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Lyndon words, polylogarithms and the Riemann ζ function☆

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Abstract

The algebra of polylogarithms (iterated integrals over two differential forms $\omega_0 = \mathrm{d}z/z$ and $\omega_1 = \mathrm{d}z/(1-z)$) is isomorphic to the shuffle algebra of polynomials on non-commutative variables x_0 and x_1. The *multiple zeta values* (MZVs) are obtained by evaluating the polylogarithms at $z = 1$. From a second shuffle product, we compute a Gröbner basis of the kernel of this evaluation morphism. The completeness of this Gröbner basis up to order 12 is equivalent to the classical conjecture about MZVs. We also show that certain known relations on MZVs hold for polylogarithms. © 2000 Published by Elsevier Science B.V. All rights reserved.

Résumé

L'algèbre des polylogarithmes (intégrales itérées des deux formes différentielles $\omega_0 = \mathrm{d}z/z$ et $\omega_1 = \mathrm{d}z/(1-z)$) est isomorphe à l'algèbre des polynômes en variables non commutatives x_0 et x_1, munie du produit de mélange. Les MZV s'obtiennent en évaluant les polylogarithmes en $z = 1$. Nous calculons, à partir d'un deuxième produit de mélange, une base de Gröbner du noyau de ce morphisme d'évaluation. La complétude de la base de Gröbner à l'ordre 12 est équivalente à une conjecture classique sur les MZV. Nous montrons aussi que certaines relations connues sur les MZV sont en fait valables pour les polylogarithmes. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Polylogarithms

For any multi-index $s=(s_1,s_2,\dots,s_k)$, $k\geqslant 0$, we define the *polylogarithms* on the real interval $]-1,1[$ as follows:

$$\mathrm{Li}_s(t)=\sum_{n_1>n_2>\cdots>n_k>0}\frac{t^{n_1}}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}. \tag{1}$$

This series convergents for $|t|<1$. We associate the function 1 with the empty multi-index. We used these functions to express the output of dynamical systems with rational entries [8,10] and to integrate differential equations with meromorphic coefficients [9]. We define the Q-algebra of polylogarithms as being the smallest sub-algebra of real analytical functions over $]-1,+1[$ which contains the Li_s's and is closed by sum, product and linear combinations with coefficients in Q.

In this work, we show how to generate the *algebraic relations* between polylogarithms by encoding them by words over the non-commutative variables x_0 and x_1. With any multi-index $s=(s_1,s_2,\dots,s_k)$, we associate the word

$$w=x_0^{s_1-1}x_1x_0^{s_2-1}x_1\cdots x_0^{s_k-1}x_1. \tag{2}$$

In this way, any polylogarithm can be encoded by some word ending with x_1.

We show that the Q-algebra of the polylogarithms is *freely* generated by functions which are encoded by some special words called *Lyndon words*. Section 2 deals with the decomposition algorithm in this basis. It is basically the application of the `xtaylor` algorithm, used in [16] to solve a control theory problem in a particular case.

1.2. Multiple zeta values and their relations

At $t=1$, the previous polylogarithm gives the *multiple zeta values* (MZVs)

$$\zeta(s)=\sum_{n_1>n_2>\cdots>n_k>0}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_k^{s_k}}, \tag{3}$$

which converges when $s_1>1$ and diverges for $s_1=1$. These MZVs [12–14,4,3] first appeared in the knot theory, in relation to Feynman diagrams in quantum physics [1,20]. They also appear in the study of the fundamental probabilities derived from quadtrees [5]. Flajolet and Salvy approached these Euler sums using their representation by *contour integrals* [6].

Euler showed that, when $n+p\leqslant 13$ and $n+p$ is odd, $\zeta(n,p)$ can be expressed as a function of the $\zeta(s)$, where s is an integer index (that is to say, a positive integer greater than 2). In 1904, Nielsen (see [15, pp. 192, 194]), in a deep work, proves in particular the following relations:[1]

$$\zeta(n)\zeta(p)=\zeta(n+p)+\zeta(n,p)+\zeta(p,n)\quad(\text{for } n>1,\ p>1), \tag{4}$$

[1] Nielsen notes $\zeta(n)$ as s_n.

$$\zeta(p,q)=(-1)^q\sum_{v=0}^{p-2}\binom{q+v-1}{q-1}\zeta(p-v,q+v)$$
$$+\sum_{v=0}^{q-2}(-1)^v\binom{p+v-1}{p-1}\zeta(q-v)\zeta(p+v)$$
$$-(-1)^q\binom{p+q-2}{p-1}[\zeta(p+q)+\zeta(p+q-1,1)]\,(\text{for } p>1,\ q>1). \tag{5}$$

Relation (4), which is called *reflection formula*, does not hold for polylogarithms: by (1), when $t\to 0$, the product $\mathrm{Li}_n(t)\mathrm{Li}_p(t)$ and of the sum $\mathrm{Li}_{n,p}(t)+\mathrm{Li}_{p,n}(t)$ are both equal to $O(t^2)$, whereas $\mathrm{Li}_{n+p}(t)$ is equal to $O(t)$. This formula (4) can be deduced from the second shuffle product [13], which formulates in a different way the computation of the product of two *quasi-symmetric* functions [19,7].

Working on relation (5), Borwein et al. [2] find explicit formulae giving $\zeta(n,p)$ when $n+p$ is *odd*. We still do not know any explicit expression for $\zeta(n,p)$ when $n+p$ is *even*, except when $n+p=4$ or 6. We prove that the *decomposition formula* (see Section 23), dealing with MZV, holds for polylogarithms.

Many authors have studied the *linear dependences* between MZVs. Many experimental results use numerical approximate computation to confirm the Zagier's dimension conjecture [20]:

Conjecture 1.1. The Q-linear relations between $\zeta(s)$ are homogenous w.r.t. the weight $|s|=s_1+s_2+\cdots+s_k$. The dimension d_n of the Q-vector space is generated by the $\zeta(s)$ for multi-indices s such that $|s|=n$ is given by the recurrence:

$$\begin{aligned} &d_n=d_{n-2}+d_{n-3},\quad n\geqslant 4,\\ &d_1=0,\quad d_2=d_3=1, \end{aligned} \tag{6}$$

In this paper, we study the ideal of *polynomial relations* between convergent $\zeta(s)$. Theorem 3.1 gives a *complete* system of relations between the Li_s and shows that the algebra polylogarithms is *freely* generated by the multi-indexes s which are Lyndon words. The Q-algebra $\mathscr{H}$ generated by $\zeta(s)$ is the image in R of the algebra of polylogarithms by the morphism of evaluation: $\mathrm{Li}_s\to\zeta(s)=\mathrm{Li}_s(1)$. The kernel of this morphism is then completely defined by the ideal of polynomial relations between $\zeta(s)$ when s is a Lyndon word. We propose a simple combinatorial process based on the quasi-symmetric functions [19,13] to generate this ideal of relations.

Here, we compute the Gröbner basis of this ideal up to weights 10.[2] The enumeration of monomials (on $\zeta(s)$) of weight n, which are irreducible by this Gröbner basis satisfies formula (6). Therefore, admitting Conjecture 1.1, this Gröbner basis is *complete*. We can also notice that this Gröbner basis is special: there is no monomial

[2] M. Bigotte (bigotte@lifl.fr) has performed the computation up to order 12.

on $\zeta(s)$ in the left parts of the rules. Hence, such a basis defines a Q-algebra which is freely generated by the irreducible $\zeta(s)$. Owing to this computation Gröbner basis, we show:

Theorem 1.1. *If Conjecture* 1.1 *holds then the Q-algebra generated by the $\zeta(s)$ of weight $|s| \leqslant 12$ is free.*

We can then:

Conjecture 1.2. The Q-algebra of the $\zeta(s)$ is a polynomial algebra.

2. Recalls on shuffle algebra

2.1. The shuffle product

Let X be a finite ordered alphabet. Let us denote by X^* the free monoid generated by X. An element w of X^* is a word, the letters of which are in X. The *length* of w, which is the number of its letters, is denoted by $|w|$ and the *empty word* is denoted by ε. We write $X^+ = X^* \backslash \varepsilon$.

We consider the free associative algebra $Q\langle X\rangle$ of the polynomials, with coefficients in Q and with non-commutative variables $x \in X$. The duality, defined on these words by

$$(u|v) = \delta_u^v, \quad u, v \in X^*$$

(where δ denotes the Kronecker symbol) is linearly extended to the polynomials of $Q\langle X\rangle$. A polynomial $f \in Q\langle X\rangle$ is a *finite* linear combination of words. It can be written as

$$f = \sum_{w \in X^*} (f|w)\, w, \quad (f|w) \in Q.$$

The *concatenation* product can be obtained by linearly extending the combinations of words:

$$\forall f, g \in Q\langle X\rangle, \quad fg = \sum_{u,v \in X^*} (f|u)(g|v)\, uv.$$

The degree of f, denoted by $\deg(f)$, is the length of a longest word $w \in X^*$ such that $(f|w) \neq 0$.

The *shuffle product* is first defined on words by the following recursive formulae:

$$\begin{aligned} &\forall w \in X^*, \quad \varepsilon \sqcup\!\sqcup w = w \sqcup\!\sqcup \varepsilon = w, \\ &\forall x, y \in X, \ u, v \in X^*, \quad xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v) \end{aligned} \tag{7}$$

and then linearly extended to the polynomials of $Q\langle X\rangle$. With the shuffle product, $Q\langle X\rangle$ is a *commutative* and *associative* Q-algebra denoted by $\mathrm{Sh}_Q(X)$.

2.2. Transcendence bases of the shuffle algebra

Let X be an alphabet equipped with a total order '$<$'. The *lexicographical order* is a total order on words of X^* satisfying:

$$\begin{aligned}&\forall u, w \in X^*, \quad w \neq \varepsilon \Rightarrow u < uw,\\&\forall u, v_1, v_2 \in X^*, \forall x, y \in X, \quad x < y \Rightarrow uxv_1 < uyv_2.\end{aligned} \tag{8}$$

By definition, a *Lyndon word* is a non-empty word of X^* which is less (for the lexicographical order) than any of its proper right factors: $\forall u, v \in X^+$, $l = uv \Rightarrow l < v$. $\mathscr{L}$yndon(X) denotes the set of all Lyndon words on the alphabet X.

Example 1. Let $X = \{x_0, x_1\}$ with $x_0 < x_1$, the Lyndon words of length 5 or less on X^* are the following 14 words, in increasing order:

$$\{x_0, x_0^4x_1, x_0^3x_1, x_0^3x_1^2, x_0^2x_1, x_0^2x_1x_0x_1, x_0^2x_1^2, x_0^2x_1^3, x_0x_1, x_0x_1x_0x_1^2, x_0x_1^2, x_0x_1^3, x_0x_1^4, x_1\}.$$

We know many *transcendence bases* of the commutative algebra $\mathrm{Sh}_Q(X)$ (with the shuffle product). The Lyndon words are one of them, which we call *Radford basis*. They are algebraically independent and generate $\mathrm{Sh}_Q(X)$. More precisely [17,19]:

Theorem 2.1 (Radford). *Let X be a finite ordered set and L be the set of all Lyndon words over X. The algebra $\mathrm{Sh}_Q(X)$ is a polynomial algebra which is isomorphic to $Q[L]$.*

2.3. Lie polynomials and differentiations

We know that the *free Lie algebra* $\mathscr{L}\mathrm{ie}_Q\langle X\rangle$ is the smallest sub-vector space of $Q\langle X\rangle$ containing X and closed under the Lie bracket operation:

$$[p, q] = pq - qp, \quad \forall p, q \in Q\langle X\rangle.$$

With any Lyndon word $l \in \mathscr{L}$yndon(X), we associate the *bracketed form* denoted by $[l] \in \mathscr{L}\mathrm{ie}_Q\langle X\rangle$, which can be defined as follows:

$$\begin{aligned}&[x] = x, \quad \forall x \in X,\\&[l] = [[u], [v]], \quad l, u, v \in \mathscr{L}\mathrm{yndon}(X),\end{aligned} \tag{9}$$

where v denotes the longest Lyndon word such that $l = uv$ with $u \neq \varepsilon$. For example, the bracketed form of $x_0x_1x_0x_1x_1$ is $[[x_0, x_1], [[x_0, x_1], x_1]]$. We know that the set $\{[l] \mid l \in L\}$ is a Q-basis of the vector space $\mathscr{L}\mathrm{ie}_Q\langle X\rangle$.

The *right residual* of a polynomial p by another polynomial q, denoted by $p \triangleright q$, is defined by the following formula:

$$(p \triangleright q | w) = (p | qw), \quad \forall p, q \in Q\langle X\rangle, \ \forall w \in X^*. \tag{10}$$

As a special case, we have

$$\begin{aligned}(xw)\triangleright x &= w, \quad \forall x\in X,\ \forall w\in X^*,\\ (xw)\triangleright y &= 0, \quad \forall x,y\in X,\ \forall w\in X^*,\ x\neq y.\end{aligned} \tag{11}$$

We can easily check that this simplification is a *right action* over $Q\langle X\rangle$, that is to say

$$p\triangleright(qr) = (p\triangleright q)\triangleright r, \quad \forall p,q,r\in Q\langle X\rangle.$$

Lemma 2.1 (Reutenauer [19]). *The right residual by a Lie polynomial is a differentiation for the shuffle product*:

$$(f ⧢ g)\triangleright p = (f\triangleright p) ⧢ g + f ⧢ (g\triangleright p), \quad \forall f,g\in Q\langle X\rangle,\ \forall p\in \mathscr{L}\mathrm{ie}_Q\langle X\rangle. \tag{12}$$

Proof. We first prove this lemma when p is a letter of the alphabet X. Then we notice that the Lie bracket of two differentiations is a differentiation. □

2.3.1. Decomposition algorithm in the Radford basis

The decomposition `xtaylor` algorithm [16] of a polynomial $f\in Q\langle X\rangle$ in the Radford basis lies on the following *triangular* property:

Lemma 2.2. *Let l be a Lyndon word and $[l]$ be its bracketed form. l is the smallest word that appears in the support of $[l]$, and has coefficient 1, that is to say*:

$$[l] = l + \sum_{w\in X^*} \alpha_w w \quad \textit{with } |w| = |l|,\ w > l,\ \alpha_w\in Z.$$

Corollary 2.1. *The Lyndon words, according to the lexicographical by length ordering,*[3] *satisfy the following triangular property*:

$$\forall l,l'\in \mathscr{L}\mathrm{yndon}(X), \quad l\triangleright[l'] = \begin{cases} 1 & \textit{if } l = l',\\ 0 & \textit{if } l < l'.\end{cases} \tag{13}$$

Let d be the degree of polynomial $f\in Q\langle X\rangle$. We compute the set L_d of all Lyndon words of length less than d. The decomposition of f in the Radford basis will need words only from L_d. Let $l_{\max}$ be the *greatest* word of L_d. We are looking for a decomposition of the form

$$f = \sum_{i=0}^{n} A_i ⧢ l_{\max}^{⧢ i}, \quad A_i\in Q\langle X\rangle.$$

The decomposition of all A_i will take place in $L'_d = L_d\setminus\{l_{\max}\}$, without the appearence of $l_{\max}$. To simplify the discussion, let us choose $n=2$ for example. From

$$f = A_2 ⧢ l_{\max}^{⧢ 2} + A_1 ⧢ l_{\max} + A_0,$$

[3] Lyndon words are first sorted by increasing length, then, when they have equal lengths, they are sorted according to the lexicographical order.

we get the unknown polynomials A_0, A_1, A_2 iterating on f the differentiation $\triangleright[l_{\max}]$. Taking into account Corollary 2.1 and applying this differentiation to A_i gives 0. We then obtain:

$$\begin{aligned} f_1 &= f \triangleright [l_{\max}] = 2A_2 \sqcup\!\sqcup\, l_{\max} + A_1, \\ f_2 &= f_1 \triangleright [l_{\max}] = 2A_2, \\ f_3 &= f_2 \triangleright [l_{\max}] = 0. \end{aligned} \tag{14}$$

Knowing A_2 owing to f_2, we get A_1 owing to f_1, and A_0 owing to f. The decomposition of the A_i in the Radford basis will be computed by a recursive call of the same algorithm using the list L'_d. Given that the length of L'_d is strictly less than the length of L_d, this computation eventually comes to an end.

3. Polylogarithms

3.1. Notations

Let $X = \{x_0, x_1\}$ be an ordered encoding alphabet such that $x_0 < x_1$. Let $Y = \{y_i \mid i > 0\}$ be ordered such that $y_i < y_j$ when $i > j$. To any multi-index s, we associate, in a one-to-one correspondence, the words $w \in X^* x_1$ and $w' \in Y^* \setminus \varepsilon$:

$$\begin{aligned} w &= x_0^{s_1-1} x_1 x_0^{s_2-1} x_1 \cdots x_0^{s_k-1} x_1, \\ w' &= y_{s_1} y_{s_2} \ldots y_{s_k}. \end{aligned} \tag{15}$$

Note that in this way we get an *increasing* isomorphism between the free monoids $\varepsilon + X^* x_1$ and Y^* posing $y_i = x_0^{i-1} x_1$, which will allow us to identify the words s, w and w'. This increasing isomorphism maps the Lyndon words in $X^* x_1$ to the Lyndon words in Y^+. In this sequel, we will use both notations Li_s and Li_w (respectively, $\zeta(s)$ and $\zeta(w)$).

3.2. Principle of the encoding

Let $s = (s_1, s_2, \ldots, s_k)$. The computation of the derivative of $\mathrm{Li}_s(t)$ in formula (1) gives

$$\begin{aligned} t \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Li}_s(t) &= \mathrm{Li}_{(s_1-1, s_2, \ldots, s_k)}(t) \quad \text{if } s_1 > 1, \\ (1-t) \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Li}_s(t) &= \mathrm{Li}_{(s_2, \ldots, s_k)}(t) \quad \text{if } s_1 = 1. \end{aligned} \tag{16}$$

Then we deduce that the polylogarithms are the *iterated integrales* w.r.t. the two differential forms $\omega_0 = \mathrm{d}t/t$ and $\omega_1 = \mathrm{d}t/(1-t)$. The encoding by the alphabet X gives the following recursive formulae:

$$\begin{aligned} \mathrm{Li}_{x_0 w}(t) &= \int_0^t \frac{\mathrm{d}z}{z} \mathrm{Li}_w(z), \quad \forall w \in X^* x_1, \\ \mathrm{Li}_{x_1 w}(t) &= \int_0^t \frac{\mathrm{d}z}{1-z} \mathrm{Li}_w(z), \quad \forall w \in X^* x_1. \end{aligned} \tag{17}$$

Put

$$\begin{aligned}\mathscr{H}_1 &= Q \oplus Q\langle X\rangle x_1 \simeq Q\langle Y\rangle,\\ \mathscr{H}_2 &= Q \oplus x_0 Q\langle X\rangle x_1,\\ L_1 &= \mathscr{L}\text{yndon}(X)\backslash\{x_0\},\\ L_2 &= \mathscr{L}\text{yndon}(X)\backslash\{x_0,x_1\}.\end{aligned} \tag{18}$$

Then $\mathscr{H}_1$ and $\mathscr{H}_2$ are stable concatenation product and shuffle product.

Lemma 3.1. *With conventions* (18), *we have the isomorphism* $\mathscr{H}_1 \simeq Q[L_1]$ *and* $\mathscr{H}_2 \simeq Q[L_2]$.

Proof. Let us show that polynomial $p \in Q \oplus Q\langle X\rangle x_1$ can be decomposed into the linear combinations of shuffle of words in L_1, by induction on the greatest word w in support of p. The factorization [19] (Corollary 4.7, p. 89) on decreasing Lyndon words $w = l_1^{\alpha_1} \cdots l_k^{\alpha_k}$ cannot contain the Lyndon word x_0. Let $Q_w = (1/\alpha_1! \cdots \alpha_k!) l_1^{ш \alpha_1} ш \cdots ш \, l_k^{ш \alpha_k}$. From [19, Theorem 6.1, p. 127], one has $w = Q_w + p_1$, the new polynomial p_1 contains the words which are strictly less than w. Since w, $Q_w \in \mathscr{H}_1$, we deduce that $p_1 \in \mathscr{H}_1$. By hypothesis, $p_1 \in Q[L_1]$ and thus $w \in Q[L_1]$. The unicity of the decomposition follows the Radford Theorem 2.1. □

Extending Q-linearly the maps $s \to \mathrm{Li}_s$, owing to the isomorphism $\mathscr{H}_1 \simeq Q[L_1]$, we build the maps:

$$\mathrm{Li} : p \in Q[L_1] \to \mathrm{Li}_p \in \mathscr{C}^{\omega}(]-1,1[;R). \tag{19}$$

Theorem 3.1. *The application* Li, *defined in* (19), *is a* Q*-algebra isomorphism.*

Proof. The classical theorem of Ree [18] shows that Li is a shuffle algebra morphism, i.e.

$$\forall p,q \in \mathscr{H}_1, \quad \mathrm{Li}_{p ш q} = \mathrm{Li}_p \mathrm{Li}_q.$$

We have proved in [11] that $\ker \mathrm{Li} = (0)$ by computing monodromy of polylogarithms considered as multi-valued functions over the complex plane $C\backslash\{0,1\}$. □

3.3. Decomposition formulae

Lemma 3.2. *For any letters* a, x, y *and for any integer* p, q, *we have*:

$$a^n ш a^p = \binom{n+p}{n} a^{n+p}, \tag{20}$$

$$a^n ш (a^p y) = \sum_{i=0}^{n} (a^i ш a^p) y a^{n-i}, \tag{21}$$

$$(a^n x) ш (a^p y) = \sum_{i=0}^{n} \binom{p+i}{i} a^{p+i} y a^{n-i} x + \sum_{i=0}^{p} \binom{n+i}{i} a^{n+i} x a^{p-i} y. \tag{22}$$

Proof. First two identities are easy; to show the third one, we can use the recursive definition $a^n x ш a^p y = (a^n ш a^p y)x + (a^n x ш a^p) y$. $\square$

Applying formulae (22) in the case $a = x_0$, $x = y = x_1$ and replacing n with $n-1$ and p with $p-1$, we can deduce the formulae:

$$\mathrm{Li}_n(t)\mathrm{Li}_p(t) = \sum_{i=0}^{n-1} \binom{p+i-1}{i} \mathrm{Li}_{p+i,n-i}(t) + \sum_{i=0}^{p-1} \binom{n+i-1}{i} \mathrm{Li}_{n+i,p-i}(t). \tag{23}$$

Evaluating this last equality at $t = 1$ gives a formula analogous — see [2] — to decompose $\zeta(n)\zeta(p)$.

4. Relations between the MZVs

Using isomorphism (18), the evaluation morphism $\mathrm{Li}_s \overset{z=1}{\rightarrow} \zeta(s)$ become

$$\zeta : p \in Q[L_2] \rightarrow \zeta_p \in R. \tag{24}$$

The goal of this section is to compute the Gröbner basis of the ideal $\ker \zeta$.

4.1. Obtention of the ideal of relations

4.1.1. Quasi-symmetric functions

Let $T = \{t_1, t_2, \ldots\}$ be the infinite commutative alphabet indexed by positive integers and $Q[[T]]$ the algebra of formal series (in commutative variables) defined over T and with coefficients in Q. A series $F(t) \in Q[[T]]$ is called *quasi-symmetric* [19,13] if for all multi-indices $s = (s_1, s_2, \ldots, s_k)$, the coefficient in $F(t)$ of any monomial of the form $t_{i_1}^{s_1} t_{i_2}^{s_2} \cdots t_{i_k}^{s_k}$ with $i_1 > i_2 > \cdots > i_k$ does not depend on the multi-index $(i_1, i_2, \ldots, i_k)$. Stanley 1972 proves that the product of two quasi-symmetric series is quasi-symmetric.

Let us consider the Q-linear map $F : Q\langle Y\rangle \rightarrow Q[[T]]$ which maps any word $w = y_{s_1} y_{s_2} \cdots y_{s_k} \in Y^*$ to the formal quasi-symmetric series:

$$F_w(t) = \sum_{n_1 > n_2 > \cdots > n_k > 0} t_{n_1}^{s_1} t_{n_2}^{s_2} \cdots t_{n_k}^{s_k}. \tag{25}$$

4.1.2. Second shuffle product

The second shuffle product $* : Q\langle Y\rangle \times Q\langle Y\rangle \to Q\langle Y\rangle$ is recursively defined on the words of Y^* with:

$$\begin{aligned}&\forall w\in Y^*,\quad \varepsilon * w = w * \varepsilon = w,\\ &\forall u,v\in Y^*,\ \forall i,j>0,\ (y_iu)*(y_jv) = y_i(u*y_jv)+y_j(y_iu*v)+y_{i+j}(u*v).\end{aligned} \tag{26}$$

We prove by induction [13] that this product is associative and commutative.

Example 2. $\forall i,j>0,\ y_i * y_j = y_iy_j + y_jy_i + y_{i+j}$ — see formulae (4).

Theorem 4.1.

$$\forall p,q\in Q\langle Y\rangle,\quad F_{p*q}(t) = F_p(t)F_q(t). \tag{27}$$

Proof. See [13]. □

Corollary 4.1. *Up to isomorphism* (18),

$$\forall p,q\in Q[L_2],\quad \zeta(p*q)=\zeta(p)\zeta(q). \tag{28}$$

Proof. $\zeta(p)$ can be computed substituting $(1,1/2,1/3,\ldots)$ into the indeterminate variables $(t_1,t_2,t_3,\ldots)$ in the series $F_p(t)$ and $F_q(t)$. □

4.1.3. Differentiation operator

Let us consider the Q-linear operator $D : Q[L_1]\to Q[L_1]$ by

$$D(p) = x_1 ш p - x_1 * p,\quad p\in Q[L_1]. \tag{29}$$

In fact, this operator corresponds to the operator introduced by Hoffmann [12,13]. This one is a derivation $Q\langle X\rangle \to Q\langle X\rangle$ (for the concatenation) i.e. $\forall p,q\in Q\langle X\rangle$, $D(pq) = (Dp)q + p(Dq)$ and such that $D(x_0)=x_0x_1$ and $D(x_1) = -x_0x_1$.

Theorem 4.2. *For any polynomial* $p\in Q[L_2]$, $D(p)\subset Q[L_2]$ *and* $D(p)$ *are elements of the kernel of* ζ, *i.e.* $\zeta_{D(p)}=0$.

Sketch of Proof. Nielsen has already known this theorem in the particular case.[4] The facts $D(x_0)=x_0x_1$ and $D(x_1)=-x_0x_1$ prove that $D(p)\subset Q[L_2]$. Thus, the divergent terms on $\zeta_{x_1}=\zeta(1)=\sum_{n\geqslant 1}1/n$ do not appear in $D(p)$ (see [12,13]). Corollary 4.1 holds. □

[4] One has, for $p\geqslant 2$, $D(x_0^{p-1}x_1)=\sum_{i=1}^{p-1}x_0^ix_1x_0^{p-1-i}x_1 - x_0^px_1$; this corresponds to the formula: $\zeta(p+1)=\sum_{i=1}^{p-1}\zeta(i+1,p-i)$ — see [15], p. 195 (14).

4.2. Computation of a Gröbner basis of relations

4.2.1. Recall on Gröbner basis

Let k be a field, $R=k[X_1,X_2,\dots,X_n]$ be a polynomial ring and I be an ideal of R. Roughly speaking, a Gröbner basis of I is a rewriting system which reduces to 0 each element of I. More precisely, we call *term* any product of the form $X^\alpha=X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n}$ where $\alpha_i\geqslant 0$ for $1\leqslant i\leqslant n$. The set of terms T is a *commutative monoid* equipped with the usual product. A total order over T is said *admissible* if it is compatible with the monoid structure of T. The *leading term* of the polynomial $p\in R$ is the greatest term which appears in p. With any polynomial $p\in R$ of the form $p=X^\alpha-q$ where $q\in R$ and such that $X^\alpha\in T$ is a leading term of p, one associates the rewriting rule $X^\alpha\to q$. Any term $X^\beta\in T$ is said *reducible* by this rule if X^α divides X^β; in this case, its reduced form is $X^{\beta-\alpha}q$. The *reduced form* of a polynomial is obtained by reducing each of its terms. A rewriting system is a set of rewriting rules. It is said *confluent* if and only if the reduced form of any polynomial is independant of the order of the use of rules. A *Gröbner* basis G of an ideal I is a set of polynomials of R with which we associate a confluent rewriting system which reduces only the polynomials of I to 0 i.e.:

$$p\in I \;\Leftrightarrow\; p\to 0 \bmod G. \tag{30}$$

The dimension of the k-vector space R/I is given by the number of irreducible terms $X^\alpha\in T$ by the Grobner basis.

In the following, we will use the following elementary lemma:

Lemma 4.1. *Let $R=k[X_1,X_2,\dots,X_n]$ be a polynomial ring, I be an ideal of R and $G\subset R$ be a Gröbner basis of I for any admissible order. If the leading terms of polynomials of G are* single *indeterminates then the algebra R/I is free and the ideal I is prime.*

Proof. The indeterminates on the left part of the rule are the combination of the irreducible indeterminates appearing on the right part. Thus it is clear that the images by the canonical morphism of these irreducible indeterminates generate the k-algebra R/I. The irreducible indeterminates are not linked by any relation $p\in I$: such a relation p will be reduced to 0 by G which is impossible. The algebra R/I is free and it does not contain any divisor of zero. It is then a prime ideal. □

4.3. Gröbner basis of relations between MZV

Owing to the `xtaylor` algorithm which decomposes any polynomial of $Q\langle X\rangle$ in the Radford basis, we can perform the second product as a product defined over $Q[L_2]$. We use the following algorithm:

- Initially $G:=\emptyset$.

- For any two Lyndon words $l_1, l_2 \in L_2$, $l_1 \leqslant l_2$ and $|l_1|+|l_2| \leqslant N$ add the polynomial $l_1 l_2 - l_1 * l_2$ to G. — see Corollary 4.1.
- For any Lyndon word $l \in L_2$, $|l| < N$, add the polynomial $D(l)$ to G. — see Theorem 4.2.
- Using the Buchberger algorithm, return the Gröbner basis of $G \subset Q[L_2]$ where the set L_2 is considered as a set of commutative indeterminates.

In the basis computed up to $N = 10$ in the annex, the set L_2 of Lyndon words over the alphabet $X = \{x_0, x_1\}$ is first sorted according to the lexicographical order by length. To any term of the form $l_1^{\alpha_1} ⧢ l_2^{\alpha_2} \cdots$ corresponds the multi-index $(\alpha_1, \alpha_2, \ldots)$. The terms are compared according to an *elimination order*. In other words, two terms are compared using the lexicographical order on the multi-indices which correspond to them. Let (G) be the ideal of $Q[L_2]$ generated by G. The irreducible terms form a basis of the Q-vector space $\mathbb{Q}[L_2]/(G)$ graduated by weight. We can check (computing the *Hilbert function* associated to G) that the number of homogenous irreducible terms of weight k satisfies the dimension conjecture (6). This basis up to the order is then complete.

4.4. Experimental result

Conjecture 4.1. The ideal (G) is equal to the kernel of the morphism $\mathrm{Li}_s \overset{z=1}{\rightarrow} \zeta(s)$. The Q-algebra of MZVs is free.

This conjecture is verified up to order 12 if one admits Zagier's dimension conjecture (6). We notice that the Gröbner basis ($N=12$) defines a polynomial Q-algebra generated by the numbers $\zeta(2)$, $\zeta(6,2)$, $\zeta(8,2)$, $\zeta(10,2)$, $\zeta(8,2,1)$, $\zeta(8,2,1,1)$, and by the $\zeta(s)$ for the odd integer $s \geqslant 3$.

5. Conclusion

In this paper, we have computed the Gröbner basis by a pure symbolic method. This basis plays an essential rôle on the effective computation of monodromy and functional identities of polylogarithms. This will be developed in our next works.

6. General basis up to order 10

Order 3

$$\zeta(2,1) = \zeta(3), \tag{31}$$

Order 4

$$\zeta(4) = \tfrac{2}{5}\zeta(2)^2, \tag{32}$$

$$\zeta(3,1) = \tfrac{1}{10}\zeta(2)^2, \tag{33}$$

$$\zeta(2,1,1) = \tfrac{2}{5}\zeta(2)^2. \tag{34}$$

Order 5

$$\zeta(4,1) = 2\zeta(5) - \zeta(2)\zeta(3), \tag{35}$$

$$\zeta(3,2) = -\tfrac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3), \tag{36}$$

$$\zeta(3,1,1) = 2\zeta(5) - \zeta(2)\zeta(3), \tag{37}$$

$$\zeta(2,2,1) = -\tfrac{11}{2}\zeta(5) + 3\zeta(2)\zeta(3), \tag{38}$$

$$\zeta(2,1,1,1) = \zeta(5). \tag{39}$$

Order 6

$$\zeta(6) = \tfrac{8}{35}\zeta(2)^3, \tag{40}$$

$$\zeta(5,1) = -\tfrac{1}{2}\zeta(3)^2 + \tfrac{6}{35}\zeta(2)^3, \tag{41}$$

$$\zeta(4,2) = \zeta(3)^2 - \tfrac{32}{105}\zeta(2)^3, \tag{42}$$

$$\zeta(4,1,1) = -\zeta(3)^2 + \tfrac{23}{70}\zeta(2)^3, \tag{43}$$

$$\zeta(3,2,1) = 3\zeta(3)^2 - \tfrac{29}{30}\zeta(2)^3, \tag{44}$$

$$\zeta(3,1,2) = -\tfrac{3}{2}\zeta(3)^2 + \tfrac{53}{105}\zeta(2)^3, \tag{45}$$

$$\zeta(3,1,1,1) = -\tfrac{1}{2}\zeta(3)^2 + \tfrac{6}{35}\zeta(2)^3, \tag{46}$$

$$\zeta(2,2,1,1) = \zeta(3)^2 - \tfrac{32}{105}\zeta(2)^3, \tag{47}$$

$$\zeta(2,1,1,1,1) = \tfrac{8}{35}\zeta(2)^3. \tag{48}$$

Order 7

$$\zeta(6,1) = 3\zeta(7) - \zeta(2)\zeta(5) - \tfrac{2}{5}\zeta(2)^2\zeta(3), \tag{49}$$

$$\zeta(5,2) = -11\zeta(7) + 5\zeta(2)\zeta(5) + \tfrac{4}{5}\zeta(2)^2\zeta(3), \tag{50}$$

$$\zeta(5,1,1) = 5\zeta(7) - 2\zeta(2)\zeta(5) - \tfrac{1}{2}\zeta(2)^2\zeta(3), \tag{51}$$

$$\zeta(4,3) = 17\zeta(7) - 10\zeta(2)\zeta(5), \tag{52}$$

$$\zeta(4,2,1) = -\tfrac{221}{16}\zeta(7) + \tfrac{11}{2}\zeta(2)\zeta(5) + \tfrac{7}{5}\zeta(2)^2\zeta(3), \tag{53}$$

$$\zeta(4,1,2) = \tfrac{5}{8}\zeta(7) + \tfrac{5}{2}\zeta(2)\zeta(5) - \tfrac{3}{2}\zeta(2)^2\zeta(3), \tag{54}$$

$$\zeta(4,1,1,1) = 5\zeta(7) - 2\zeta(2)\zeta(5) - \tfrac{1}{2}\zeta(2)^2\zeta(3), \tag{55}$$

$$\zeta(3,3,1) = \tfrac{61}{8}\zeta(7) - \tfrac{9}{2}\zeta(2)\zeta(5), \tag{56}$$

$$\zeta(3,2,2) = \tfrac{157}{16}\zeta(7) - \tfrac{15}{2}\zeta(2)\zeta(5) + \tfrac{9}{10}\zeta(2)^2\zeta(3), \tag{57}$$

$$\zeta(3,2,1,1) = -\tfrac{221}{16}\zeta(7) + \tfrac{11}{2}\zeta(2)\zeta(5) + \tfrac{7}{5}\zeta(2)^2\zeta(3), \tag{58}$$

$$\zeta(3,1,2,1) = \tfrac{61}{8}\zeta(7) - \tfrac{9}{2}\zeta(2)\zeta(5), \tag{59}$$

$$\zeta(3,1,1,2) = -\tfrac{109}{16}\zeta(7) + 5\zeta(2)\zeta(5) - \tfrac{1}{2}\zeta(2)^2\zeta(3), \tag{60}$$

$$\zeta(3,1,1,1,1) = 3\zeta(7) - \zeta(2)\zeta(5) - \tfrac{2}{5}\zeta(2)^2\zeta(3), \tag{61}$$

$$\zeta(2,2,2,1) = \tfrac{157}{16}\zeta(7) - \tfrac{15}{2}\zeta(2)\zeta(5) + \tfrac{9}{10}\zeta(2)^2\zeta(3), \tag{62}$$

$$\zeta(2,2,1,1,1) = -11\zeta(7) + 5\zeta(2)\zeta(5) + \tfrac{4}{5}\zeta(2)^2\zeta(3), \tag{63}$$

$$\zeta(2,1,2,1,1) = 17\zeta(7) - 10\zeta(2)\zeta(5), \tag{64}$$

$$\zeta(2,1,1,1,1,1) = \zeta(7). \tag{65}$$

Order 8

$$\zeta(8) = \tfrac{24}{175}\zeta(2)^4, \tag{66}$$

$$\zeta(7,1) = -\zeta(3)\zeta(5) + \tfrac{6}{35}\zeta(2)^4, \tag{67}$$

$$\zeta(6,1,1) = -3\zeta(3)\zeta(5) + \tfrac{1}{2}\zeta(2)\zeta(3)^2 + \tfrac{61}{175}\zeta(2)^4, \tag{68}$$

$$\zeta(5,3) = -\tfrac{5}{2}\zeta(6,2) + 5\zeta(3)\zeta(5) - \tfrac{21}{25}\zeta(2)^4, \tag{69}$$

$$\zeta(5,2,1) = \tfrac{7}{4}\zeta(6,2) + \tfrac{7}{2}\zeta(3)\zeta(5) - \zeta(2)\zeta(3)^2 - \tfrac{289}{1050}\zeta(2)^4, \tag{70}$$

$$\zeta(5,1,2) = -\zeta(6,2) + \tfrac{9}{2}\zeta(3)\zeta(5) - \tfrac{3}{2}\zeta(2)\zeta(3)^2 - \tfrac{29}{105}\zeta(2)^4, \tag{71}$$

$$\zeta(5,1,1,1) = -4\zeta(3)\zeta(5) + \zeta(2)\zeta(3)^2 + \tfrac{499}{1400}\zeta(2)^4, \tag{72}$$

$$\zeta(4,3,1) = -\tfrac{25}{4}\zeta(6,2) + \tfrac{21}{2}\zeta(3)\zeta(5) + \tfrac{1}{2}\zeta(2)\zeta(3)^2 - \tfrac{677}{350}\zeta(2)^4, \tag{73}$$

$$\zeta(4,2,2) = \tfrac{9}{2}\zeta(6,2) - 20\zeta(3)\zeta(5) + 3\zeta(2)\zeta(3)^2 + \tfrac{1271}{525}\zeta(2)^4, \tag{74}$$

$$\zeta(4,2,1,1) = \tfrac{3}{2}\zeta(6,2) + 5\zeta(3)\zeta(5) - 2\zeta(2)\zeta(3)^2 - \tfrac{863}{4200}\zeta(2)^4, \tag{75}$$

$$\zeta(4,1,3) = \tfrac{5}{2}\zeta(6,2) - \tfrac{15}{2}\zeta(3)\zeta(5) + \tfrac{1}{2}\zeta(2)\zeta(3)^2 + \tfrac{583}{525}\zeta(2)^4, \tag{76}$$

$$\zeta(4,1,2,1) = -\tfrac{15}{4}\zeta(6,2) + 13\zeta(3)\zeta(5) - \tfrac{3}{2}\zeta(2)\zeta(3)^2 - \tfrac{7211}{4200}\zeta(2)^4, \tag{77}$$

$$\zeta(4,1,1,2)=3\zeta(6,2)-9\zeta(3)\zeta(5)+\tfrac{1}{2}\zeta(2)\zeta(3)^2+\tfrac{229}{168}\zeta(2)^4, \quad (78)$$

$$\zeta(4,1,1,1,1)=-3\zeta(3)\zeta(5)+\tfrac{1}{2}\zeta(2)\zeta(3)^2+\tfrac{61}{175}\zeta(2)^4, \quad (79)$$

$$\zeta(3,3,2)=\tfrac{13}{4}\zeta(6,2)-23\zeta(3)\zeta(5)+\tfrac{9}{2}\zeta(2)\zeta(3)^2+\tfrac{857}{350}\zeta(2)^4, \quad (80)$$

$$\zeta(3,3,1,1)=-\tfrac{15}{4}\zeta(6,2)+13\zeta(3)\zeta(5)-\frac{3}{2}\zeta(2)\zeta(3)^2-\frac{7211}{4200}\zeta(2)^4, \quad (81)$$

$$\zeta(3,2,2,1)=9\zeta(6,2)-51\zeta(3)\zeta(5)+9\zeta(2)\zeta(3)^2+\tfrac{1607}{280}\zeta(2)^4, \quad (82)$$

$$\zeta(3,2,1,2)=-\tfrac{27}{4}\zeta(6,2)+27\zeta(3)\zeta(5)-3\zeta(2)\zeta(3)^2-\tfrac{15143}{4200}\zeta(2)^4, \quad (83)$$

$$\zeta(3,2,1,1,1)=\tfrac{7}{4}\zeta(6,2)+\tfrac{7}{2}\zeta(3)\zeta(5)-\zeta(2)\zeta(3)^2-\tfrac{289}{1050}\zeta(2)^4, \quad (84)$$

$$\zeta(3,1,2,2)=-\tfrac{9}{2}\zeta(6,2)+\tfrac{51}{2}\zeta(3)\zeta(5)-\tfrac{9}{2}\zeta(2)\zeta(3)^2-\tfrac{803}{280}\zeta(2)^4, \quad (85)$$

$$\zeta(3,1,2,1,1)=-\tfrac{25}{4}\zeta(6,2)+\tfrac{21}{2}\zeta(3)\zeta(5)+\tfrac{1}{2}\zeta(2)\zeta(3)^2-\tfrac{677}{350}\zeta(2)^4, \quad (86)$$

$$\zeta(3,1,1,2,1)=\tfrac{15}{4}\zeta(6,2)-2\zeta(2)\zeta(3)^2+\tfrac{673}{1050}\zeta(2)^4, \quad (87)$$

$$\zeta(3,1,1,1,2)=\tfrac{7}{4}\zeta(6,2)-12\zeta(3)\zeta(5)+2\zeta(2)\zeta(3)^2+\tfrac{487}{350}\zeta(2)^4, \quad (88)$$

$$\zeta(3,1,1,1,1,1)=-\zeta(3)\zeta(5)+\tfrac{6}{35}\zeta(2)^4, \quad (89)$$

$$\zeta(2,2,2,1,1)=\tfrac{9}{2}\zeta(6,2)-20\zeta(3)\zeta(5)+3\zeta(2)\zeta(3)^2+\tfrac{1271}{525}\zeta(2)^4, \quad (90)$$

$$\zeta(2,2,1,2,1)=\tfrac{13}{4}\zeta(6,2)-23\zeta(3)\zeta(5)+\tfrac{9}{2}\zeta(2)\zeta(3)^2+\tfrac{857}{350}\zeta(2)^4, \quad (91)$$

$$\zeta(2,2,1,1,1,1)=\zeta(6,2), \quad (92)$$

$$\zeta(2,1,2,1,1,1)=-\tfrac{5}{2}\zeta(6,2)+5\zeta(3)\zeta(5)-\tfrac{21}{25}\zeta(2)^4, \quad (93)$$

$$\zeta(2,1,1,1,1,1,1)=\tfrac{24}{175}\zeta(2)^4. \quad (94)$$

Order 9

$$\zeta(8,1)=4\zeta(9)-\zeta(2)\zeta(7)-\tfrac{2}{5}\zeta(2)^2\zeta(5)-\tfrac{8}{35}\zeta(2)^3\zeta(3), \quad (95)$$

$$\zeta(7,2)=-\tfrac{37}{2}\zeta(9)+7\zeta(2)\zeta(7)+\tfrac{8}{5}\zeta(2)^2\zeta(5)+\tfrac{16}{35}\zeta(2)^3\zeta(3), \quad (96)$$

$$\zeta(7,1,1)=\tfrac{28}{3}\zeta(9)-3\zeta(2)\zeta(7)-\tfrac{9}{10}\zeta(2)^2\zeta(5)+\tfrac{1}{6}\zeta(3)^3-\tfrac{2}{5}\zeta(2)^3\zeta(3), \quad (97)$$

$$\zeta(6,3)=\tfrac{83}{2}\zeta(9)-21\zeta(2)\zeta(7)-\tfrac{12}{5}\zeta(2)^2\zeta(5), \quad (98)$$

$$\zeta(6,2,1)=-\tfrac{2189}{72}\zeta(9)+11\zeta(2)\zeta(7)+\tfrac{13}{5}\zeta(2)^2\zeta(5) -\tfrac{1}{3}\zeta(3)^3+\tfrac{36}{35}\zeta(2)^3\zeta(3), \quad (99)$$

$$\zeta(6,1,2) = -\tfrac{313}{36}\zeta(9) + 7\zeta(2)\zeta(7) - \tfrac{1}{10}\zeta(2)^2\zeta(5) - \tfrac{1}{3}\zeta(3)^3 - \tfrac{8}{21}\zeta(2)^3\zeta(3), \tag{100}$$

$$\zeta(6,1,1,1) = 14\zeta(9) - 5\zeta(2)\zeta(7) - \tfrac{7}{5}\zeta(2)^2\zeta(5) + \tfrac{1}{2}\zeta(3)^3 - \tfrac{1}{2}\zeta(2)^3\zeta(3), \tag{101}$$

$$\zeta(5,4) = -\tfrac{127}{2}\zeta(9) + 35\zeta(2)\zeta(7) + 2\zeta(2)^2\zeta(5), \tag{102}$$

$$\zeta(5,3,1) = \tfrac{845}{24}\zeta(9) - 17\zeta(2)\zeta(7) - \tfrac{23}{10}\zeta(2)^2\zeta(5) + \tfrac{1}{6}\zeta(3)^3 - \tfrac{6}{35}\zeta(2)^3\zeta(3), \tag{103}$$

$$\zeta(5,2,2) = \tfrac{2513}{72}\zeta(9) - 21\zeta(2)\zeta(7) + \tfrac{7}{10}\zeta(2)^2\zeta(5) + \tfrac{2}{3}\zeta(3)^3 - \tfrac{64}{105}\zeta(2)^3\zeta(3), \tag{104}$$

$$\zeta(5,2,1,1) = -\tfrac{1909}{48}\zeta(9) + \tfrac{221}{16}\zeta(2)\zeta(7) + \tfrac{19}{4}\zeta(2)^2\zeta(5) - \zeta(3)^3 + \zeta(2)^3\zeta(3), \tag{105}$$

$$\zeta(5,1,3) = \tfrac{121}{12}\zeta(9) - 7\zeta(2)\zeta(7) + \tfrac{1}{5}\zeta(2)^2\zeta(5) - \tfrac{1}{3}\zeta(3)^3 + \tfrac{2}{7}\zeta(2)^3\zeta(3), \tag{106}$$

$$\zeta(5,1,2,1) = \tfrac{361}{144}\zeta(9) - \tfrac{5}{8}\zeta(2)\zeta(7) - \tfrac{7}{4}\zeta(2)^2\zeta(5) - \zeta(3)^3 + \tfrac{29}{30}\zeta(2)^3\zeta(3), \tag{107}$$

$$\zeta(5,1,1,2) = -\tfrac{2765}{144}\zeta(9) + \tfrac{189}{16}\zeta(2)\zeta(7) + \tfrac{3}{5}\zeta(2)^2\zeta(5) + \tfrac{1}{2}\zeta(3)^3 - \tfrac{19}{35}\zeta(2)^3\zeta(3), \tag{108}$$

$$\zeta(5,1,1,1,1) = 14\zeta(9) - 5\zeta(2)\zeta(7) - \tfrac{7}{5}\zeta(2)^2\zeta(5) + \tfrac{1}{2}\zeta(3)^3 - \tfrac{1}{2}\zeta(2)^3\zeta(3), \tag{109}$$

$$\zeta(4,4,1) = -\tfrac{328}{9}\zeta(9) + 18\zeta(2)\zeta(7) + 2\zeta(2)^2\zeta(5) - \tfrac{1}{3}\zeta(3)^3 + \tfrac{32}{105}\zeta(2)^3\zeta(3), \tag{110}$$

$$\zeta(4,3,2) = -\tfrac{53}{36}\zeta(9) + 14\zeta(2)\zeta(7) - 7\zeta(2)^2\zeta(5) + \tfrac{2}{3}\zeta(3)^3 - \tfrac{64}{105}\zeta(2)^3\zeta(3), \tag{111}$$

$$\zeta(4,3,1,1) = \tfrac{3227}{144}\zeta(9) - \tfrac{61}{8}\zeta(2)\zeta(7) - \tfrac{69}{20}\zeta(2)^2\zeta(5) - \tfrac{1}{42}\zeta(2)^3\zeta(3), \tag{112}$$

$$\zeta(4,2,3) = -59\zeta(9) + 28\zeta(2)\zeta(7) + 4\zeta(2)^2\zeta(5) - \tfrac{1}{3}\zeta(3)^3 + \tfrac{8}{21}\zeta(2)^3\zeta(3), \tag{113}$$

$$\zeta(4,2,2,1) = \tfrac{845}{24}\zeta(9) - \tfrac{157}{16}\zeta(2)\zeta(7) - \tfrac{16}{5}\zeta(2)^2\zeta(5) + 3\zeta(3)^3 - \tfrac{299}{105}\zeta(2)^3\zeta(3), \tag{114}$$

$$\zeta(4,2,1,2) = \tfrac{1501}{36}\zeta(9) - \tfrac{483}{16}\zeta(2)\zeta(7) + \tfrac{33}{20}\zeta(2)^2\zeta(5) - \zeta(3)^3 + \tfrac{106}{105}\zeta(2)^3\zeta(3), \tag{115}$$

$$\zeta(4,2,1,1,1) = -\tfrac{1909}{48}\zeta(9) + \tfrac{221}{16}\zeta(2)\zeta(7) + \tfrac{19}{4}\zeta(2)^2\zeta(5) - \zeta(3)^3 + \zeta(2)^3\zeta(3), \tag{116}$$

$$\zeta(4,1,3,1)=\tfrac{103}{144}\zeta(9)+\tfrac{1}{4}\zeta(2)\zeta(7)+\tfrac{1}{4}\zeta(2)^2\zeta(5)+\tfrac{1}{2}\zeta(3)^3 - \tfrac{53}{105}\zeta(2)^3\zeta(3), \tag{117}$$

$$\zeta(4,1,2,2)=-\tfrac{1187}{36}\zeta(9)+\tfrac{189}{16}\zeta(2)\zeta(7)+\tfrac{15}{4}\zeta(2)^2\zeta(5)-\zeta(3)^3 + \tfrac{61}{70}\zeta(2)^3\zeta(3), \tag{118}$$

$$\zeta(4,1,2,1,1)=\tfrac{3227}{144}\zeta(9)-\tfrac{61}{8}\zeta(2)\zeta(7)-\tfrac{69}{20}\zeta(2)^2\zeta(5)-\tfrac{1}{42}\zeta(2)^3\zeta(3), \tag{119}$$

$$\zeta(4,1,1,3)=\tfrac{3721}{144}\zeta(9)-\tfrac{231}{16}\zeta(2)\zeta(7)-\zeta(2)^2\zeta(5)-\tfrac{1}{2}\zeta(3)^3 + \tfrac{34}{105}\zeta(2)^3\zeta(3), \tag{120}$$

$$\zeta(4,1,1,2,1)=-\tfrac{4067}{144}\zeta(9)+\tfrac{109}{16}\zeta(2)\zeta(7)+\tfrac{43}{10}\zeta(2)^2\zeta(5)-\tfrac{3}{2}\zeta(3)^3 + \tfrac{148}{105}\zeta(2)^3\zeta(3), \tag{121}$$

$$\zeta(4,1,1,1,2)=\tfrac{265}{144}\zeta(9)+7\zeta(2)\zeta(7)-\tfrac{13}{5}\zeta(2)^2\zeta(5)+\tfrac{3}{2}\zeta(3)^3 - \tfrac{172}{105}\zeta(2)^3\zeta(3), \tag{122}$$

$$\zeta(4,1,1,1,1,1)=\tfrac{28}{3}\zeta(9)-3\zeta(2)\zeta(7)-\tfrac{9}{10}\zeta(2)^2\zeta(5)+\tfrac{1}{6}\zeta(3)^3 - \tfrac{2}{5}\zeta(2)^3\zeta(3), \tag{123}$$

$$\zeta(3,3,2,1)=\tfrac{1009}{36}\zeta(9)-\tfrac{75}{8}\zeta(2)\zeta(7)-\tfrac{9}{20}\zeta(2)^2\zeta(5)+\tfrac{7}{2}\zeta(3)^3 - \tfrac{227}{70}\zeta(2)^3\zeta(3), \tag{124}$$

$$\zeta(3,3,1,2)=-\tfrac{1205}{72}\zeta(9)+\tfrac{189}{16}\zeta(2)\zeta(7)-\tfrac{27}{20}\zeta(2)^2\zeta(5)-\tfrac{1}{2}\zeta(3)^3 + \tfrac{12}{35}\zeta(2)^3\zeta(3), \tag{125}$$

$$\zeta(3,3,1,1,1)=\tfrac{361}{144}\zeta(9)-\tfrac{5}{8}\zeta(2)\zeta(7)-\tfrac{7}{4}\zeta(2)^2\zeta(5)-\zeta(3)^3+\tfrac{29}{30}\zeta(2)^3\zeta(3), \tag{126}$$

$$\zeta(3,2,3,1)=-\tfrac{367}{8}\zeta(9)+\tfrac{291}{16}\zeta(2)\zeta(7)-\tfrac{9}{2}\zeta(3)^3+\tfrac{309}{70}\zeta(2)^3\zeta(3), \tag{127}$$

$$\zeta(3,2,2,2)=-\tfrac{223}{16}\zeta(9)+\tfrac{189}{16}\zeta(2)\zeta(7)-\tfrac{9}{4}\zeta(2)^2\zeta(5)+\tfrac{9}{70}\zeta(2)^3\zeta(3), \tag{128}$$

$$\zeta(3,2,2,1,1)=\tfrac{845}{24}\zeta(9)-\tfrac{157}{16}\zeta(2)\zeta(7)-\tfrac{16}{5}\zeta(2)^2\zeta(5)+3\zeta(3)^3 - \tfrac{299}{105}\zeta(2)^3\zeta(3), \tag{129}$$

$$\zeta(3,2,1,2,1)=\tfrac{1009}{36}\zeta(9)-\tfrac{75}{8}\zeta(2)\zeta(7)-\tfrac{9}{20}\zeta(2)^2\zeta(5)+\tfrac{7}{2}\zeta(3)^3 - \tfrac{227}{70}\zeta(2)^3\zeta(3), \tag{130}$$

$$\zeta(3,2,1,1,2)=\tfrac{989}{72}\zeta(9)-21\zeta(2)\zeta(7)+\tfrac{33}{20}\zeta(2)^2\zeta(5)-4\zeta(3)^3 + \tfrac{153}{35}\zeta(2)^3\zeta(3), \tag{131}$$

$$\zeta(3,2,1,1,1,1)=-\tfrac{2189}{72}\zeta(9)+11\zeta(2)\zeta(7)+\tfrac{13}{5}\zeta(2)^2\zeta(5)-\tfrac{1}{3}\zeta(3)^3 + \tfrac{36}{35}\zeta(2)^3\zeta(3), \tag{132}$$

$$\zeta(3,1,3,1,1)=\tfrac{103}{144}\zeta(9)+\tfrac{1}{4}\zeta(2)\zeta(7)+\tfrac{1}{4}\zeta(2)^2\zeta(5)+\tfrac{1}{2}\zeta(3)^3 - \tfrac{53}{105}\zeta(2)^3\zeta(3), \qquad (133)$$

$$\zeta(3,1,2,2,1)=-\tfrac{367}{8}\zeta(9)+\tfrac{291}{16}\zeta(2)\zeta(7)-\tfrac{9}{2}\zeta(3)^3+\tfrac{309}{70}\zeta(2)^3\zeta(3), \qquad (134)$$

$$\zeta(3,1,2,1,2)=-\tfrac{91}{6}\zeta(9)+14\zeta(2)\zeta(7)-\tfrac{9}{10}\zeta(2)^2\zeta(5)+\tfrac{3}{2}\zeta(3)^3 - \tfrac{53}{35}\zeta(2)^3\zeta(3), \qquad (135)$$

$$\zeta(3,1,2,1,1,1)=\tfrac{845}{24}\zeta(9)-17\zeta(2)\zeta(7)-\tfrac{23}{10}\zeta(2)^2\zeta(5)+\tfrac{1}{6}\zeta(3)^3 - \tfrac{6}{35}\zeta(2)^3\zeta(3), \qquad (136)$$

$$\zeta(3,1,1,2,2)=\tfrac{803}{36}\zeta(9)-14\zeta(2)\zeta(7)+\tfrac{17}{10}\zeta(2)^2\zeta(5)+\zeta(3)^3 - \tfrac{221}{210}\zeta(2)^3\zeta(3), \qquad (137)$$

$$\zeta(3,1,1,2,1,1)=-\tfrac{328}{9}\zeta(9)+18\zeta(2)\zeta(7)+2\zeta(2)^2\zeta(5)-\tfrac{1}{3}\zeta(3)^3 + \tfrac{32}{105}\zeta(2)^3\zeta(3), \qquad (138)$$

$$\zeta(3,1,1,1,2,1)=\tfrac{341}{24}\zeta(9)-10\zeta(2)\zeta(7)+\tfrac{1}{2}\zeta(2)^2\zeta(5)-\tfrac{1}{3}\zeta(3)^3 + \tfrac{2}{7}\zeta(2)^3\zeta(3), \qquad (139)$$

$$\zeta(3,1,1,1,1,2)=-\tfrac{461}{72}\zeta(9)+7\zeta(2)\zeta(7)-\tfrac{7}{10}\zeta(2)^2\zeta(5)+\tfrac{2}{3}\zeta(3)^3 - \tfrac{86}{105}\zeta(2)^3\zeta(3), \qquad (140)$$

$$\zeta(3,1,1,1,1,1,1)=4\zeta(9)-\zeta(2)\zeta(7)-\tfrac{2}{5}\zeta(2)^2\zeta(5)-\tfrac{8}{35}\zeta(2)^3\zeta(3), \qquad (141)$$

$$\zeta(2,2,2,2,1)=-\tfrac{223}{16}\zeta(9)+\tfrac{189}{16}\zeta(2)\zeta(7)-\tfrac{9}{4}\zeta(2)^2\zeta(5)+\tfrac{9}{70}\zeta(2)^3\zeta(3), \qquad (142)$$

$$\zeta(2,2,2,1,1,1)=\tfrac{2513}{72}\zeta(9)-21\zeta(2)\zeta(7)+\tfrac{7}{10}\zeta(2)^2\zeta(5)+\tfrac{2}{3}\zeta(3)^3 - \tfrac{64}{105}\zeta(2)^3\zeta(3), \qquad (143)$$

$$\zeta(2,2,1,2,1,1)=-\tfrac{53}{36}\zeta(9)+14\zeta(2)\zeta(7)-7\zeta(2)^2\zeta(5)+\tfrac{2}{3}\zeta(3)^3 - \tfrac{64}{105}\zeta(2)^3\zeta(3), \qquad (144)$$

$$\zeta(2,2,1,1,2,1)=\tfrac{593}{36}\zeta(9)-14\zeta(2)\zeta(7)+3\zeta(2)^2\zeta(5)+\tfrac{2}{3}\zeta(3)^3 - \tfrac{8}{15}\zeta(2)^3\zeta(3), \qquad (145)$$

$$\zeta(2,2,1,1,1,1,1)=-\tfrac{37}{2}\zeta(9)+7\zeta(2)\zeta(7)+\tfrac{8}{5}\zeta(2)^2\zeta(5)+\tfrac{16}{35}\zeta(2)^3\zeta(3), \qquad (146)$$

$$\zeta(2,1,2,1,1,1,1)=\tfrac{83}{2}\zeta(9)-21\zeta(2)\zeta(7)-\tfrac{12}{5}\zeta(2)^2\zeta(5), \qquad (147)$$

$$\zeta(2,1,1,2,1,1,1)=-\tfrac{127}{2}\zeta(9)+35\zeta(2)\zeta(7)+2\zeta(2)^2\zeta(5), \qquad (148)$$

$$\zeta(2,1,1,1,1,1,1,1)=\zeta(9). \qquad (149)$$

Order 10

$$\zeta(10) = \tfrac{32}{385}\zeta(2)^5, \tag{150}$$

$$\zeta(9,1) = -\zeta(3)\zeta(7) - \tfrac{1}{2}\zeta(5)^2 + \tfrac{8}{55}\zeta(2)^5, \tag{151}$$

$$\zeta(8,1,1) = -4\zeta(3)\zeta(7) - 2\zeta(5)^2 + \zeta(2)\zeta(3)\zeta(5) + \tfrac{1}{5}\zeta(2)^2\zeta(3)^2 + \tfrac{666}{1925}\zeta(2)^5, \tag{152}$$

$$\zeta(7,3) = -\tfrac{7}{2}\zeta(8,2) + 7\zeta(3)\zeta(7) + 4\zeta(5)^2 - \tfrac{408}{385}\zeta(2)^5, \tag{153}$$

$$\zeta(7,2,1) = \tfrac{9}{4}\zeta(8,2) - \zeta(2)\zeta(6,2) + \tfrac{9}{2}\zeta(3)\zeta(7) + \tfrac{9}{4}\zeta(5)^2 - \tfrac{2}{5}\zeta(2)^2\zeta(3)^2 - \tfrac{3016}{5775}\zeta(2)^5, \tag{154}$$

$$\zeta(7,1,2) = -\zeta(8,2) + \zeta(2)\zeta(6,2) + 10\zeta(3)\zeta(7) + 5\zeta(5)^2 - 7\zeta(2)\zeta(3)\zeta(5) - \tfrac{2}{5}\zeta(2)^2\zeta(3)^2 - \tfrac{2}{15}\zeta(2)^5, \tag{155}$$

$$\zeta(7,1,1,1) = -8\zeta(3)\zeta(7) - 4\zeta(5)^2 + 3\zeta(2)\zeta(3)\zeta(5) + \tfrac{9}{20}\zeta(2)^2\zeta(3)^2 + \tfrac{139}{275}\zeta(2)^5, \tag{156}$$

$$\zeta(6,4) = \tfrac{7}{2}\zeta(8,2) - 7\zeta(3)\zeta(7) - 5\zeta(5)^2 + \tfrac{2216}{1925}\zeta(2)^5, \tag{157}$$

$$\vdots$$

$$\zeta(2,1,1,2,1,1,1,1) = \tfrac{7}{2}\zeta(8,2) - 7\zeta(3)\zeta(7) - 5\zeta(5)^2 + \tfrac{2216}{1925}\zeta(2)^5, \tag{158}$$

$$\zeta(2,1,1,1,1,1,1,1,1) = \tfrac{32}{385}\zeta(2)^5. \tag{159}$$

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