

# Norm estimates for a Keakeya-type maximal operator

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We prove  $L^p \rightarrow L^q$  estimates for the 2-dimensional analog of the Keakeya maximal function.

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## 1 Introduction

Let  $\mathcal{G}_{n,2}$  be the Grassmannian manifold of all 2-dimensional linear subspaces of  $\mathbb{R}^n$  equipped with the unique probability measure  $\varphi_{n,2}$  which is invariant under the action of the orthogonal group (see, for example, [2]). For any locally integrable function  $f$  and  $0 < \delta \ll 1$  we define

$$\mathcal{M}_\delta f : \mathcal{G}_{n,2} \longrightarrow \mathbb{R}$$

by

$$\mathcal{M}_\delta f(\Pi) = \sup_{a \in \mathbb{R}^n} \frac{1}{|\Pi^\delta(a)|} \int_{\Pi^\delta(a)} |f(y)| dy,$$

where  $\Pi^\delta(a)$  is the  $\delta/2$ -neighborhood of the intersection of the plane  $\Pi + a$  with the ball of radius  $1/2$  centered at  $a$ , and  $|\Pi^\delta(a)|$  is the volume of  $\Pi^\delta(a)$ . Thus,  $\mathcal{M}_\delta f(\Pi)$  is, essentially, the maximal average of  $f$  over the neighborhoods of all 2-dimensional discs of unit diameter which are parallel to the plane  $\Pi$ . This operator is a natural variant of the Keakeya maximal function introduced by Bourgain [1]. In that context, one averages over neighborhoods of line segments of unit length. The mapping properties of the Keakeya maximal function have been the subject of a great deal of research. We refer the reader to the expository articles [5] and [6] for extensive accounts.

In this paper we are interested in proving  $L^p \rightarrow L^q(\mathcal{G}_{n,2}, \varphi_{n,2})$  estimates for  $\mathcal{M}_\delta$ . To find the optimal range for  $p$  and  $q$  we argue as follows.

If  $f$  is the characteristic function of a ball of radius  $\delta$ , then  $\|f\|_p$  is comparable to  $\delta^{n/p}$ , and  $\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2}, \varphi_{n,2})}$  is comparable to  $\delta^2$ . Therefore, it seems reasonable to expect a bound of the form

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2}, \varphi_{n,2})} \leq C_{n,p,q} \delta^{2-n/p} \|f\|_p, \quad p \leq n/2, \quad q \geq 1.$$

On the other hand, if  $f$  is the characteristic function of a rectangle of dimensions  $1 \times 1 \times \delta \times \dots \times \delta$ , then  $\|f\|_p = \delta^{(n-2)/p}$  and  $\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2}, \varphi_{n,2})}$  is, up to a multiplicative constant, greater than  $\delta^{2(n-2)/q}$ . It follows that if the above estimate is true, we must have

$$\delta^{2(n-2)/q} \leq C_{n,p,q} \delta^{2-n/p} \delta^{(n-2)/p},$$

which implies  $q \leq (n-2)p'$ , where  $p'$  is the conjugate exponent of  $p$ .

These examples would reasonably lead to the following.

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**Conjecture 1.1** For every  $\epsilon > 0$  there exists a constant  $C_{\epsilon,n,p,q} > 0$  such that

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2}, \varphi_{n,2})} \leq C_{\epsilon,n,p,q} \delta^{2-n/p-\epsilon} \|f\|_p,$$

where

$$1 \leq p \leq \frac{n}{2}, \quad q \leq (n-2)p'.$$

We remark that this conjecture, if true, would imply the following geometric result which was proved by the author in [3]. Suppose  $E$  is a subset of  $\mathbb{R}^n$  containing a translate of every plane, then the Hausdorff dimension of  $E$  equals  $n$ .

The purpose of this paper is to verify the conjecture when the range of  $p$  is a smaller interval. Namely, we use geometric-combinatorial ideas in the spirit of [1], to prove the following.

**Theorem 1.2** For every  $\epsilon > 0$  there exists a constant  $C_{\epsilon,n,p,q} > 0$  such that

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2}, \varphi_{n,2})} \leq C_{\epsilon,n,p,q} \delta^{2-n/p-\epsilon} \|f\|_p,$$

where

$$1 \leq p < \frac{n+1}{3}, \quad q \leq (n-2)p'.$$

## 2 Preliminaries

For the rest of the paper, the capital letter  $C$  will denote various positive constants whose values may change from line to line. Similarly  $C_\epsilon$  will denote constants depending on  $\epsilon$ .  $x \lesssim y$  means  $x \leq Cy$ , and  $x \simeq y$  means ( $x \lesssim y$  and  $y \lesssim x$ ). Also, we will use the notation  $\Pi^\delta$  for any set  $\Pi^\delta(a)$ , since the basepoint  $a$  is irrelevant in all our arguments. Further notational conventions follow below.

$S^{n-1}$  is the  $(n-1)$ -dimensional unit sphere.

$B(a, r)$  is the ball of radius  $r$  centered at  $a$ .

$A(a, r)$  is the annulus  $B(a, 2r) \setminus B(a, r)$ .

$L_e(a)$  is the line in the direction  $e \in S^{n-1}$  passing through the point  $a$ , i.e.,

$$L_e(a) = \{a + te : t \in \mathbb{R}\}.$$

$T_e^{(r)(\beta)}(a)$  is the tube of length  $r$ , cross-section radius  $\beta$ , centered at  $a$ , and with axis in the direction  $e \in S^{n-1}$ , i.e.,

$$T_e^{(r)(\beta)}(a) = \{x \in \mathbb{R}^n : \text{dist}(x, L_e(a)) \leq \beta \text{ and } |\text{proj}_{L_e(a)}(x) - a| \leq r/2\},$$

where  $\text{proj}_{L_e(a)}(x)$  is the orthogonal projection of  $x$  onto  $L_e(a)$ .

$\chi_E$  is the characteristic function of the set  $E$ .

$|\cdot|$  denotes Lebesgue measure or cardinality, depending on the context.

If  $\Pi_1, \Pi_2 \in \mathcal{G}_{n,2}$ , then their distance  $\theta$  is defined by

$$\theta(\Pi_1, \Pi_2) = \|\text{proj}_{\Pi_1} - \text{proj}_{\Pi_2}\|,$$

where  $\|\cdot\|$  is the operator norm.  $\varphi_{n,2}$  is a  $2(n-2)$ -dimensional Ahlfors-regular measure with respect to this distance (see [2]), in the sense that

$$\varphi_{n,2}(\{\Pi \in \mathcal{G}_{n,2} : \theta(\Pi, \Pi_0) \leq r\}) \simeq r^{2(n-2)} \quad \text{for all } \Pi_0 \in \mathcal{G}_{n,2}, \quad r < 1.$$

A finite subset of  $\mathcal{G}_{n,2}$  is called  $\delta$ -separated if the distance between any two of its elements is at least  $\delta$ . So, if  $\mathcal{B}$  is a maximal  $\delta$ -separated subset of  $\mathcal{A} \subset \mathcal{G}_{n,2}$ , then

$$\varphi_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}| \delta^{2(n-2)}.$$

Moreover, if  $\mathcal{A} \subset \mathcal{G}_{n,2}$  is  $\delta$ -separated, and  $\mathcal{B}$  is a maximal  $\eta$ -separated subset of  $\mathcal{A}$  with  $\eta \geq \delta$ , then

$$|\mathcal{B}| \gtrsim |\mathcal{A}| (\delta/\eta)^{2(n-2)}.$$

For technical reasons, we introduce the following subset of  $\mathcal{G}_{n,2}$ :

$$\mathcal{A}_{n,2} := \{ \Pi \in \mathcal{G}_{n,2} : \theta(\Pi, x_1x_2\text{-plane}) \leq 1/4 \}.$$

We note that by invariance, it is enough to prove Theorem 1.2 for  $\mathcal{M}_\delta$  restricted to  $\mathcal{A}_{n,2}$ .

We close this section with two lemmas that will allow us to control the intersection properties and the cardinality of a family of sets  $\Pi^\delta$  containing a fixed line segment. They can be proved by fairly elementary arguments, so we omit the proofs.

**Lemma 2.1** *Let  $\Pi_1, \Pi_2 \in \mathcal{G}_{n,2}$  be such that  $\theta(\Pi_1, \Pi_2) \leq 1/2$ . Fix  $a, b \in \mathbb{R}^n$ ,  $\rho > 0$  with  $\rho \leq |a - b| \leq 2\rho$ . Then for any  $\Pi_1^\delta, \Pi_2^\delta$  with  $a, b \in \Pi_1^\delta \cap \Pi_2^\delta$  we have*

$$\Pi_1^\delta \cap \Pi_2^\delta \cap B(a, 2\rho) \subset T_e^{(4\rho)^\beta}(a),$$

where  $e = (a - b)/|a - b|$  and  $\beta = C\delta/\theta(\Pi_1, \Pi_2)$ .

**Lemma 2.2** *Let  $\{\Pi_j\}_{j=1}^M$  be a  $\delta$ -separated set in  $\mathcal{A}_{n,2}$ . Fix  $a, b \in \mathbb{R}^n$ ,  $\rho \geq 4\delta$  with  $|a - b| \geq \rho$ . Suppose that for each  $j$  there exists  $\Pi_j^\delta$  with  $a, b \in \Pi_j^\delta$ . Then, for any maximal  $\zeta$ -separated subset  $\mathcal{B} \subset \{\Pi_j\}_{j=1}^M$  with  $\zeta \geq C\delta/\rho$ , we have*

$$|\mathcal{B}| \gtrsim M(\rho\delta/\zeta)^{n-2}.$$

### 3 Proof of the Theorem 1.2

Let  $E \subset \mathbb{R}^n$ ,  $0 < \lambda \leq 1$ ,  $\epsilon > 0$ , and

$$A_\lambda = \{ \Pi \in \mathcal{A}_{n,2} : \mathcal{M}_\delta \chi_E(\Pi) \geq \lambda \}.$$

By the standard interpolation theorems (see [4]), it is enough to prove the following restricted weak-type estimate at the endpoint

$$\varphi_{n,2}(A_\lambda) \leq C_\epsilon \left(\frac{1}{\delta}\right)^\epsilon \left(\frac{|E|}{\lambda^{(n+1)/3} \delta^{(n-2)/3}}\right)^3. \tag{3.1}$$

Now, let  $\{\Pi_j\}_{j=1}^M$  be a maximal  $\delta$ -separated subset of  $A_\lambda$ . Then proving (3.1) amounts to proving

$$|E| \geq C_\epsilon \delta^\epsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}. \tag{3.2}$$

Since  $\Pi_j \in A_\lambda$ , there exists  $\Pi_j^\delta$  such that

$$|\Pi_j^\delta \cap E| \geq \frac{3}{4} \lambda |\Pi_j^\delta|. \tag{3.3}$$

Put  $\gamma = \lambda^{1/2} (\log(1/\delta))^{-1/2}$  and note that (3.2) is trivial if  $4\delta \geq \gamma$ . Indeed, (3.3) implies

$$\begin{aligned} |E| &\gtrsim \lambda \delta^{n-2} = \lambda^{(n+1)/3} \lambda^{-(n-2)/3} \delta^{n-2} \\ &\gtrsim \lambda^{(n+1)/3} (\delta^2 \log(1/\delta))^{-(n-2)/3} \delta^{n-2} \\ &= (\log(1/\delta))^{-(n-2)/3} \lambda^{(n+1)/3} ((1/\delta)^{2(n-2)})^{1/3} \delta^{n-2} \\ &\geq C_\epsilon \delta^\epsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}. \end{aligned}$$

We may therefore assume that  $4\delta \leq \gamma$ .

Now, let  $\delta_0$  be a small constant to be determined later. Then for  $\delta \geq \delta_0$ , we have  $M \lesssim 1$ , and so (3.3) trivially implies (3.2) as before. Hence we can also assume that  $\delta \leq \delta_0$ .

After these preliminary reductions, we can proceed with the proof of (3.2). First, we find a large number of sets  $\Pi_j^\delta$  so that the measure of their intersection with  $E$  is concentrated in annuli of fixed dimensions. More precisely, we claim that there exist a number  $\rho \geq \gamma$  and a set  $\mathcal{C} \subset \{\Pi_j^\delta\}_{j=1}^M$  with

$$|\mathcal{C}| \gtrsim (\log(C/\gamma))^{-2}M, \tag{3.4}$$

so that for each  $\Pi_j^\delta \in \mathcal{C}$  there is a set  $P_j \subset \Pi_j^\delta$  of measure

$$|P_j| \gtrsim (\log(C/\gamma))^{-2}\lambda |\Pi_j^\delta| \tag{3.5}$$

such that for each  $z \in P_j$

$$\left| \Pi_j^\delta \cap E \cap B(z, 2\rho) \cap (T_e^{(4r)(\gamma^2/r)}(z))^c \right| \gtrsim \lambda |\Pi_j^\delta| \quad \text{for all } e \in S^{n-1}, r \in [\gamma, 1], \tag{3.6}$$

and

$$|\Pi_j^\delta \cap E \cap A(z, \rho)| \gtrsim (\log(C/\gamma))^{-1}\lambda |\Pi_j^\delta|. \tag{3.7}$$

To see this, note that for all  $z \in \mathbb{R}^n, e \in S^{n-1}$  and  $r \in [\gamma, 1]$  we have

$$\begin{aligned} \left| \Pi_j^\delta \cap E \cap (T_e^{(4r)(\gamma^2/r)}(z))^c \right| &= |\Pi_j^\delta \cap E| - |\Pi_j^\delta \cap E \cap T_e^{(4r)(\gamma^2/r)}(z)| \\ &\geq \frac{3}{4} \left( \lambda |\Pi_j^\delta| - Cr \frac{\gamma^2}{r} |\Pi_j^\delta| \right) \\ &= \frac{3}{4} \lambda (1 - C(\log(1/\delta))^{-1}) |\Pi_j^\delta| \\ &\geq \frac{\lambda}{2} |\Pi_j^\delta|, \end{aligned} \tag{3.8}$$

for  $\delta_0$  small enough.

Now, for each  $1 \leq j \leq M, z \in \Pi_j^\delta \cap E, i \in \mathbb{N}$ , consider the quantity

$$Q(j, z, i) = \inf_{r \in [\gamma, 1], e \in S^{n-1}} \left| \Pi_j^\delta \cap E \cap B(z, \gamma 2^i) \cap (T_e^{(4r)(\gamma^2/r)}(z))^c \right|.$$

Then

$$Q(j, z, 0) \leq C\gamma^2\delta^{n-2} = C\lambda(\log(1/\delta))^{-1}\delta^{n-2} \leq \frac{\lambda}{10} |\Pi_j^\delta|,$$

provided that  $\delta_0$  has been chosen small enough. On the other hand

$$Q(j, z, \log(C/\gamma)) \geq \frac{\lambda}{2} |\Pi_j^\delta|.$$

Therefore, there exists  $i_{j,z}$  with  $1 \leq i_{j,z} \leq \log(C/\gamma)$ , such that

$$Q(j, z, i_{j,z}) \geq \frac{\lambda}{4} |\Pi_j^\delta| \quad \text{and} \quad Q(j, z, i_{j,z} - 1) < \frac{\lambda}{4} |\Pi_j^\delta|.$$

Now, we use the pigeonhole principle: Since there are at most  $\log(C/\gamma)$  possible  $i_{j,z}$ , there is an  $i_j$  and a set  $P'_j \subset \Pi_j^\delta \cap E$  of measure

$$|P'_j| \gtrsim (\log(C/\gamma))^{-1}\lambda |\Pi_j^\delta|$$

such that for each  $z \in P'_j$

$$Q(j, z, i_j) \geq \frac{\lambda}{4} |\Pi_j^\delta| \quad \text{and} \quad Q(j, z, i_j - 1) < \frac{\lambda}{4} |\Pi_j^\delta|.$$

By the pigeonhole principle again, since there are  $M$  sets  $\Pi_j^\delta$  and at most  $\log(C/\gamma)$  possible  $i_j$ , there is an  $i_0$  and a subset  $\mathcal{C}' \subset \{\Pi_j^\delta\}_{j=1}^M$  such that

$$|\mathcal{C}'| \gtrsim (\log(C/\gamma))^{-1}M,$$

and for each  $\Pi_j^\delta \in \mathcal{C}'$  and each  $z \in P'_j$

$$\left| \Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0}) \cap (T_e^{(4r)}(\gamma^2/r)(z))^{\mathbb{G}} \right| \geq \frac{\lambda}{4} |\Pi_j^\delta| \quad \text{for all } e \in S^{n-1}, r \in [\gamma, 1],$$

and

$$\left| \Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0-1}) \cap (T_{e_{j,z}}^{(4r_{j,z})}(\gamma^2/r_{j,z})(z))^{\mathbb{G}} \right| < \frac{\lambda}{4} |\Pi_j^\delta|, \tag{3.9}$$

for some  $e_{j,z} \in S^{n-1}, r_{j,z} \in [\gamma, 1]$ .

Now, (3.8) and (3.9) imply that

$$\begin{aligned} \frac{\lambda}{4} |\Pi_j^\delta| &\leq \left| \Pi_j^\delta \cap E \cap (T_{e_{j,z}}^{(4r_{j,z})}(\gamma^2/r_{j,z})(z))^{\mathbb{G}} \right| \\ &\quad - \left| \Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0-1}) \cap (T_{e_{j,z}}^{(4r_{j,z})}(\gamma^2/r_{j,z})(z))^{\mathbb{G}} \right| \\ &= \left| \Pi_j^\delta \cap E \cap (B(z, \gamma 2^{i_0-1}))^{\mathbb{G}} \cap (T_{e_{j,z}}^{(4r_{j,z})}(\gamma^2/r_{j,z})(z))^{\mathbb{G}} \right| \\ &\leq \left| \Pi_j^\delta \cap E \cap (B(z, \gamma 2^{i_0-1}))^{\mathbb{G}} \right| \\ &= \sum_{k=0}^{\log(C/\gamma)} |\Pi_j^\delta \cap E \cap A(z, \gamma 2^{i_0+k-1})|. \end{aligned}$$

Therefore, there is a  $k_{j,z}$  such that

$$\left| \Pi_j^\delta \cap E \cap A(z, \gamma 2^{i_0+k_{j,z}-1}) \right| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\delta|.$$

Repeatedly using the pigeonhole principle as before, we conclude that there exist an integer  $k_0$  and a set  $\mathcal{C} \subset \mathcal{C}'$  with

$$|\mathcal{C}| \gtrsim (\log(C/\gamma))^{-2}M,$$

so that for each  $\Pi_j^\delta \in \mathcal{C}$ , there is a subset  $P_j \subset P'_j$  of measure

$$|P_j| \gtrsim (\log(C/\gamma))^{-2} \lambda |\Pi_j^\delta|$$

such that for each  $z \in P_j$

$$\left| \Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0}) \cap (T_e^{(4r)}(\gamma^2/r)(z))^{\mathbb{G}} \right| \gtrsim \lambda |\Pi_j^\delta| \quad \text{for all } e \in S^{n-1}, r \in [\gamma, 1],$$

and

$$\left| \Pi_j^\delta \cap E \cap A(z, \gamma 2^{i_0+k_0-1}) \right| \gtrsim (\log(C/\gamma))^{-1} \lambda |\Pi_j^\delta|.$$

This proves the claim with  $\rho := \gamma 2^{i_0+k_0-1}$ .

Now we are in a position to carry out a ‘‘high-low multiplicity segment’’ argument (see [1]), as follows.

We fix a number  $N$  and consider two cases.

Case I. For every  $a \in \mathbb{R}^n$  we have  $|\{j : a \in P_j\}| \leq N$ .

Case II. There exists  $a \in \mathbb{R}^n$  such that  $|\{j : a \in P_j\}| \geq N$ .

In case I we have

$$\begin{aligned}
 |E| &\geq \left| \bigcup_{j:\Pi_j^\delta \in \mathcal{C}} P_j \right| \\
 &\geq \frac{1}{N} \sum_{j:\Pi_j^\delta \in \mathcal{C}} |P_j| \gtrsim \frac{1}{N} |\mathcal{C}| (\log(C/\gamma))^{-2} \lambda \delta^{n-2} \gtrsim \frac{M}{N} (\log(C/\gamma))^{-4} \lambda \delta^{n-2},
 \end{aligned}
 \tag{3.10}$$

where the last two inequalities follow from (3.4) and (3.5).

In case II, we fix a number  $\mu$  and consider two subcases.

(II)<sub>1</sub>. For every  $b \in A(a, \rho)$  we have  $|\{j : a \in P_j, b \in \Pi_j^\delta\}| \leq \mu$ .

(II)<sub>2</sub>. There exists  $b \in A(a, \rho)$  such that  $|\{j : a \in P_j, b \in \Pi_j^\delta\}| \geq \mu$ .

In subcase (II)<sub>1</sub> we have

$$\begin{aligned}
 |E| &\geq \left| \bigcup_{j:a \in P_j} \Pi_j^\delta \cap E \cap A(a, \rho) \right| \\
 &\geq \frac{1}{\mu} \sum_{j:a \in P_j} |\Pi_j^\delta \cap E \cap A(a, \rho)| \gtrsim \frac{N}{\mu} (\log(C/\gamma))^{-1} \lambda \delta^{n-2},
 \end{aligned}
 \tag{3.11}$$

where the last inequality follows from (3.7).

In subcase (II)<sub>2</sub> let  $\mathcal{B}$  be a maximal  $C\rho\delta/\gamma^2$ -separated subset of  $\{\Pi_j : a \in P_j, b \in \Pi_j^\delta\}$ . Then for  $C$  large enough, Lemma 2.2 implies

$$|\mathcal{B}| \gtrsim \mu \gamma^{2(n-2)}.
 \tag{3.12}$$

Note that if  $\Pi_j, \Pi_k \in \mathcal{B}$  then by Lemma 2.1

$$\Pi_j^\delta \cap \Pi_k^\delta \cap B(a, 2\rho) \subset T_e^{(4\rho)(\gamma^2/\rho)}(a),$$

where  $e = (a - b)/|a - b|$ , provided that  $C$  has been chosen large enough. Therefore the family

$$\left\{ \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^c : \Pi_j \in \mathcal{B} \right\}$$

is disjoint. Consequently

$$\begin{aligned}
 |E| &\geq \left| \bigcup_{j:\Pi_j \in \mathcal{B}} \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^c \right| \\
 &= \sum_{j:\Pi_j \in \mathcal{B}} \left| \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^c \right| \\
 &\gtrsim |\mathcal{B}| \lambda \delta^{n-2} \\
 &\gtrsim \gamma^{2(n-2)} \mu \lambda \delta^{n-2},
 \end{aligned}
 \tag{3.13}$$

where the last two inequalities follow from (3.6) and (3.12). So, in case II we see that choosing

$$\mu = N^{1/2} (\log(C/\gamma))^{-1/2} \gamma^{-(n-2)},$$

(3.11) and (3.13) imply that

$$|E| \gtrsim (\log(C/\gamma))^{-1/2} \gamma^{n-2} \lambda N^{1/2} \delta^{n-2}.
 \tag{3.14}$$

In conclusion, we see that in case I, (3.10) holds, whereas in case II, (3.14) holds. So choosing

$$N = M^{2/3} (\log(C/\gamma))^{-7/3} \gamma^{-2(n-2)/3},$$

the right-hand sides of (3.10) and (3.14) become equal. So, in both cases we have

$$\begin{aligned} |E| &\gtrsim (\log(C/\gamma))^{-5/3} \gamma^{2(n-2)/3} \lambda M^{1/3} \delta^{n-2} \\ &= (\log(C/\gamma))^{-5/3} (\lambda^{1/2} (\log(1/\delta))^{-1/2})^{2(n-2)/3} \lambda M^{1/3} \delta^{n-2} \\ &\geq (\log(C/\delta))^{-5/3} (\lambda^{1/2} (\log(1/\delta))^{-1/2})^{2(n-2)/3} \lambda M^{1/3} \delta^{n-2} \\ &\geq C_\epsilon \delta^\epsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}, \end{aligned}$$

where the inequality before the last one is true because  $4\delta \leq \gamma$ . The proof is complete.  $\square$

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