $D=(V(D),A(D))$ be a connected digraph (possibly multiple arcs and loops) with the set $V(D)$ of vertices and the set $A(D)$ of oriented edges. For $u,v\in V(D)$, an arc (u,v) is the oriented edge from u to v. For a graph G, set $D(G)=\{(u,v),(v,u)\mid uv\in E(G)\}$. A digraph $D_G=(V(G),D(G))$ is called the *symmetric digraph* corresponding to G. For $e=(u,v)\in D(G)$, set $u=o(e)$ and $v=t(e)$. Furthermore, let $e^{-1}=(v,u)$ be the *inverse* of $e=(u,v)$.

Let G be a connected graph and D a connected digraph. A *path P of length n* in G (or D) is a sequence $P=(e_1,\dots,e_n)$ of n arcs such that $e_i\in D(G)$ (or $A(D)$), $t(e_i)=o(e_{i+1})$ $(1\leqslant i\leqslant n-1)$, where indices are treated mod n. Set $|P|=n$, $o(P)=o(e_1)$ and $t(P)=t(e_n)$. Also, P is called an $(o(P),t(P))$*-path*. We say that a path $P=(e_1,\dots,e_n)$ has a *backtracking* or *bump* if $e_{i+1}=e_i^{-1}$ for some i $(1\leqslant i\leqslant n-1)$. A (v,w)-path is called a v*-cycle* (or v*-closed path*) if $v=w$. The *inverse cycle* of a cycle $C=(e_1,\dots,e_n)$ is the cycle $C^{-1}=(e_n^{-1},\dots,e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1=(e_1,\dots,e_m)$ and $C_2=(f_1,\dots,f_m)$ are called *equivalent* if there exists k such that $f_j=e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. The *cyclic bump count* $cbc(C)$ of a cycle $C=(e_1,\dots,e_n)$ in G (or D) is

$$cbc(C)=\left|\left\{i=1,\dots,n\;\middle|\;e_{i+1}=e_i^{-1}\right\}\right|,$$

where $e_{n+1}=e_1$.

Now, we state a new zeta function of a digraph by Mizuno and Sato [13].

Let D be a connected digraph with n vertices $v_1,\dots,v_n$ and m arcs. Then we consider an $n\times n$ matrix $\mathbf{W}=\mathbf{W}(D)=(w_{ij})_{1\leqslant i,j\leqslant n}$ with ij entry the complex variable w_{ij} if $(v_i,v_j)\in A(D)$, and $w_{ij}=0$ otherwise. The matrix $\mathbf{W}=\mathbf{W}(D)$ is called the *weighted matrix* of D. Furthermore, let $w(v_i,v_j)=w_{ij}$, $v_i,v_j\in V(D)$ and $w(e)=w_{ij}$, $e=(v_i,v_j)\in A(D)$. Then $w:A(D)\longrightarrow\mathbf{C}$ is called a *weight* of D. For each path $P=(e_1,\dots,e_r)$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P)=w(e_1)\cdots w(e_r)$.

Let D be a connected digraph with n vertices and m arcs, and $\mathbf{W}=\mathbf{W}(D)$ a weighted matrix of D. Two $m\times m$ matrices $\mathbf{B}=\mathbf{B}(D)=(\mathbf{B}_{e,f})_{e,f\in A(D)}$ and $\mathbf{J}=\mathbf{J}(D)=(\mathbf{J}_{e,f})_{e,f\in A(D)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise}, \end{cases} \qquad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that ${}^t\mathbf{J} = \mathbf{J}$. Then a *weighted Bartholdi zeta function* of D is defined by

$$\zeta_1(D, w, u, t) = \det\bigl(\mathbf{I}_m - t\bigl(\mathbf{B} - (1-u)\mathbf{J}\bigr)\bigr)^{-1}.$$

If $u = 0$, then the weighted Bartholdi zeta function of D is called *weighted Ihara zeta function* of D:

$$\mathbf{Z}_1(D, w, t) = \zeta_1(D, w, 0, t) = \det\bigl(\mathbf{I}_m - t(\mathbf{B} - \mathbf{J})\bigr)^{-1}.$$

If $w = \mathbf{1}$, i.e., $w(e) = 1$ for any $e \in A(D)$, then the weighted Bartholdi zeta function of D is the Bartholdi zeta function of D (see [12]). If $w = \mathbf{1}$ and $D = D_G$ is the symmetric digraph corresponding to a graph G, then the weighted Bartholdi zeta function of D is the Bartholdi zeta function $\zeta(G, u, t)$ of G (see [2]), where the symmetric digraph D_G corresponding to G is the digraph with vertex set $V(G)$ and arc set $D(G)$. Furthermore, in the case of $D = D_G$, $\zeta_1(D_G, \mathbf{1}, 0, t)$ is the Ihara zeta function $\mathbf{Z}(G, t)$ of G (see [3,9,10,17,18]).

We define two $n \times n$ matrices $\mathbf{W}_1 = \mathbf{W}_1(D) = (a_{uv})$ and $\mathbf{W}_0$ as follows:

$$a_{uv} = \begin{cases} w(u, v) & \text{if both } (u, v) \text{ and } (v, u) \in A(D), \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

and

$$\mathbf{W}_0 = \mathbf{W}_0(D) = \mathbf{W}(D) - \mathbf{W}_1. \tag{2}$$

Let $V(D) = \{v_1, \dots, v_n\}$. Then an $n \times n$ matrix $\mathbf{S}_w = (s_{ij})$ is the diagonal matrix defined by

$$s_{ii} = \sum_{e, e^{-1} \in A(D);\, o(e) = v_i} w(e). \tag{3}$$

Set

$$s(v_i) = s_{ii}, \quad 1 \leqslant i \leqslant n.$$

Theorem 1 *(Mizuno and Sato). Let D be a connected digraph, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Furthermore, let $m_1 = |\{e \in A(D) \mid e^{-1} \in A(D)\}|/2$. Then the reciprocal of the weighted Bartholdi zeta function of D is given by*

$$\zeta_1(D, w, u, t)^{-1} = \bigl(1 - (1-u)^2 t^2\bigr)^{m_1 - n} \det\bigl(\mathbf{I}_n - t\mathbf{W}_1(D) - \bigl(1 - (1-u)^2 t^2\bigr) t\mathbf{W}_0(D) + (1-u)t^2\bigl(\mathbf{S}_w - (1-u)\mathbf{I}_n\bigr)\bigr),$$

where $n = |V(D)|$.

Next, we state a scattering matrix of a graph by Smilansky [16].

Let G be a connected graph with n vertices and m unoriented edges, and $V(G) = \{v_1, \dots, v_n\}$. Assume that $\mathbf{W} = \mathbf{W}(G)$ is a symmetric weighted matrix of G with all non-negative elements. Then $\mathbf{W}(G)$ is called a *non-negative* symmetric weighted matrix of G. Set $D(G) = \{e_1, \dots, e_m, e_1^{-1}, \dots, e_m^{-1}\}$, and

$$d_j = \sum_{o(e) = v_j} w(e) \quad \text{for } j = 1, \dots, n.$$

The *weighted bond scattering matrix* $\mathbf{U}(\lambda) = (U_{ef})_{e, f \in D(G)}$ of G is defined by

$$U_{ef} = \begin{cases} i\bigl(\delta_{e^{-1} f} - x_{o(e)} \sqrt{w(e)} \sqrt{w(f)}\bigr) & \text{if } t(f) = o(e), \\ 0 & \text{otherwise}, \end{cases}$$

where

$$x_j = \frac{2}{d_j} \frac{1}{1 - i(1 - \lambda/d_j)}$$

for each $j = 1, \dots, n$.

Smilansky [16] stated a formula for some determinant on the weighted scattering matrix of a graph G without a proof.

Theorem 2 *(Smilansky). Let G be a connected graph with n vertices and m unoriented edges and $\mathbf{W}(G)$ a non-negative symmetric weighted matrix of G. Then, for the weighted scattering matrix of G,*

$$\det\bigl(\mathbf{I}_{2m} - \mathbf{U}(\lambda)\bigr) = \frac{2^m i^n \det(\lambda \mathbf{I}_n + \mathbf{W}(G) - \mathbf{D}_w)}{\prod_{j=1}^n (d_j - id_j + \lambda i)},$$

where $\mathbf{D}_w = (d_{ij})$ is the diagonal matrix with $d_{ii} = \sum_{o(e)=v_i} w(e)$ where $V(G) = \{v_1, \dots, v_n\}$.

A new zeta function of a graph by Oren, Godel and Smilansky [14] is defined as follows.

Let G be a connected graph with n vertices and m edges, and let $\mathbf{W}(G)$ be a non-negative symmetric weighted matrix of G. Then a $2m \times 2m$ matrix $\mathbf{B}^{(W)} = (B^{(W)}_{ef})_{e,f \in D(G)}$ of G is given by

$$B^{(W)}_{ef} = \begin{cases} \sqrt{w(e)}\sqrt{w(f)} & \text{if } t(f) = o(e), \\ 0 & \text{otherwise.} \end{cases}$$

A *new Bartholdi zeta function* of G is defined as follows:

$$\zeta(G, w, u, t) = \det\bigl(\mathbf{I}_{2m} - t\bigl(\mathbf{B}^{(W)} - (1-u)\mathbf{J}\bigr)\bigr)^{-1}.$$

Theorem 3 *(Oren, Godel and Smilansky).*

$$\zeta(G, w, u, t)^{-1} = \bigl(1 - (1-u)^2 t^2\bigr)^{m-n} \det\bigl(\mathbf{I}_n - t\mathbf{W}(G) + (1-u)t^2\bigl(\mathbf{D}_w - (1-u)\mathbf{I}_n\bigr)\bigr).$$

In this paper, we define some weighted Bartholdi zeta function and some weighted Bartholdi L-function of a digraph under Theorems 1, 2, and 3, and give determinant expressions for them.

In Section 2, we define some weighted Bartholdi zeta function of a digraph D, and give a determinant expression for it. In Section 3, we present a decomposition formula for the weighted Bartholdi zeta function of a group covering of D. In Section 4, we define a Bartholdi L-function of D, and give a determinant expression for it. As a corollary, we express the weighted Bartholdi zeta function of a group covering of D by means of its L-functions. In Section 5, we give the Euler product for the above L-function of D.

A general theory of the representation of groups and graph coverings, the reader is referred to [15] and [7], respectively.

3. Some weighted Bartholdi zeta function of a digraph

We define a weighted zeta function of a digraph by using not an infinite product but a determinant.

Let D be a connected digraph with n vertices and m arcs and $V(D) = \{v_1, \dots, v_n\}$. Furthermore, let $A(D) = \{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{m_0+m_1}, e^{-1}_{m_0+1}, \dots, e^{-1}_{m_0+m_1}\}$ be such that $e_i^{-1} \notin A(D)$ $(1 \leqslant i \leqslant m_0)$. Let $\mathbf{W}(D)$ be the weighted matrix of D. Assume that $w(v, u) = w(u, v)$ for symmetric arc $(u, v) \in D(G)$. Then $\mathbf{W}(D)$ is called *pseudo-symmetric*.

Let $x : V(D) \longrightarrow \mathbf{C}$ be the function from $V(D)$ to the complex number field $\mathbf{C}$. Set $x_v = x(v)$ for each $v \in V(D)$. An $m \times m$ matrix $\mathbf{B} = \mathbf{B}(D) = (\mathbf{B}_{e,f})_{e,f \in A(D)}$ is defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} x(t(e))\sqrt{w(e)}\sqrt{w(f)} & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then the *weighted zeta function* of D is defined by

$$\zeta_2(D, w, x, u, t) = \det\big(\mathbf{I}_m - t\big(\mathbf{B} - (1-u)\mathbf{J}\big)\big)^{-1}.$$

If $D = D_G$, $w = \mathbf{1}$ and $x = \mathbf{1}$, i.e., $x(v) = 1$ for any $v \in V(D)$, then the weighted zeta function of D is the Bartholdi zeta function of G. If $D = D_G$ and $x = \mathbf{1}$, then the weighted zeta function of D is the new Bartholdi zeta function of G in Theorem 3.

Theorem 4. *Let D be a connected digraph, $\mathbf{W} = \mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D and $x : V(D) \longrightarrow \mathbf{C}$ a function. Set $m_1 = |\{e \in A(D) \mid e^{-1} \in A(D)\}|/2$. Then the reciprocal of the zeta function of D is given by*

$$\zeta_2(D, w, x, u, t)^{-1} = \big(1 - (1-u)^2t^2\big)^{m_1 - n} \det\big(\mathbf{I}_n - t\mathbf{X}\mathbf{W}_1(D) - \big(1 - (1-u)^2t^2\big)t\mathbf{X}\mathbf{W}_0(D) + (1-u)t^2\big(\mathbf{X}\mathbf{S}_w - (1-u)\mathbf{I}_n\big)\big),$$

where $n = |V(D)|$ and $m = |A(D)|$. Furthermore, $\mathbf{W}_1(D)$, $\mathbf{W}_0(D)$ and $\mathbf{S}_w$ are given in (1), (2) and (3), and $\mathbf{X}$ is given by

$$\mathbf{X} = \begin{bmatrix} x(v_1) & & 0 \\ & \ddots & \\ 0 & & x(v_n) \end{bmatrix}.$$

Proof. By the argument in the proof of Theorem 6. □

If $D = D_G$ and $x = \mathbf{1}$, then Theorem 4 implies Theorem 3.
In the case of $x = \mathbf{1}$, we obtain the following result from Theorems 1 and 4.

Corollary 1. *Let D be a connected digraph, $\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D and $x = \mathbf{1}$. Then*

$$\zeta_2(D, w, \mathbf{1}, u, t) = \zeta_1(D, w, u, t).$$

4. Weighted zeta function of a group covering of a digraph

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v, w) \in D(G)\}$ denote the neighborhood of a vertex v in G. A graph H is called a *covering* of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the *quotient graph* G/Π is a graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $Aut\,H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and Γ a finite group. Then a mapping $\alpha : D(G) \longrightarrow \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The pair (G, α) is called an *ordinary voltage graph*. The *derived graph* G^α of the ordinary voltage graph (G, α) is defined as follows: $V(G^\alpha) = V(G) \times \Gamma$ and $((u, h), (v, k)) \in D(G^\alpha)$ if and only if $(u, v) \in D(G)$ and $k = h\alpha(u, v)$. The *natural projection* $\pi : G^\alpha \longrightarrow G$ is defined by $\pi(u, h) = u$. The graph G^α is called a *derived graph covering* of G with voltages in Γ or a *Γ-covering* of G. The natural projection π commutes with the right multiplication action of the $\alpha(e)$, $e \in D(G)$, and the left action of Γ on the fibers: $g(u, h) = (u, gh)$, $g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^α is a $|\Gamma|$-fold regular covering of G with covering transformation group Γ. Furthermore, every regular covering of a graph G is a Γ-covering of G for some group Γ (see [7]).

We can generalize the notion of a Γ-covering of a graph to a simple digraph. Let D be a connected digraph and Γ a finite group. Then a mapping $\alpha : A(D) \longrightarrow \Gamma$ is called a *pseudo ordinary voltage* *assignment* if $\alpha(v,u)=\alpha(u,v)^{-1}$ for each $(u,v)\in A(D)$ such that $(v,u)\in A(D)$. The pair (D,α) is called an *ordinary voltage digraph*. The *derived digraph* D^α of the ordinary voltage digraph (D,α) is defined as follows: $V(D^\alpha)=V(D)\times\Gamma$ and $((u,h),(v,k))\in A(D^\alpha)$ if and only if $(u,v)\in A(D)$ and $k=h\alpha(u,v)$. The digraph D^α is called a *Γ-covering* of D. Note that a Γ-covering of the symmetric digraph corresponding to a graph G is a Γ-covering of G (cf., [7]).

Let D be a connected graph with n vertices and m arcs, Γ a finite group and $\alpha: A(D)\longrightarrow\Gamma$ a pseudo ordinary voltage assignment. In the Γ-covering D^α, set $v_g=(v,g)$ and $e_g=(e,g)$, where $v\in V(D)$, $e\in A(D)$, $g\in\Gamma$. For $e=(u,v)\in A(D)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1}=(e^{-1})_{g\alpha(e)}$.

Let $\mathbf{W}=\mathbf{W}(D)$ be a non-negative pseudo-symmetric weighted matrix and $x: V(D)\longrightarrow\mathbf{C}$ a function. Then we define a *weighted matrix* $\tilde{\mathbf{W}}=\mathbf{W}(D^\alpha)=(\tilde{w}(u_g,v_h))$ of D^α *derived from* $\mathbf{W}$ as follows:

$$\tilde{w}(u_g,v_h):=\begin{cases} w(u,v) & \text{if } (u,v)\in A(D) \text{ and } h=g\alpha(u,v),\\ 0 & \text{otherwise.}\end{cases}$$

Moreover, we define a matrix $\tilde{\mathbf{X}}=\mathbf{X}(D^\alpha)=(\tilde{x}(u_g,v_h))$ of G^α *derived from* $\mathbf{X}$ as follows:

$$\tilde{w}(u_g,v_h):=\begin{cases} x(u) & \text{if } u=v \text{ and } g=h,\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, let $r=|\Gamma|$. Then we define two $mr\times mr$ matrices $\tilde{\mathbf{B}}=(B(e_g,f_h))$ and $\tilde{\mathbf{J}}=(J(e_g,f_h))$ of D^α *derived from* $\mathbf{B}$ and $\mathbf{J}$, respectively, as follows:

$$B(e_g,f_h):=\begin{cases} x(t(e))\sqrt{w(e)}\sqrt{w(f)} & \text{if } o(e_g)=t(f_h),\\ 0 & \text{otherwise,}\end{cases}\qquad J(e_g,f_h):=\begin{cases} 1 & \text{if } f_h=e_g^{-1},\\ 0 & \text{otherwise.}\end{cases}$$

For $g\in\Gamma$, let the matrices $\mathbf{W}_{1,g}=(w_{uv}^{(g)})$ and $\mathbf{W}_{0,g}=(z_{uv}^{(g)})$ be defined by

$$w_{uv}^{(g)}:=\begin{cases} w(u,v) & \text{if } \alpha(u,v)=g \text{ and } (u,v)\in A(D),\ (v,u)\in A(D),\\ 0 & \text{otherwise,}\end{cases}$$

and

$$z_{uv}^{(g)}:=\begin{cases} w(u,v) & \text{if } \alpha(u,v)=g \text{ and } (u,v)\in A(D),\ (v,u)\notin A(D),\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, two $m\times m$ matrices $\mathbf{B}_g=(\mathbf{B}_{ef}^{(g)})_{e,f\in A(D)}$ and $\mathbf{J}_g=(\mathbf{J}_{ef}^{(g)})_{e,f\in A(D)}$ are defined as follows:

$$\mathbf{B}_{ef}^{(g)}=\begin{cases} x_{t(e)}\sqrt{w(e)w(f)} & \text{if } t(e)=o(f) \text{ and } \alpha(e)=g,\\ 0 & \text{otherwise,}\end{cases}$$
$$\mathbf{J}_{ef}^{(g)}=\begin{cases} 1 & \text{if } f=e^{-1} \text{ and } \alpha(e)=g,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\mathbf{M}_1\oplus\cdots\oplus\mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1,\dots,\mathbf{M}_s$. If $\mathbf{M}_1=\mathbf{M}_2=\cdots=\mathbf{M}_s=\mathbf{M}$, then we write $s\circ\mathbf{M}=\mathbf{M}_1\oplus\cdots\oplus\mathbf{M}_s$. The *Kronecker product* $\mathbf{A}\otimes\mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

Theorem 5. *Let D be a connected digraph with n vertices and m arcs, $\mathbf{W}=\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D, $x: V(D)\longrightarrow\mathbf{C}$ a function, Γ a finite group and $\alpha: A(D)\longrightarrow\Gamma$ a pseudo ordinary voltage assignment. Set $|\Gamma|=r$ and $m_1=|\{e\in A(D)\mid e^{-1}\in A(D)\}|/2$. Furthermore, let $\rho_1=1,\rho_2,\dots,\rho_k$ be the irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1=1$. Suppose that the Γ-covering D^α of D is connected. Then,*

$$\begin{aligned}
&\zeta_2\big(D^\alpha, \tilde{w}, \tilde{x}, u, t\big)^{-1} \\
&\quad = \zeta_2(D, w, x, u, t)^{-1} \prod_{i=2}^{k} \det\Big(\mathbf{I}_{mf_i} - t\sum_h \rho_i(h) \otimes \big(\mathbf{B}_h - (1-u)\mathbf{J}_h\big)\Big)^{f_i} \\
&\quad = \zeta_2(D, w, x, u, t)^{-1} \prod_{i=2}^{k} \Big\{\big(1-(1-u)^2t^2\big)^{(m_1-n)f_i} \\
&\qquad \times \det\Big(\mathbf{I}_{nf_i} - t\sum_{h\in\Gamma} \rho_i(h) \otimes \mathbf{XW}_{1,h} - \big(1-(1-u)^2t^2\big)t\sum_{h\in\Gamma} \rho_i(h) \otimes \mathbf{XW}_{0,h} \\
&\qquad + (1-u)t^2\mathbf{I}_{f_i} \otimes \big(\mathbf{XS}_w - (1-u)\mathbf{I}_n\big)\Big)\Big\}^{f_i}.
\end{aligned}$$

Proof. Let $|\Gamma| = r$. By Theorem 4, we have

$$\begin{aligned}
\zeta_2\big(D^\alpha, \tilde{w}, \tilde{x}, u, t\big)^{-1} &= \det\big(\mathbf{I}_{mr} - t\big(\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}}\big)\big) \\
&= \big(1-(1-u)^2t^2\big)^{(m_1-n)r} \det\big(\mathbf{I}_{nr} - t\tilde{\mathbf{X}}\mathbf{W}_1\big(D^\alpha\big) \\
&\quad - \big(1-(1-u)^2t^2\big)t\tilde{\mathbf{X}}\mathbf{W}_0\big(D^\alpha\big) + (1-u)t^2\big(\tilde{\mathbf{X}}\mathbf{S}_{\tilde{w}} - (1-u)\mathbf{I}_{nr}\big)\big).
\end{aligned}$$

Let $A(D) = \{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{m_0+m_1}, e_{m_0+1}^{-1}, \dots, e_{m_0+m_1}^{-1}\}$ such that $e_i^{-1} \notin A(D)$ $(1 \leqslant i \leqslant m_0)$. Furthermore, let $\Gamma = \{1 = g_1, g_2, \dots, g_r\}$. Arrange arcs of G^α in m blocks: $(e_1, 1), \dots, (e_{m_0+m_1}^{-1}, 1)$; $(e_1, g_2), \dots, (e_{m_0+m_1}^{-1}, g_2); \dots; (e_1, g_r), \dots, (e_{m_0+m_1}^{-1}, g_r)$. We consider the matrix $\tilde{\mathbf{B}} - \tilde{\mathbf{J}}$ under this order. For $h \in \Gamma$, the matrix $\mathbf{P}_h = (p_{ij}^{(h)})$ is defined as follows:

$$p_{ij}^{(h)} = \begin{cases} 1 & \text{if } g_ih = g_j, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_ih$. Then $B(e_{g_i}, f_{g_j}) \neq 0$, i.e., $t(e, g_i) = o(f, g_j)$ if and only if $(o(f), g_j) = o(f, g_j) = t(e, g_i) = (t(e), g_i\alpha(e))$, i.e., $t(e) = o(f)$ and $\alpha(e) = g_i^{-1}g_j = g_i^{-1}g_ih = h$. Furthermore, $J(e_{g_i}, f_{g_j}) = 1$, i.e., $(f, g_j) = (e, g_i)^{-1} = (e^{-1}, g_i\alpha(e))$ if and only if $f = e^{-1}$ and $\alpha(e) = g_i^{-1}g_j = h$. Thus we have

$$\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}} = \sum_{h\in\Gamma} \mathbf{P}_h \otimes \big(\mathbf{B}_h - (1-u)\mathbf{J}_h\big).$$

Let ρ be the right regular representation of Γ. Furthermore, let $\rho_1 = 1, \rho_2, \dots, \rho_k$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_k \circ \rho_k(h)$ for each $h \in \Gamma$ (see [15]). Putting $\mathbf{F} = (\mathbf{P}^{-1} \otimes \mathbf{I}_{2m})(\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}}_0)(\mathbf{P} \otimes \mathbf{I}_{2m})$, we have

$$\mathbf{F} = \sum_{h\in\Gamma} \big\{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_k \circ \rho_k(h)\big\} \otimes \big(\mathbf{B}_h - (1-u)\mathbf{J}_h\big).$$

Note that $\mathbf{B} - (1-u)\mathbf{J}_0 = \sum_{h\in\Gamma}(\mathbf{B}_h - (1-u)\mathbf{J}_h)$ and $1 + f_2^2 + \cdots + f_k^2 = r$. Therefore it follows that

$$\begin{aligned}
\zeta_2\big(D^\alpha, \tilde{w}, \tilde{x}, u, t\big)^{-1} &= \det\big(\mathbf{I}_m - t\big(\mathbf{B} - (1-u)\mathbf{J}\big)\big) \\
&\quad \times \prod_{i=2}^{k} \det\Big(\mathbf{I}_{mf_i} - t\sum_h \rho_i(h) \otimes \big(\mathbf{B}_h - (1-u)\mathbf{J}_h\big)\Big)^{f_i}.
\end{aligned}$$

Next, let $V(D)=\{v_1,\dots,v_n\}$. Arrange vertices of D^α in m blocks: $(v_1,1),\dots,(v_n,1);(v_1,g_2),\dots,(v_n,g_2);\dots;(v_1,g_r),\dots,(v_n,g_r)$. We consider the weighted matrix $\mathbf{W}(D^\alpha)$ under this order.

At first, we have

$$\tilde{\mathbf{X}}=\mathbf{I}_r\otimes\mathbf{X}.$$

Suppose that $p_{ij}^{(h)}=1$, i.e., $g_j=g_ih$. Then $((u,g_i),(v,g_j))\in A(D^\alpha)$ if and only if $(u,v)\in A(D)$ and $g_j=g_i\alpha(u,v)$, i.e., $\alpha(u,v)=g_i^{-1}g_j=g_i^{-1}g_ih=h$. Thus we have

$$\mathbf{W}_i\big(D^\alpha\big)=\sum_{h\in\Gamma}\mathbf{P}_h\otimes\mathbf{W}_{i,h},\quad \text{i.e.,}\quad \tilde{\mathbf{X}}\mathbf{W}_i\big(D^\alpha\big)=\sum_{h\in\Gamma}\mathbf{P}_h\otimes\mathbf{X}\mathbf{W}_{i,h}$$

for $i=0,1$.

Putting $\mathbf{E}_i=(\mathbf{P}^{-1}\otimes\mathbf{I}_u)\tilde{\mathbf{X}}\mathbf{W}_i(D^\alpha)(\mathbf{P}\otimes\mathbf{I}_n)$ $(i=0,1)$, we have

$$\mathbf{E}_i=\sum_{h\in\Gamma}\big\{(1)\oplus f_2\circ\rho_2(h)\oplus\cdots\oplus f_k\circ\rho_k(h)\big\}\otimes\mathbf{X}\mathbf{W}_{i,h}.$$

Note that $\mathbf{W}_i(D)=\sum_{h\in\Gamma}\mathbf{W}_{i,h}$ $(i=0,1)$. Furthermore, we have

$$\tilde{\mathbf{X}}\mathbf{S}_{\tilde{w}}=(\mathbf{I}_r\otimes\mathbf{X})(\mathbf{I}_r\otimes\mathbf{S}_w)=\mathbf{I}_r\otimes\mathbf{X}\mathbf{S}_w.$$

Therefore it follows that

$$\begin{aligned}
&\det\big(\mathbf{I}_{nr}-t\tilde{\mathbf{X}}\mathbf{W}_1\big(D^\alpha\big)-\big(1-(1-u)^2t^2\big)t\tilde{\mathbf{X}}\mathbf{W}_1\big(D^\alpha\big)+(1-u)t^2\big(\tilde{\mathbf{X}}\mathbf{S}_{\tilde{w}}-(1-u)\mathbf{I}_{nr}\big)\big)\\
&\quad=\det\big(\mathbf{I}_n-t\mathbf{X}\mathbf{W}_1(D)-\big(1-(1-u)^2t^2\big)t\mathbf{X}\mathbf{W}_0(D)+(1-u)t^2\big(\mathbf{X}\mathbf{S}_w-(1-u)\mathbf{I}_n\big)\big)\\
&\qquad\times\prod_{i=2}^{k}\det\Bigg(\mathbf{I}_{nf_i}-t\sum_{h\in\Gamma}\rho_i(h)\otimes\mathbf{X}\mathbf{W}_{1,h}-\big(1-(1-u)^2t^2\big)t\sum_{h\in\Gamma}\rho_i(h)\otimes\mathbf{X}\mathbf{W}_{0,h}\\
&\qquad+(1-u)t^2\mathbf{I}_{f_i}\otimes\big(\mathbf{X}\mathbf{S}_w-(1-u)\mathbf{I}_n\big)\Bigg)^{f_i}.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\zeta_2\big(D^\alpha,\tilde{w},\tilde{x},u,t\big)^{-1}\\
&\quad=\zeta_2(D,w,x,u,t)^{-1}\prod_{i=2}^{k}\det\Bigg(\mathbf{I}_{mf_i}-t\sum_{h}\rho_i(h)\otimes\big(\mathbf{B}_h-(1-u)\mathbf{J}_h\big)\Bigg)^{f_i}\\
&\quad=\zeta_2(D,w,x,u,t)^{-1}\prod_{i=2}^{k}\Bigg\{\big(1-(1-u)^2t^2\big)^{(m_1-n)f_i}\det\Bigg(\mathbf{I}_{nf_i}-t\sum_{h\in\Gamma}\rho_i(h)\otimes\mathbf{X}\mathbf{W}_{1,h}\\
&\qquad-\big(1-(1-u)^2t^2\big)t\sum_{h\in\Gamma}\rho_i(h)\otimes\mathbf{X}\mathbf{W}_{0,h}+(1-u)t^2\mathbf{I}_{f_i}\otimes\big(\mathbf{X}\mathbf{S}_w-(1-u)\mathbf{I}_n\big)\Bigg)\Bigg\}^{f_i}.\qquad\square
\end{aligned}$$

5. *L*-functions of digraphs

Let D be a connected digraph with n vertices and m arcs, $\mathbf{W}=\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of G, $x:V(D)\longrightarrow\mathbf{C}$ a function, Γ a finite group and $\alpha:A(D)\longrightarrow\Gamma$ a pseudo ordinary voltage assignment. Furthermore, let ρ be a unitary representation of Γ and d its degree. The *L-function* of D associated with ρ and α is defined by

$$\zeta_2(D,w,x,u,t,\rho,\alpha)=\det\Bigg(\mathbf{I}_{md}-t\sum_{h\in\Gamma}\rho(h)\otimes\big(\mathbf{B}_h-(1-u)\mathbf{J}_h\big)\Bigg)^{-1}.$$

A determinant expression for the L-function of D associated with ρ and α is given as follows. Let $1 \leqslant i, j \leqslant n$. Then, the (i, j)-*block* $\mathbf{F}_{i,j}$ of a $dn \times dn$ matrix $\mathbf{F}$ is the submatrix of $\mathbf{F}$ consisting of $d(i-1)+1, \dots, di$ rows and $d(j-1)+1, \dots, dj$ columns.

Theorem 6. *Let D be a connected digraph with n vertices and m arcs, $\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D, $x : V(D) \longrightarrow \mathbf{C}$ a function, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment. Set $m_1 = |\{e \in A(D) \mid e^{-1} \in A(D)\}|/2$. Furthermore, let ρ be a representation of Γ and d the degree of ρ. Then the reciprocal of the L-function of D associated with ρ and α is*

$$\begin{aligned}\zeta_2(D, w, x, u, t, \rho, \alpha)^{-1} &= \left(1-(1-u)^2t^2\right)^{(m_1-n)d} \det\Bigg(\mathbf{I}_{nd} - t\sum_{g\in\Gamma}\rho(g)\otimes \mathbf{XW}_{1,g} \\ &\quad - \left(1-(1-u)^2t^2\right)t\sum_{g\in\Gamma}\rho(g)\otimes\mathbf{XW}_{0,g} + (1-u)t^2\mathbf{I}_d\otimes\left(\mathbf{XS}_w-(1-u)\mathbf{I}_n\right)\Bigg).\end{aligned}$$

Proof. The argument is an analogue of Bass' method [3].

Let ρ be a unitary representation of Γ, and d the degree of ρ. Furthermore, let $V(D) = \{v_1, \dots, v_n\}$ and $A(D) = \{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{m_0+m_1}, e^{-1}_{m_0+1}, \dots, e^{-1}_{m_0+m_1}\}$ be such that $e_j^{-1} \notin A(D)$ $(1 \leqslant j \leqslant m_0)$ and $e_{m_0+m_1+i} = e^{-1}_{m_0+i}$ $(1 \leqslant i \leqslant m_1)$. Then, note that the (e, f)-block $(\sum_{g\in\Gamma}(\mathbf{B}_g - \mathbf{J}_g)\otimes\rho(g))_{ef}$ of $\sum_{g\in\Gamma}(\mathbf{B}_g - \mathbf{J}_g)\otimes\rho(g)$ is given by

$$\begin{aligned}&\left(\sum_{g\in\Gamma}\left(\mathbf{B}_g-(1-u)\mathbf{J}_g\right)\otimes\rho(g)\right)_{ef} \\ &\quad = \begin{cases}\rho(\alpha(e))(x(t(e))\sqrt{w(e)}\sqrt{w(f)} - (1-u)\delta_{e^{-1}f}\epsilon_{e^{-1}}) & \text{if } o(e)=t(f),\\ \mathbf{0}_d & \text{otherwise,}\end{cases}\end{aligned}$$

where

$$\epsilon_{e^{-1}} = \begin{cases}1 & \text{if there exists an arc } e^{-1} \text{ in } D,\\ 0 & \text{otherwise.}\end{cases}$$

Now, let $\mathbf{K} = (\mathbf{K}_{ev})_{e\in A(D); v\in V(D)}$ be the $md \times nd$ matrix defined as follows:

$$\mathbf{K}_{ev} := \begin{cases}\sqrt{w(e)}\rho(\alpha(e)) & \text{if } t(e)=v,\\ \mathbf{0}_d & \text{otherwise.}\end{cases}$$

Furthermore, we define two $md \times nd$ matrices $\mathbf{L} = (\mathbf{L}_{ev})_{e\in A(D); v\in V(D)}$ and $\mathbf{H} = (\mathbf{H}_{ev})_{e\in A(D); v\in V(D)}$ as follows:

$$\mathbf{L}_{ev} := \begin{cases}x(v)\sqrt{w(e)}\mathbf{I}_d & \text{if } o(e)=v,\\ \mathbf{0}_d & \text{otherwise,}\end{cases}$$

and

$$\mathbf{H}_{ev} := \begin{cases}(1-u)t\sqrt{w(e)}\rho(\alpha(e)) & \text{if } t(e)=v \text{ and } e\in A(D),\ e^{-1}\notin A(D),\\ \sqrt{w(e)}\mathbf{I}_d & \text{if } o(e)=v \text{ and } e, e^{-1}\in A(D),\\ \mathbf{0}_d & \text{otherwise.}\end{cases}$$

Then we have

$$^t\mathbf{KL} = {}^t\mathbf{W}^{(x)}_{1,\rho} + {}^t\mathbf{W}^{(x)}_{0,\rho}, \tag{4}$$

where two matrices $\mathbf{W}^{(x)}_{1,\rho} = ((w_x)_{uv})_{u,v\in V(D)}$, $\mathbf{W}^{(x)}_{0,\rho} = ((z_x)_{uv})_{u,v\in V(D)}$ are given by

$$(w_x)_{uv} := \begin{cases} w(u,v)x_u\rho(\alpha(u,v)) & \text{if } (u,v) \in A(D) \text{ and } (v,u) \in A(D), \\ 0 & \text{otherwise,} \end{cases}$$
$$(z_x)_{uv} := \begin{cases} w(u,v)x_u\rho(\alpha(u,v)) & \text{if } (u,v) \in A(D) \text{ and } (v,u) \notin A(D), \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\mathbf{W}_{i,\rho}^{(x)} = \sum_{g\in\Gamma} \mathbf{X}\mathbf{W}_{i,g} \otimes \rho(g) \quad (i=1,0). \tag{5}$$

Furthermore,

$${}^t\mathbf{H}\mathbf{L} = \mathbf{S}_x \otimes \mathbf{I}_d + (1-u)t{}^t\mathbf{W}_{0,\rho}^{(x)}, \tag{6}$$

where $\mathbf{S}_x = (s_{uv})_{u,v\in V(D)}$ is given by

$$s_{uv} := \begin{cases} s_u x_u & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Here

$$s_j = s_{v_j} = s(v_j) = \sum_{o(e)=v_j, e, e^{-1}\in A(D)} w(e).$$

Now, let

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0 \\ \mathbf{K}_1 \\ \mathbf{K}_2 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} \mathbf{L}_0 \\ \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix},$$

where $\mathbf{K}_0, \mathbf{K}_1$ and $\mathbf{K}_2$ is an $m_0 d \times nd$, $m_1 d \times nd$ and $m_1 d \times nd$ matrix, respectively, etc. By the definition of $\mathbf{H}$,

$$\mathbf{H}_0 = (1-u)t\mathbf{K}_0. \tag{7}$$

Furthermore, by the definition of $\mathbf{L}$,

$$\mathbf{L}_1 = \mathbf{H}_1\mathbf{X}_d \quad \text{and} \quad \mathbf{L}_2 = \mathbf{H}_2\mathbf{X}_d,$$

where $\mathbf{X}_d = \mathbf{X} \otimes \mathbf{I}_d$.

Let $f_j = e_{m_0+j}$ $(1 \leqslant j \leqslant m_1)$. Furthermore, let $\mathbf{N}_1 = \rho(\alpha(f_1)) \oplus \cdots \oplus \rho(\alpha(f_{m_1}))$ and $\mathbf{N}_2 = \rho(\alpha(f_1^{-1})) \oplus \cdots \oplus \rho(\alpha(f_{m_1}^{-1}))$. By the definitions of $\mathbf{K}$ and $\mathbf{H}$, we have

$$\mathbf{H}_1 = \mathbf{N}_1\mathbf{K}_2 \quad \text{and} \quad \mathbf{H}_2 = \mathbf{N}_2\mathbf{K}_1. \tag{8}$$

But, we have

$$\mathbf{L}{}^t\mathbf{K} = \begin{bmatrix} \mathbf{L}_0{}^t\mathbf{K}_0 & \mathbf{L}_0{}^t\mathbf{K}_1 & \mathbf{L}_0{}^t\mathbf{K}_2 \\ \mathbf{L}_1{}^t\mathbf{K}_0 & \mathbf{L}_1{}^t\mathbf{K}_1 & \mathbf{L}_1{}^t\mathbf{K}_2 \\ \mathbf{L}_2{}^t\mathbf{K}_0 & \mathbf{L}_2{}^t\mathbf{K}_1 & \mathbf{L}_2{}^t\mathbf{K}_2 \end{bmatrix} = {}^t\mathbf{B}_\rho, \tag{9}$$

where the matrix $\mathbf{B}_\rho = (B_{ef})_{e,f\in A(D)}$ is given by

$$B_{ef} := \begin{cases} x_{t(e)}\sqrt{w(e)w(f)}\rho(\alpha(e)) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbf{B}_\rho = \sum_{g\in\Gamma} \mathbf{B}_g \otimes \rho(g)$.

We introduce two $(m+n)d \times (m+n)d$ matrices as follows:

$$\mathbf{P} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{nd} & -{}^t\mathbf{K} + (1-u)t^t\mathbf{H} \\ \mathbf{0} & \mathbf{I}_{md} \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{I}_{nd} & {}^t\mathbf{K} - (1-u)t^t\mathbf{H} \\ t\mathbf{L} & (1-(1-u)^2t^2)\mathbf{I}_{md} \end{bmatrix}$$

By (4) and (6), we have

$$\mathbf{PQ} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{nd} - t^t\mathbf{KL} + (1-u)t^{2t}\mathbf{HL} & \mathbf{0} \\ t\mathbf{L} & (1-(1-u)^2t^2)\mathbf{I}_{md} \end{bmatrix}$$
$$= \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{nd} - t({}^t\mathbf{W}_{1,\rho}^{(x)} + {}^t\mathbf{W}_{0,\rho}^{(x)}) + (1-u)t^2(\mathbf{S}_x \otimes \mathbf{I}_d + (1-u)t^t\mathbf{W}_{0,\rho}^{(x)}) & \mathbf{0} \\ t\mathbf{L} & (1-(1-u)^2t^2)\mathbf{I}_{md} \end{bmatrix}.$$

Furthermore,

$$\mathbf{QP} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{nd} & \mathbf{0} \\ t(1-(1-u)^2t^2)\mathbf{L} & -t\mathbf{L}^t\mathbf{K} + (1-u)t^2\mathbf{L}^t\mathbf{H} + (1-(1-u)^2t^2)\mathbf{I}_{md} \end{bmatrix}.$$

Then, by (7), we have

$$\mathbf{L}^t\mathbf{H} = \begin{bmatrix} (1-u)t\mathbf{L}_0{}^t\mathbf{K}_0 & \mathbf{L}_0{}^t\mathbf{H}_1 & \mathbf{L}_0{}^t\mathbf{H}_2 \\ (1-u)t\mathbf{L}_1{}^t\mathbf{K}_0 & \mathbf{L}_1{}^t\mathbf{H}_1 & \mathbf{L}_1{}^t\mathbf{H}_2 \\ (1-u)t\mathbf{L}_2{}^t\mathbf{K}_0 & \mathbf{L}_2{}^t\mathbf{H}_1 & \mathbf{L}_2{}^t\mathbf{H}_2 \end{bmatrix}. \tag{10}$$

Now, let

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_1 \\ \mathbf{0} & \mathbf{M}_2 & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{N} = \mathbf{B}_\rho - \mathbf{M},$$

where $\mathbf{M}_1 = w(f_1)x_{o(f_1^{-1})}\rho(\alpha(f_1)) \oplus \cdots \oplus w(f_{m_1})x_{o(f_{m_1}^{-1})}\rho(\alpha(f_{m_1}))$ and $\mathbf{M}_2 = w(f_1)x_{o(f_1)}\rho(\alpha(f_1^{-1})) \oplus \cdots \oplus w(f_{m_1})x_{o(f_{m_1})}\rho(\alpha(f_{m_1}^{-1}))$. Furthermore, let

$$\mathbf{M}_0 = (1-u)t\mathbf{I}_{m_0d} \oplus \mathbf{0}_{2m_1d} + \mathbf{J}_\rho,$$

where

$$\mathbf{J}_\rho = \sum_{g \in \Gamma} \mathbf{J}_g \otimes \rho(g) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{N}_1 \\ \mathbf{0} & \mathbf{N}_2 & \mathbf{0} \end{bmatrix}.$$

Then, by (8) and (9), we have

$${}^t\mathbf{N}^t\mathbf{M}_0 = \begin{bmatrix} (1-u)t\mathbf{L}_0{}^t\mathbf{K}_0 & \mathbf{L}_0{}^t\mathbf{K}_2{}^t\mathbf{N}_1 & \mathbf{L}_0{}^t\mathbf{K}_1{}^t\mathbf{N}_2 \\ (1-u)t\mathbf{L}_1{}^t\mathbf{K}_0 & \mathbf{L}_1{}^t\mathbf{K}_2{}^t\mathbf{N}_1 - {}^t\mathbf{M}_2{}^t\mathbf{N}_1 & \mathbf{L}_1{}^t\mathbf{K}_1{}^t\mathbf{N}_2 \\ (1-u)t\mathbf{L}_2{}^t\mathbf{K}_0 & \mathbf{L}_2{}^t\mathbf{K}_2{}^t\mathbf{N}_1 & \mathbf{L}_2{}^t\mathbf{K}_1{}^t\mathbf{N}_2 - {}^t\mathbf{M}_1{}^t\mathbf{N}_2 \end{bmatrix}$$
$$= \begin{bmatrix} (1-u)t\mathbf{L}_0{}^t\mathbf{K}_0 & \mathbf{L}_0{}^t\mathbf{H}_1 & \mathbf{L}_0{}^t\mathbf{H}_2 \\ (1-u)t\mathbf{L}_1{}^t\mathbf{K}_0 & \mathbf{L}_1{}^t\mathbf{H}_1 - {}^t\mathbf{M}_2{}^t\mathbf{N}_1 & \mathbf{L}_1{}^t\mathbf{H}_2 \\ (1-u)t\mathbf{L}_2{}^t\mathbf{K}_0 & \mathbf{L}_2{}^t\mathbf{H}_1 & \mathbf{L}_2{}^t\mathbf{H}_2 - {}^t\mathbf{M}_1{}^t\mathbf{N}_2 \end{bmatrix}.$$

Since

$${}^t\mathbf{M}_2{}^t\mathbf{N}_1 = w(f_1)x_{o(f_1)}\mathbf{I}_d \oplus \cdots \oplus w(f_{m_1})x_{o(f_{m_1})}\mathbf{I}_d$$

and

$$ {}^t\mathbf{M}_1{}^t\mathbf{N}_2 = w(f_1)x_{o(f_1^{-1})}\mathbf{I}_d \oplus \cdots \oplus w(f_{m_1})x_{o(f_{m_1}^{-1})}\mathbf{I}_d. $$

Thus, by (10),

$$ \mathbf{L}{}^t\mathbf{H} = {}^t\mathbf{N}{}^t\mathbf{M}_0 + \left(\mathbf{0}_{m_0d} \oplus w(f_1)x_{o(f_1)}\mathbf{I}_d \oplus \cdots \oplus w(f_{m_1})x_{o(f_{m_1}^{-1})}\mathbf{I}_d\right). \tag{11} $$

But, we have

$$ {}^t\mathbf{M}{}^t\mathbf{M}_0 = \mathbf{0}_{m_0d} \oplus w(f_1)x_{o(f_1)}\mathbf{I}_d \oplus \cdots \oplus w(f_{m_1})x_{o(f_{m_1}^{-1})}\mathbf{I}_d $$

and

$$ {}^t\mathbf{J}_\rho{}^t\mathbf{M}_0 = \mathbf{0}_{m_0d} \oplus \mathbf{I}_{2m_1d}. $$

By (9) and (11), we have

$$ \begin{aligned} &-t\mathbf{L}{}^t\mathbf{K} + (1-u)t^2\mathbf{L}{}^t\mathbf{H} + \left(1-(1-u)^2t^2\right)\mathbf{I}_{md} \\ &\quad = \mathbf{I}_{md} - t\left({}^t\mathbf{N} + {}^t\mathbf{M}\right) + (1-u)t^2\left({}^t\mathbf{N}{}^t\mathbf{M}_0 + {}^t\mathbf{M}{}^t\mathbf{M}_0\right) \\ &\qquad - (1-u)t\left({}^t\mathbf{M}_0 - {}^t\mathbf{J}_\rho\right) - (1-u)^2t^{2t}\mathbf{J}_\rho{}^t\mathbf{M}_0 \\ &\quad = \left(\mathbf{I}_{md} - t\left({}^t\mathbf{N} + {}^t\mathbf{M} - (1-u){}^t\mathbf{J}_\rho\right)\right)\left(\mathbf{I}_{md} - (1-u)t{}^t\mathbf{M}_0\right) \\ &\quad = \left(\mathbf{I}_{md} - t\left({}^t\mathbf{B}_\rho - (1-u){}^t\mathbf{J}_\rho\right)\right)\left(\mathbf{I}_{md} - (1-u)t{}^t\mathbf{M}_0\right). \end{aligned} $$

Thus,

$$ \mathbf{QP} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{nd} & \mathbf{0} \\ t(1-(1-u)^2t^2)\mathbf{L} & (\mathbf{I}_{md} - t({}^t\mathbf{B}_\rho - (1-u){}^t\mathbf{J}_\rho))(\mathbf{I}_{md} - (1-u)t{}^t\mathbf{M}_0) \end{bmatrix}. $$

Since $\det(\mathbf{PQ}) = \det(\mathbf{QP})$, we have

$$ \begin{aligned} &\left(1-(1-u)^2t^2\right)^{md}\det\left(\mathbf{I}_{nd} - t{}^t\mathbf{W}_{1,\rho}^{(x)} - \left(1-(1-u)^2t^2\right)t{}^t\mathbf{W}_{0,\rho}^{(x)}\right. \\ &\qquad \left. + (1-u)t^2\left(\mathbf{S}_x \otimes \mathbf{I}_d - (1-u)\mathbf{I}_{nd}\right)\right) \\ &\quad = \left(1-(1-u)^2t^2\right)^{nd}\det\left(\mathbf{I}_{md} - t\left({}^t\mathbf{B}_\rho - (1-u){}^t\mathbf{J}_\rho\right)\right)\det\left(\mathbf{I}_{md} - (1-u)t{}^t\mathbf{M}_0\right). \end{aligned} $$

But,

$$ \begin{aligned} &\det\left(\mathbf{I}_{md} - (1-u)t{}^t\mathbf{M}_0\right) \\ &\quad = \det\left(\begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{m_0d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_1d} & -(1-u)t{}^t\mathbf{N}_2 \\ \mathbf{0} & -(1-u)t{}^t\mathbf{N}_1 & \mathbf{I}_{m_1d} \end{bmatrix}\right) \\ &\qquad \times \det\left(\begin{bmatrix} \mathbf{I}_{m_0d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_1d} & (1-u)t{}^t\mathbf{N}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{m_1d} \end{bmatrix}\right) \\ &\quad = \det\left(\begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{m_0d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_1d} & \mathbf{0} \\ \mathbf{0} & -(1-u)t{}^t\mathbf{N}_1 & (1-(1-u)^2t^2)\mathbf{I}_{m_1d} \end{bmatrix}\right) \\ &\quad = \left(1-(1-u)^2t^2\right)^{(m_0+m_1)d}. \end{aligned} $$

Therefore it follows that

$$\begin{aligned}&\bigl(1-(1-u)^2t^2\bigr)^{(m_0+2m_1)d}\det\bigl(\mathbf{I}_{nd}-t^t\mathbf{W}^{(x)}_{1,\rho}-\bigl(1-(1-u)^2t^2\bigr)t^t\mathbf{W}^{(x)}_{0,\rho}\\&\quad+(1-u)t^2\bigl(\mathbf{S}_x\otimes\mathbf{I}_d-(1-u)\mathbf{I}_{nd}\bigr)\bigr)\\&\quad=\bigl(1-(1-u)^2t^2\bigr)^{(m_0+m_1+n)d}\det\bigl(\mathbf{I}_{md}-t\bigl({}^t\mathbf{B}_\rho-(1-u)^t\mathbf{J}_\rho\bigr)\bigr).\end{aligned}$$

But, we have

$$\mathbf{S}_x=\mathbf{X}\mathbf{S}_w.$$

By (5), it follows that

$$\begin{aligned}&\zeta_2(D,w,x,u,t,\rho,\alpha)^{-1}\\&\quad=\bigl(1-(1-u)^2t^2\bigr)^{(m_1-n)d}\det\Biggl(\mathbf{I}_{nd}-t\sum_{g\in\Gamma}\rho(g)\otimes\mathbf{X}\mathbf{W}_{1,g}\\&\qquad-\bigl(1-(1-u)^2t^2\bigr)t\sum_{g\in\Gamma}\rho(g)\otimes\mathbf{X}\mathbf{W}_{0,g}+(1-u)t^2\bigl(\mathbf{I}_d\otimes\bigl(\mathbf{X}\mathbf{S}_w-(1-u)\mathbf{I}_n\bigr)\bigr)\Biggr).\qquad\square\end{aligned}$$

By Theorems 5, 6, the following result holds.

Corollary 2. *Let D be a connected digraph, $\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D, Γ a finite group and $\alpha: A(D)\longrightarrow\Gamma$ a pseudo ordinary voltage assignment. Then we have*

$$\zeta_2\bigl(D^\alpha,\tilde{w},\tilde{x},u,t\bigr)=\prod_\rho\zeta_2(D,w,x,u,t,\rho,\alpha)^{\deg\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

6. The Euler product for the L-function $\zeta_2(D,w,x,u,t,\rho,\alpha)$ of a digraph

We present the Euler product for the L-function of a digraph introduced in Section 5.

Foata and Zeilberger [5] gave a new proof of Bass' Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ a total order in X, and X^* the free monoid generated by X. Then the total order $<$ on X derive the lexicographic order $<^*$ on X^*. A *Lyndon word* in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l for any $r\geqslant 2$, and which is also minimal in the class of its cyclic rearrangements under $<^*$ (see [11]). Let L denote the set of all Lyndon words in X.

Foata and Zeilberger [5] gave a short proof of Amitsur's identity [1].

Theorem 7 *(Amitsur). For square matrices $\mathbf{A}_1,\dots,\mathbf{A}_k$,*

$$\det\bigl(\mathbf{I}-(\mathbf{A}_1+\cdots+\mathbf{A}_k)\bigr)=\prod_{l\in L}\det(\mathbf{I}-\mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1,\dots,k\}$, and $\mathbf{A}_l=\mathbf{A}_{i_1}\cdots\mathbf{A}_{i_p}$ for $l=i_1\cdots i_p$.

Theorem 8. *Let D be a connected digraph with m arcs, $\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D, $x: V(D)\longrightarrow\mathbf{C}$ a function, Γ a finite group and $\alpha: A(D)\longrightarrow\Gamma$ a pseudo ordinary voltage assignment. For each path $P=(e_1,\dots,e_r)$ of G, set $\alpha(P)=\alpha(e_1)\cdots\alpha(e_r)$. Furthermore, let ρ be a representation of Γ and d the degree of ρ. Then*

$$\zeta_2(D,w,x,u,t,\rho,\alpha)=\prod_{[C]}\det\bigl(\mathbf{I}_d-\rho\bigl(\alpha(C)\bigr)a_{w,x}(C)t^{|C|}\bigr)^{-1},$$

where $[C]$ *runs over all equivalence classes of prime cycles of* D, *and*

$$a_{w,x}(C)=\sigma_{e_1e_2}\sigma_{e_2e_3}\cdots\sigma_{e_{n-1}e_n}\sigma_{e_ne_1},\quad C=(e_1,e_2,\dots,e_n).$$

Furthermore,

$$\sigma_{ef}=x\bigl(t(e)\bigr)\sqrt{w(e)}\sqrt{w(f)}-(1-u)\delta_{e^{-1}f}\epsilon_{e^{-1}}.$$

Proof. At first, let $A(D)=\{e_1,\dots,e_{m_0},e_{m_0+1},\dots,e_{m_0+m_1},e_{m_0+m_1+1},\dots,e_{m_0+2m_1}\}$ such that $e_i^{-1}\notin A(D)$ and $e_{m_0+m_1+j}=e_{m_0+j}^{-1}$ $(1\leqslant i\leqslant m_0;\ 1\leqslant j\leqslant m_1)$. We consider the lexicographic order on $A(D)\times A(D)$ derived from a total order of $A(D)$: $e_1<e_2<\cdots<e_m$. If (e_i,e_j) is the r-th pair under the above order, then we define the $md\times md$ matrix $\mathbf{T}_r=((\mathbf{T}_r)_{p,q})_{1\leqslant p,q\leqslant m}$ as follows:

$$(\mathbf{T}_r)_{p,q}=\begin{cases}\rho(\alpha(e_i))\sigma_{e_ie_j}t & \text{if } p=e_i,\ q=e_j \text{ and } t(e_i)=o(e_j),\\ \mathbf{0} & \text{otherwise,}\end{cases}$$

where

$$\sigma_{ef}=x_{t(e)}\sqrt{w(e)}\sqrt{w(f)}-(1-u)\delta_{e^{-1}f}\epsilon_{e^{-1}}.$$

Let $\mathbf{F}=\mathbf{T}_1+\cdots+\mathbf{T}_k$, $k=m^2$. Then we have

$$\mathbf{F}=t\sum_h\bigl(\mathbf{B}_h-(1-u)\mathbf{J}_h\bigr)\otimes\rho(h).$$

Let L be the set of all Lyndon words in $A(D)\times A(D)$. Then we can also consider L as the set of all Lyndon words in $\{1,\dots,k\}$: $(e_{i_1},e_{j_1})\cdots(e_{i_q},e_{j_q})$ corresponds to $m_1m_2\cdots m_q$, where (e_{i_r},e_{j_r}) $(1\leqslant r\leqslant q)$ is the m_r-th pair. Theorem 7 implies that

$$\det(\mathbf{I}_{md}-\mathbf{F})=\prod_{t\in L}\det(\mathbf{I}_{md}-\mathbf{T}_t),$$

where

$$\mathbf{T}_t=\mathbf{T}_{i_1}\cdots\mathbf{T}_{i_p}$$

for $t=i_1\cdots i_p$. Note that $\det(\mathbf{I}_{md}-\mathbf{T}_t)$ is the alternating sum of the diagonal minors of $\mathbf{T}_t$. Thus, we have

$$\det(\mathbf{I}-\mathbf{T}_t)=\begin{cases}\det(\mathbf{I}-\rho(\alpha(C))a_{w,x}(C)t^{|C|}) & \text{if } t \text{ is a prime cycle } C,\\ 1 & \text{otherwise,}\end{cases}$$

where

$$a_{w,x}(C)=\sigma_{e_1e_2}\sigma_{e_2e_3}\cdots\sigma_{e_{n-1}e_n}\sigma_{e_ne_1},\quad C=(e_1,e_2,\dots,e_n).$$

Therefore, it follows that

$$\begin{aligned}\zeta_2(D,w,x,u,t,\rho,\alpha)^{-1}&=\det\Bigl(\mathbf{I}_{md}-t\sum_{h\in\Gamma}\rho(h)\otimes\bigl(\mathbf{B}_h-(1-u)\mathbf{J}_h\bigr)\Bigr)\\&=\det\Bigl(\mathbf{I}_{md}-t\sum_{h\in\Gamma}\bigl(\mathbf{B}_h-(1-u)\mathbf{J}_h\bigr)\otimes\rho(h)\Bigr)\\&=\prod_{[C]}\det\bigl(\mathbf{I}_d-\rho\bigl(\alpha(C)\bigr)a_{w,x}(C)t^{|C|}\bigr),\end{aligned}$$

where $[C]$ runs over all equivalence classes of prime cycles of D. □

Mizuno and Sato [13] defined a new L-function of a digraph by using not an infinite product but a determinant. Let D be a connected digraph with m arcs, $\mathbf{W}=\mathbf{W}(D)$ a pseudo-symmetric weighted matrix of G, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment. Furthermore, let ρ be a unitary representation of Γ and d its degree. The *L-function* of D associated with ρ and α is defined by

$$\zeta_1(D, w, u, t, \rho, \alpha) = \det\left(\mathbf{I}_{md} - t\sum_{h\in\Gamma}\rho(h)\otimes\left(\mathbf{B}'_h - (1-u)\mathbf{J}_h\right)\right)^{-1},$$

where $\mathbf{B}'_g = (\mathbf{B}^{(\prime,g)}_{ef})_{e,f\in A(D)}$ is the $m\times m$ matrix given by

$$\mathbf{B}^{(\prime,g)}_{ef} = \begin{cases} w(f) & \text{if } t(e)=o(f) \text{ and } \alpha(e)=g, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the Euler product for $\zeta_1(D, w, u, t, \rho, \alpha)$ is given as follows:

Theorem 9. *Let D be a connected digraph, $\mathbf{W}(D)$ a pseudo-symmetric weighted matrix of D, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment. For each path $P=(e_1,\dots,e_r)$ of G, set $\alpha(P)=\alpha(e_1)\cdots\alpha(e_r)$. Furthermore, let ρ be a representation of Γ and d the degree of ρ. Then*

$$\zeta_1(D, w, u, t, \rho, \alpha) = \prod_{[C]}\det\left(\mathbf{I}_d - \rho\left(\alpha(C)\right)b_w(C)t^{|C|}\right)^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D, and

$$b_w(C) = \tau_{e_1e_2}\tau_{e_2e_3}\cdots\tau_{e_{n-1}e_n}\tau_{e_ne_1}, \quad C=(e_1,e_2,\dots,e_n).$$

Furthermore,

$$\tau_{ef} = w(f) - (1-u)\delta_{e^{-1}f}\epsilon_{e^{-1}}.$$

By Theorems 8 and 9, we obtain the following result.

Corollary 3. *Let D be a connected digraph, $\mathbf{W}(D)$ a non-negative pseudo-symmetric weighted matrix of D, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo ordinary voltage assignment. For each path $P=(e_1,\dots,e_r)$ of G, set $\alpha(P)=\alpha(e_1)\cdots\alpha(e_r)$. Furthermore, let ρ be a representation of Γ and d the degree of ρ. If $x=\mathbf{1}$, then*

$$\zeta_2(D, w, \mathbf{1}, u, t, \rho, \alpha) = \zeta_1(D, w, u, t, \rho, \alpha).$$

Proof. Let $C=(e_1,\dots,e_n)$ be any prime cycle of D. Then we have

$$a_{w,\mathbf{1}}(C) = \sigma_{e_1e_2}\cdots\sigma_{e_{n-1}e_n}\sigma_{e_ne_1} \quad \text{and} \quad b_w(C) = \tau_{e_1e_2}\cdots\tau_{e_{n-1}e_n}\tau_{e_ne_1},$$

where $\sigma_{e_ie_{i+1}} = \sqrt{w(e_i)}\sqrt{w(e_{i+1})} - (1-u)\delta_{e_i^{-1}e_{i+1}}\epsilon_{e_i^{-1}}$, $\tau_{e_ie_{i+1}} = w(e_{i+1}) - (1-u)\delta_{e_i^{-1}e_{i+1}}\epsilon_{e_i^{-1}}$ and indices are read mod n.

If $e_{j+1}=e_j^{-1}$ for some j $(1\leqslant j\leqslant n)$, then we have

$$\begin{aligned} a_{w,\mathbf{1}}(C) &= \cdots\sqrt{w(e_{j-1})}\sqrt{w(e_j)}\left(\sqrt{w(e_j)}\sqrt{w(e_{j+1})} - (1-u)\right)\sqrt{w(e_{j+1})}\sqrt{w(e_{j+2})}\cdots \\ &= \cdots w(e_j)\left(w(e_j) - (1-u)\right)\cdots \end{aligned}$$

and

$$b_w(C) = \cdots w(e_j)\left(w(e_{j+1}) - (1-u)\right)\cdots = \cdots w(e_j)\left(w(e_j) - (1-u)\right)\cdots.$$

Thus,

$$a_{w,\mathbf{1}}(C) = b_w(C).$$

By Theorems 8 and 9, it follows that

$$\zeta_2(D, w, \mathbf{1}, u, t, \rho, \alpha) = \zeta_1(D, w, u, t, \rho, \alpha). \qquad \square$$

If $\rho = \mathbf{1}$, then we obtain Corollary 1.

7. Example

Finally, we give an example. Let D be the digraph with three vertices v_1, v_2, v_3 and five arcs (v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2), (v_3, v_1). Furthermore, let

$$\mathbf{W}(D) = \begin{bmatrix} 0 & a & 0 \\ a & 0 & b \\ c & b & 0 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}.$$

Then we have $n = 3$, $m = 5$, $m_1 = 2$. By Theorem 4, we have

$$\begin{aligned} &\zeta_2(D, w, x, u, t)^{-1} \\ &= \left(1 - (1-u)^2t^2\right)^{-1} \det\left(\mathbf{I}_3 - t\mathbf{X}\mathbf{W}_1 - \left(1 - (1-u)^2t^2\right)t\mathbf{X}\mathbf{W}_0 \right.\\ &\quad \left. + (1-u)t^2\left(\mathbf{X}\mathbf{S}_w - (1-u)\mathbf{I}_3\right)\right) \\ &= \left(1 - (1-u)^2t^2\right)^{-1} \\ &\quad \times \det \begin{bmatrix} 1 + (1-u)(ax_1 - 1 + u)t^2 & -ax_1t & 0 \\ -ax_2t & 1 + (1-u)(ax_2 + bx_2 - 1 + u)t^2 & -bx_2t \\ -(1 - (1-u)^2t^2)cx_3t & -bx_3t & 1 + (1-u)(bx_3 - 1 + u)t^2 \end{bmatrix} \\ &= 1 - \left(2(1-u)^2 - (1-u)\left(a(x_1 + x_2) + b(x_2 + x_3)\right) + x_2\left(a^2x_1 + b^2x_3\right)\right)t^2 \\ &\quad + (u-1)(ax_1 - 1 + u)\left((a+b)x_2 - 1 + u\right)(bx_3 - 1 + u)t^4 - abcx_1x_2x_3t^3. \end{aligned}$$

Let $\Gamma = Z_3 = \{1, \tau, \tau^2\}$ $(\tau^3 = 1)$ be the cyclic group of order 3, and let $\alpha : A(D) \longrightarrow Z_3$ be the pseudo ordinary voltage assignment such that $\alpha(v_1, v_2) = \tau$, $\alpha(v_2, v_1) = \tau^2$ and $\alpha(v_2, v_3) = \alpha(v_3, v_2) = \alpha(v_3, v_1) = 1$. The characters of $\mathbf{Z}_3$ are given as follows: $\chi_i(\tau^j) = (\xi^i)^j$, $0 \leqslant i, j \leqslant 2$, where $\xi = \frac{-1+\sqrt{-3}}{2}$.

Now, we present the L-function $\zeta_2(D, w, u, t, \chi_1, \alpha)$ of D associated with χ_1 and α. Theorem 6 implies that

$$\begin{aligned} &\zeta_2(D, w, x, u, t, \chi_1, \alpha)^{-1} \\ &= \left(1 - (1-u)^2t^2\right)^{-1} \det\left(\mathbf{I}_3 - t\sum_{i=0}^{2} \chi_1(\tau^i)\mathbf{X}\mathbf{W}_{1,\tau^i} \right.\\ &\quad \left. - \left(1 - (1-u)^2t^2\right)t\sum_{i=0}^{2} \chi_1(\tau^i)\mathbf{X}\mathbf{W}_{0,\tau^i} + (1-u)t^2\left(\mathbf{X}\mathbf{S}_w - (1-u)\mathbf{I}_3\right)\right) \\ &= \left(1 - (1-u)^2t^2\right)^{-1} \\ &\quad \times \det \begin{bmatrix} 1 + (1-u)(ax_1 - 1 + u)t^2 & -ax_1\xi t & 0 \\ -ax_2\xi^2t & 1 + (1-u)(ax_2 + bx_2 - 1 + u)t^2 & -bx_2t \\ -(1 - (1-u)^2t^2)cx_3t & -bx_3t & 1 + (1-u)(bx_3 - 1 + u)t^2 \end{bmatrix} \\ &= 1 - \left(2(1-u)^2 - (1-u)\left(a(x_1 + x_2) + b(x_2 + x_3)\right) + x_2\left(a^2x_1 + b^2x_3\right)\right)t^2 \\ &\quad + (u-1)(ax_1 - 1 + u)\left((a+b)x_2 - 1 + u\right)(bx_3 - 1 + u)t^4 - abcx_1x_2x_3\xi t^3. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\zeta_2(D, w, x, u, t, \chi_2, \alpha)^{-1} \\
&\quad = 1 - \big(2(1-u)^2 - (1-u)\big(a(x_1+x_2) + b(x_2+x_3)\big) + x_2\big(a^2x_1 + b^2x_3\big)\big)t^2 \\
&\qquad + (u-1)(ax_1 - 1 + u)\big((a+b)x_2 - 1 + u\big)(bx_3 - 1 + u)t^4 - abcx_1x_2x_3\xi^2t^3.
\end{aligned}$$

By Corollary 2, it follows that

$$\begin{aligned}
&\zeta_2\big(D^\alpha, \tilde{w}, \tilde{x}, u, t\big)^{-1} \\
&\quad = \zeta_2(D, w, x, u, t)^{-1}\zeta_2(D, w, x, u, t, \chi_1, \alpha)^{-1}\zeta_2(D, w, x, u, t, \chi_2, \alpha)^{-1} \\
&\quad = \big(1 - \big(2(1-u)^2 - (1-u)\big(a(x_1+x_2) + b(x_2+x_3)\big) + x_2\big(a^2x_1 + b^2x_3\big)\big)t^2 \\
&\qquad + (u-1)(ax_1 - 1 + u)\big((a+b)x_2 - 1 + u\big)(bx_3 - 1 + u)t^4\big)^3 - a^3b^3c^3x_1^3x_2^3x_3^3t^9.
\end{aligned}$$

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