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A NEW ITERATIVE METHOD FOR FINDING COMMON SOLUTIONS OF GENERALIZED EQUILIBRIUM PROBLEM, FIXED POINT PROBLEM OF INFINITE k-STRICT PSEUDO-CONTRACTIVE MAPPINGS, AND QUASI-VARIATIONAL INCLUSION PROBLEM*

Liu Min (刘敏)† *Zhang Shisheng* (张石生)
Department of Mathematics, Yibin university, Yibin 644007, China
E-mail: liuminybsc@yahoo.com.cn; changss@yahoo.cn

Abstract In this article, we introduce a hybrid iterative scheme for finding a common element of the set of solutions for a generalized equilibrium problems, the set of common fixed point for a family of infinite k-strict pseudo-contractive mappings, and the set of solutions of the variational inclusion problem with multi-valued maximal monotone mappings and inverse-strongly monotone mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extends the recent results in G.L.Acedo and H.K.Xu [2], Zhang, Lee and Chan [8], Takahashi and Toyoda [9], Takahashi and Takahashi [10] and S. S. Chang, H. W. Joseph Lee and C. K. Chan [11], S.Takahashi and W.Takahashi [12]. Moreover, the method of proof adopted in this article is different from those of [4] and [12].

Key words k-strict pseudo-contractive mappings; generalized equilibrium problem; viscosity approximation method; variational inclusion problem; multi-valued maximal monotone mappings; α-inverse-strongly monotone mapping

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1 Introduction

Throughout this article, we assume that H is a real Hilbert space and C is a nonempty closed convex subset of H.

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†Corresponding author.

A mapping $T : C \to H$ is said to be an k-strict pseudo-contraction if there exists a constant $0 \le k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

In the sequel, we denote the set of fixed points of T by $F(T)$.

The normal Mann's iterative process [1] generates a sequence $\{x_n\}$ in the following manner: for any $x_1 \in C$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \ge 1, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$.

If $T : C \to C$ is a nonexpansive mapping with a fixed point, and the control sequence $\{\alpha_n\}$ is chosen such that $\sum\limits_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann's iterative process (1.2) converges weakly to a fixed point of T.

Recently, Acedo and Xu [2] studied the k-strict pseudo-contractions and gave a weak convergence theorem in the framework of Hilbert spaces. More precisely, they proved the following theorem.

Theorem 1.1 [2] Let C be a closed convex subset of a Hilbert space H. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N - 1$, $T_i : C \to C$ be a k_i-strict pseudo-contraction for some $0 \le k_i < 1$ and $k = \max\{k_i : 0 \le i \le N - 1\}$. Assume that the common fixed point set $\bigcap\limits_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. For any $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{[n]}x_n, \quad \forall n \ge 0. \tag{1.3}$$

Assume that the control sequence $\{\alpha_n\}$ is chosen such that $k+\epsilon \le \alpha_n < 1-\epsilon$ for all n and some $\epsilon \in (0, 1)$. Then, $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.

In contrast, let C be a nonempty closed convex subset of H. Let F be an equilibrium bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$, such that

$$F(x, y) \ge 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(F)$.

A bounded linear operator $A : H \to H$ is said to be strongly positive if there exists a constant $\bar{\gamma}$ such that

$$\langle Ax, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.5}$$

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multi-valued mapping. The "so-called" quasi-variational inclusion problem (see Verma [3], Chang [4, 5]) is to find an $u \in H$ such that

$$\theta \in B(u) + M(u). \tag{1.6}$$

A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions (see, for example, [6]).

The set of solutions of variational inclusion (1.6) is denoted by $VI(H, B, M)$.

Special Case

If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty)$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then, the variational inclusion problem (1.6) is equivalent to find $u \in C$, such that

$$\langle B(u), v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.7}$$

This problem is called Hartman-Stampacchia variational inequality problem (see, for example, [7]). The set of solutions of (1.7) is denoted by $VI(C, B)$.

Recall that a mapping $B : H \to H$ is called α-inverse strongly monotone [8] if there exists an $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha \parallel Bx - By \parallel^2, \quad \forall x, y \in H$$

A multi-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H, u \in Mx$ and $v \in My$ imply that $\langle u - v, x - y\rangle \geq 0$. A multi-valued mapping $M : H \to 2^H$ is called maximal monotone if it is monotone and, for any $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$, $\forall (y, v) \in \text{Graph}(M)$ (the graph of mapping M) implies that $u \in Mx$.

Proposition 1.1 Let $B : H \to H$ be an α-inverse strongly monotone mapping, then,

(a) B is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;

(b) If λ is any constant in $(0, 2\alpha]$, then the mapping $I - \lambda B$ is nonexpansive, where I is the identity mapping on H.

For finding an element of $F(S) \bigcap VI(C, B)$, Takahashi and Toyoda [9] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Bx_n), \quad \forall n \geq 0. \tag{1.8}$$

Under suitable conditions, they proved that the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \bigcap VI(C, B)$.

For finding an element of $F(S) \cap EP(F)$, Takahashi and Takahashi [10] introduced the following iterative scheme using the viscosity approximation method in a Hilbert space: $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n \end{cases} \quad \forall n \geq 0 \tag{1.9}$$

For finding an common element of $F(S) \bigcap VI(H, B, M)$, in 2008, Zhang, Lee, and Chan [8] introduced an iterative scheme: $x_0 = x \in H$,

$$\begin{cases} x_{n+1} = \alpha_n x + (1 - \alpha_n)Sy_n, \\ y_n = J_{M,\lambda}(x_n - \lambda Bx_n), \end{cases} \quad \forall n \geq 0. \tag{1.10}$$

Under suitable conditions, some strong convergence theorems are proved, which extend some recent results of Takahashi and Toyoda [9].

For finding a common element of the set of solutions of the variational inequality problem (1.7) for an α-inverse strongly monotone mappings, the set of solutions for an equilibrium problems, and the set of common fixed point for a family of infinite nonexpansive mappings in a Hilbert space. Very recently, S. S. Chang, H. W. Joseph Lee, and C. K. Chan [11] introduced a new iterative sequence:

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \\ k_n = P_C(y_n - \lambda_n B y_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \end{cases} \quad \forall n \geq 1. \tag{1.11}$$

Under suitable conditions, some strong convergence theorems are proved, which extend the main results in Takahashi and Toyoda [9] and Takahashi and Takahashi [10].

Let F be an equilibrium bifunction from $C \times C$ into R and $B : C \to H$ be a nonlinear mapping. Then, we consider the following generalized equilibrium problem: find $z \in C$ such that

$$F(z, y) + \langle Bz, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.12}$$

The set of solutions of (1.12) is denoted by EP, that is,

$$EP = \{z \in C : F(z, y) + \langle Bz, y - z\rangle \geq 0, \quad \forall y \in C\}.$$

In the case of $B \equiv 0$, EP is denoted by $EP(F)$. In the case of $F \equiv 0$, EP is denoted by $VI(C, B)$.

Motivated and inspired by G.L.Acedo and H.K.Xu [2], Zhang, Lee and Chan [8], Takahashi and Toyoda [9], S. S. Chang, H. W. Joseph Lee, C. K. Chan [11] and S.Takahashi and W.Takahashi [12], the purpose of this article is to introduce a hybrid iterative scheme for finding a common element of the set of solutions for a generalized equilibrium problem, the set of common fixed point for a family of infinite k-strict pseudo-contractive mappings and the set of solutions of the variational inclusion problem with multi-valued maximal monotone mappings and inverse-strongly monotone mappings in Hilbert space. Under suitable conditions, some strong convergence theorems are proved. Our results extend the recent results in G.L.Acedo and H.K.Xu [2], Zhang, Lee and Chan [8], Takahashi and Toyoda [9], Takahashi and Takahashi [10] and S. S. Chang, H. W. Joseph Lee and C. K. Chan [11], S.Takahashi and W.Takahashi [12]. Moreover, the method of proof adopted in this article is different from those of [4] and [12].

2 Preliminaries

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

Let H be a Hilbert space, and C be a nonempty closed convex set of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

Such a mapping P_C from H onto C is called the metric projection.

Remark 1 It is well-known that the metric projection P_C has the following properties:

i) $P_C : H \to C$ is nonexpansive;

ii) P_C is firmly nonexpansive, that is, $\|P_Cx - P_Cy\|^2 \le \langle P_Cx - P_Cy, x - y\rangle,\ \forall x, y \in H$;

iii) For each $x \in H$, $z = P_Cx \Leftrightarrow \langle x - z, z - y\rangle \ge 0,\ \forall y \in C$.

It is also known that H satisfies Opial's condition [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Definition 2.1 Let $M : H \to 2^H$ be a multi-valued maximal monotone mapping, then, the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad \forall u \in H,$$

is called the resolvent operator associated with M, where λ is any positive number and I is the identity mapping.

Proposition 2.1 [8] (a) The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \le \|x - y\|, \quad \forall x, y \in H, \forall \lambda > 0.$$

(b) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone, that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \le \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \quad \forall x, y \in H.$$

Definition 2.2 A single-valued mapping $P : H \to H$ is said to be hemi-continuous if, for any $x, y, z \in H$, the function $t \mapsto \langle p(x + ty), z\rangle$ is continuous at $0+$.

It is well-known that every continuous mapping must be hemi-continuous.

Lemma 2.1 [14] Let E be a real Banach space, E^* be its dual space, $T : E \to 2^{E^*}$ be a maximal monotone mapping, and $P : E \to E^*$ be a hemi-continuous bounded monotone mapping with $D(T) = X$, then the mapping $S = T + P : E \to 2^{E^*}$ is a maximal monotone mapping.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, 1]$. For $n \ge 1$, define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I\\
&U_{n,n} = \lambda_n S_n U_{n,n+1} + (1 - \lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} S_{n-1} U_{n,n} + (1 - \lambda_{n-1})I,\\
&\qquad\vdots\\
&U_{n,k} = \lambda_k S_k U_{n,k+1} + (1 - \lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} S_{k-1} U_{n,k} + (1 - \lambda_{k-1})I,\\
&\qquad\vdots\\
&U_{n,2} = \lambda_2 S_2 U_{n,3} + (1 - \lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 S_1 U_{n,2} + (1 - \lambda_1)I,
\end{aligned} \tag{2.1}$$

such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \cdots, S_1$ and $\lambda_n, \lambda_{n-1}, \cdots, \lambda_1$ (see [15]).

Lemma 2.2 [15] Let C be a nonempty closed convex subset of a Banach space E and $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap\limits_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. For any $n \geq 1$, let W_n be the W-mapping of C into itself generated by $S_n, S_{n-1}, \cdots, S_1$, and $\lambda_n, \lambda_{n-1}, \cdots, \lambda_1$. Then, W_n is asymptotically regular and nonexpansive. Further, if E is strictly convex, then, $F(W_n) = \bigcap\limits_{i=1}^{n} F(S_i)$.

Lemma 2.3 [15] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap\limits_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim\limits_{n\to\infty} U_{n,k}x$ exists.

Using Lemma 2.3, one can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for every $x \in C$. Such a W is called the W-mapping generated by the sequence $\{S_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Throughout this article, we always assume that $\{\lambda_n\}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$.

Lemma 2.4 [15] Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap\limits_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then, W is a nonexpansive mapping and $F(W) = \bigcap\limits_{n=1}^{\infty} F(S_n)$.

Lemma 2.5 [11] Let C be a nonempty closed convex subset of a Hilbert space H and $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings, such that $\bigcap\limits_{n=1}^{\infty} F(S_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. If K is any bounded subset of C, then,

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.2}$$

Lemma 2.6 [16] Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup\limits_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim\limits_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.7 [3] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers, such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\gamma_n \in (0, 1)$ and δ_n are sequences satisfying

(1) $\sum\limits_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum\limits_{n=1}^{\infty} |\delta_n| = \infty$;

then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemama 2.8 [17] Let E be a real Banach space, $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, the following conclusion holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Especially, if $E = H$ is a real Hilbert space, then, $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x, y \in H$.

For solving the equilibrium problem for bifunction $F : C \times C \to R$, assume that F satisfies the following conditions:

(A_1) $F(x,x) = 0$ for all $x \in C$;

(A_2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A_3) for any $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x,\ y) \leq F(x,y);$$

(A_4) for each $x \in C, y \mapsto F(x,y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $F : C \times C \to R$ satisfies conditions $(A_1) - (A_4)$, then, we have the following three important results.

Lemma 2.9 [18] Let C be a nonempty closed convex subset of H and let F be an equilibrium bifunction $F : C \times C \to R$ satisfying conditions $(A_1) - (A_4)$. Let $r > 0$ and $x \in C$. Then, there exists $y \in C$, such that $F(y,z) + \frac{1}{r}\langle z-y, y-x\rangle \geq 0,\ \forall z \in C$.

Lemma 2.10 [19] Let F be the same as given in Lemma 2.9. For given $r > 0$ and $x \in C$, define a mapping $T_r : H \to C$ as

$$T_r(x) = \{y \in C : F(y,z) + \frac{1}{r}\langle z-y, y-x\rangle \geq 0, \quad \forall z \in C\}.$$

Then, the following conclusions hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is a closed and convex set.

3 The Main Results

In order to prove the main results, we first give the following lemma.

Lemma 3.1 Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to H$ be an k-strict pseudo-contraction, then, condition (1.1) is equivalent to the following condition:

$$\langle (I-T)x - (I-T)y, x-y\rangle \geq \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C,\ k \in [0,1] \qquad (3.1)$$

Proof Necessity: From the definition of k-strict pseudo-contraction, we have

$$\begin{aligned}\|Tx - Ty\|^2 &= \|(I-T)x - (I-T)y - (x-y)\|^2 \\ &= \|(I-T)x - (I-T)y\|^2 + \|x-y\|^2 - 2\langle (I-T)x - (I-T)y, x-y\rangle \\ &\leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2.\end{aligned}$$

So, we have

$$\langle (I-T)x-(I-T)y, x-y\rangle \geq \frac{1-k}{2}\|(I-T)x-(I-T)y\|^2.$$

Sufficiency: Form (3.1), we have

$$\begin{aligned}\|Tx-Ty\|^2 &= \|(I-T)x-(I-T)y-(x-y)\|^2\\ &= \|(I-T)x-(I-T)y\|^2+\|x-y\|^2-2\langle (I-T)x-(I-T)y, x-y\rangle\\ &\leq \|(I-T)x-(I-T)y\|^2+\|x-y\|^2-(1-k)\|(I-T)x-(I-T)y\|^2\\ &\leq \|x-y\|^2+k\|(I-T)x-(I-T)y\|^2.\end{aligned}$$

Lemma 3.2 Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to H$ be a k-strict pseudo-contraction, then, T has the following properties:

(1) T is a Lipschitz continuous mapping;

(2) $I-T$ is demiclosed at any $y \in H$;

(3) $F(T)$ is closed and convex;

(4) For any $\alpha \in [k, 1)$, define a mapping $S : C \to H$ as follows:

$$Sx = \alpha x + (1-\alpha)Tx, \quad \forall x \in C,$$

then, S is nonexpansive and $F(S) = F(T)$.

Proof (1) Form (3.1), we have

$$\begin{aligned}\frac{1-k}{2}\|(I-T)x-(I-T)y\|^2 &\leq \langle (I-T)x-(I-T)y, x-y\rangle\\ &\leq \|(I-T)x-(I-T)y\|\cdot\|x-y\|.\end{aligned}$$

So, we have

$$\frac{2}{1-k}\|x-y\| \geq \|(I-T)x-(I-T)y\| \geq \|Tx-Ty\|-\|x-y\|,$$

then,

$$(1+\frac{2}{1-k})\|x-y\| \geq \|Tx-Ty\|,$$

that is, $\|Tx-Ty\| \leq \frac{3-k}{1-k}\|x-y\|$.

(2) Assume $x_n \rightharpoonup x$ and $x_n - Tx_n \to y$. From (3.1), we have

$$\langle (I-T)x_n-(I-T)x, x_n-y\rangle \geq \frac{1-k}{2}\|(x_n-Tx_n)-(x-Tx)\|^2.$$

So, we have $0 \geq \frac{1-k}{2}\|y-(x-Tx)\|^2 \geq 0$, then, $(I-T)x = y$.

(3) We first prove $F(T)$ is close.

Because T is Lipschitz continuous, we have T which is continuous. So, $F(T)$ is close.

Next, we prove $F(T)$ is convex.

For any $p, q \in F(T)$, let $z_t = (1-t)p + tq, \forall t \in (0, 1)$. Form (3.1), we have

$$\langle (I-T)z_t-(I-T)p, z_t-p\rangle \geq \frac{1-k}{2}\|(I-T)z_t-(I-T)p\|^2.$$

So, we have

$$\langle z_t - Tz_t, z_t - p\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2. \tag{3.2}$$

Similarly, we have

$$\langle z_t - Tz_t, z_t - q\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2. \tag{3.3}$$

Because $z_t - p = t(q-p), z_t - q = (1-t)(p-q)$, form (3.2) and (3.3), we have

$$t\langle z_t - Tz_t, q-p\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2 \text{ and } (1-t)\langle z_t - Tz_t, p-q\rangle \geq \frac{1-k}{2}\|z_t - Tz_t\|^2.$$

So, we have $0 \geq \frac{1-k}{2}\|z_t - Tz_t\|^2$, then, $\|z_t - Tz_t\|^2 = 0$, that is, $z_t \in F(T)$.

(4) We first prove S is nonexpansive.

From T is an k-strict pseudo-contraction, for any $x, y \in C$ and $\alpha \in [k, 1)$, we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|\alpha x + (1-\alpha)Tx - [\alpha y + (1-\alpha)Ty]\|^2 \\
&= \|\alpha(x-y) + (1-\alpha)(Tx - Ty)\|^2 \\
&= \alpha\|x-y\|^2 + (1-\alpha)\|Tx - Ty\|^2 - \alpha(1-\alpha)\|x - y - (Tx - Ty)\|^2 \\
&\leq \alpha\|x-y\|^2 + (1-\alpha)\{\|x-y\|^2 + k\|x - y - (Tx-Ty)\|^2\} \\
&\quad -\alpha(1-\alpha)\|x-y-(Tx-Ty)\|^2 \\
&= \|x-y\|^2 - (1-\alpha)(\alpha-k)\|x-y-(Tx-Ty)\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Next, we prove $F(S) = F(T)$.

Because $x - Sx = (1-\alpha)(x - Tx)$, we have $F(S) = F(T)$.

Let $T_i : C \to C, i = 1, 2, \cdots$, be a family of infinite k_i-strict pseudo-contractive mappings, we can define a family of infinite nonexpansive mappings $\{S_i\}_{i=1}^{\infty} : C \to C$ by

$$S_i x = \alpha x + (1-\alpha)T_i x, \quad \alpha \in [k, 1], \tag{3.4}$$

where $k = \sup_{i\geq 1} k_i$.

It is obvious from Lemma 3.1 that

$$\bigcap_{i=1}^{\infty} F(T_i) = \bigcap_{i=1}^{\infty} F(S_i).$$

Lemma 3.3 [11] (a) $u \in H$ is a solution of variational inclusion (1.6), if and only if $u = J_{M,\lambda}(u - \lambda Bu), \forall \lambda > 0$, that is,

$$VI(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0.$$

(b) If $\lambda \in (0, 2\alpha]$, then, $VI(H, B, M)$ is a closed convex subset in H.

Theorem 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H and B, D be two different α-inverse-strongly monotone mappings from C into H. Let F be a bifunction from $C \times C \to R$ which satisfies $(A_1) - (A_4)$. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of infinite k_n-strict pseudo-contractive mappings with $0 \leq k_n < 1$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, b]$ for some $b \in (0, 1)$, $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings defined by (3.4) such that $G = F(W)\bigcap VI(H,B,M)\bigcap EP \neq \emptyset$, where $F(W) := \bigcap_{n=1}^{\infty} F(S_n)$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and f be a contraction of H into itself with contraction constant $h(0 < h < 1)$ and $0 < \gamma < \frac{\bar{\gamma}}{h}$. Let $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$, and $\{y_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} F(y_n,\eta) + \langle Dy_n, \eta - y_n\rangle + \dfrac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0, \ \forall \eta \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n, \\ \rho_n = J_{M,\lambda}(I-\lambda B)\xi_n, \\ \xi_n = J_{M,\lambda}(I-\lambda B)y_n, \end{cases} \quad \forall n \geq 1, \tag{3.5}$$

where $\{W_n : C \to C\}$ is the sequence defined by (2.1), $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [0,\infty]$. If the following conditions are satisfied:

(C_1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(C_2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C_3) $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(C_4) $\liminf\limits_{n\to\infty} r_n > 0, 0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$,

then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in G$, where $q = P_G(\gamma f + (I - A))(q)$.

Proof We define a bifunction $\phi : C \times C \to R$ by

$$\phi(z,y) = F(z,y) + \langle Dz, y - z\rangle, \quad \forall z, y \in C.$$

Next, we prove that the bifunction ϕ satisfies the following conditions $(A_1) - (A_4)$.

(A_1) $\phi(x,x) = 0$ for all $x \in C$.

Because $\phi(x,x) = F(x,x) + \langle Dx, 0\rangle = F(x,x) = 0$, for all $x \in C$.

(A_2) ϕ is monotone, that is, $\phi(z,y) + \phi(y,z) \leq 0$ for all $y, z \in C$.

Because D is α-inverse-strongly monotone, hence from the definition of ϕ, we have

$$\begin{aligned} \phi(z,y) + \phi(y,z) &= F(z,y) + \langle Dz, y - z\rangle + F(y,z) + \langle Dy, z - y\rangle \\ &= F(z,y) + F(y,z) + \langle Dz, y - z\rangle - \langle Dy, y - z\rangle \\ &\leq 0 + \langle Dz - Dy, y - z\rangle \\ &= -\langle Dy - Dz, y - z\rangle \leq 0, \end{aligned}$$

(A_3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x,y),$$

because

$$\begin{aligned} &\lim_{t\downarrow 0} \phi(tz + (1-t)x, y) \\ =& \lim_{t\downarrow 0} F(tz + (1-t)x, y) + \lim_{t\downarrow 0}\langle D(tz + (1-t)x), y - (tz + (1-t)x)\rangle \\ \leq& F(x,y) + \langle Dx, y - x\rangle = \phi(x,y). \end{aligned}$$

(A_4) for each $x \in C, y \mapsto \phi(x, y)$ is a convex and lower semicontinuous.

For each $x \in C$, $\forall t \in (0,1)$ and $\forall y, z \in C$, because F satisfies (A_4), we have

$$\begin{aligned}\phi(x, ty + (1-t)z) &= F(x, ty + (1-t)z) + \langle Dx, ty + (1-t)z - x\rangle \\ &\le t[F(x,y) + \langle Dx, y - x\rangle] + (1-t)[F(x,z) + \langle Dx, z - x\rangle] \\ &= t\phi(x,y) + (1-t)\phi(x,z).\end{aligned}$$

So, $y \mapsto \phi(x, y)$ is convex.

Similarly, we can prove that $y \mapsto \phi(x, y)$ is lower semi-continuous.

Therefore, the generalized equilibrium problem (1.12) is equivalent to the following equilibrium problem: find $z \in C$ such that $\phi(z, y) \ge 0,\ \forall y \in C$, and (3.5) can be written as:

$$\begin{cases}\phi(y_n, \eta) + \dfrac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \ge 0, \quad \forall \eta \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n, \\ \rho_n = J_{M,\lambda}(I - \lambda B)\xi_n, \\ \xi_n = J_{M,\lambda}(I - \lambda B)y_n,\end{cases} \quad \forall n \ge 1, \tag{3.6}$$

Because the bifunction ϕ satisfies conditions $(A_1) - (A_4)$, from Lemma 2.10, for a given $r > 0$ and $x \in C$, we can define a mapping $V_r : H \to C$ as follows:

$$V_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}.$$

Moreover, V_r satisfies the conclusions in Lemma 2.10 and $EP = EP(\phi)$.

We observe that from conditions (C_1) and (C_4), we can assume, without loss of generality, that $\alpha_n \le (1-\beta_n)\|A\|^{-1}$.

Because A is a linear bounded self-adjoint operator on H, then,

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}.$$

Because

$$\langle ((1-\beta_n)I - \alpha_n A)u, u\rangle = 1 - \beta_n - \alpha_n\langle Au, u\rangle \ge 1 - \beta_n - \alpha_n\|A\| \ge 0,$$

this implies that $(1-\beta_n)I - \alpha_n A$ is positive. Hence, we have

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n A\| &= \sup\{|\langle((1-\beta_n)I - \alpha_n A)u, u\rangle| : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1 - \beta_n - \alpha_n\bar{\gamma}.\end{aligned} \tag{3.7}$$

We divide the proof of Theorem 3.1 into nine steps:

Step 1 First, prove the sequences $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$, and $\{y_n\}$ are bounded.

(a) pick $p \in G$, as $y_n = V_{r_n}x_n$, we have

$$\|y_n - p\| = \|V_{r_n}x_n - p\| \le \|x_n - p\| \tag{3.8}$$

(b) Because $p \in VI(H,B,M)$ and $\rho_n = J_{M,\lambda}(I-\lambda B)\xi_n$, we have $p = J_{M,\lambda}(I-\lambda B)p$, and so

$$\begin{aligned}\|\rho_n - p\| &= \|J_{M,\lambda}(I-\lambda B)\xi_n - J_{M,\lambda}(I-\lambda B)p\| \\ &\le \|(I-\lambda B)\xi_n - (I-\lambda B)p\| \le \|\xi_n - p\| \\ &= \|J_{M,\lambda}(I-\lambda B)y_n - J_{M,\lambda}(I-\lambda B)p\| \\ &\le \|y_n - p\| \le \|x_n - p\|. \end{aligned} \tag{3.9}$$

From (3.6), (3.7), (3.8), and (3.9), we have

$$\begin{aligned}&\|x_{n+1} - p\| \\ =&\ \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n - p\| \\ =&\ \|\alpha_n\gamma(f(x_n) - f(p)) + \beta_n(x_n - p) + ((1-\beta_n)I - \alpha_n A)(W_n\rho_n - p) + \alpha_n(\gamma f(p) - Ap)\| \\ \le&\ \alpha_n\gamma h\|x_n - p\| + \beta_n\|x_n - p\| + ((1-\beta_n) - \alpha_n\bar{\gamma})\|\rho_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ \le&\ (1 - \alpha_n(\bar{\gamma} - \gamma h))\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\ \le&\ \max \|x_n - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\| \\ &\ \vdots \\ \le&\ \max \|x_1 - p\|, \frac{1}{\bar{\gamma} - \gamma h}\|\gamma f(p) - Ap\|.\end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in H. Therefore, $\{y_n\}, \{\rho_n\}, \{\xi_n\}, \{\gamma f(x_n)\}$, and $\{W_n\rho_n\}$ are all bounded.

Step 2 Next, we prove that

$$\|x_{n+1} - x_n\| \to 0(n\to\infty) \ \text{ and } \ \|\rho_{n+1} - \rho_n\| \to 0(n\to\infty). \tag{3.10}$$

In fact, let us define a sequence $\{z_n\}$ by

$$x_{n+1} = (1-\beta_n)z_n + \beta_n x_n \qquad \forall n \ge 1,$$

then, we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)W_{n+1}\rho_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)W_n\rho_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1}) - AW_{n+1}\rho_{n+1}] - \frac{\alpha_n}{1-\beta_n}[\gamma f(x_n) - AW_n\rho_n] \\ &\quad + W_{n+1}\rho_{n+1} - W_{n+1}\rho_n + W_{n+1}\rho_n - W_n\rho_n \end{aligned}$$

and so

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ \le&\ \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\rho_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AW_n\rho_n\|) \\ &+ \|W_{n+1}\rho_{n+1} - W_{n+1}\rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\| - \|x_{n+1} - x_n\|. \end{aligned} \tag{3.11}$$

Because

$$\rho_n = J_{M,\lambda}(I - \lambda B)\xi_n,$$

we have

$$\begin{aligned}\|\rho_{n+1} - \rho_n\| &= \|J_{M,\lambda}(I - \lambda B)\xi_{n+1} - J_{M,\lambda}(I - \lambda B)\xi_n\| \le \|\xi_{n+1} - \xi_n\| \\ &= \|J_{M,\lambda}(I - \lambda B)y_{n+1} - J_{M,\lambda}(I - \lambda B)y_n\| \le \|y_{n+1} - y_n\|. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le &\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\rho_{n+1}\|) \\ &+\frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AW_n\rho_n\|) + \|y_{n+1} - y_n\| \\ &+\|W_{n+1}\rho_n - W_n\rho_n\| - \|x_{n+1} - x_n\|. \end{aligned} \tag{3.13}$$

Observing $y_n = V_{r_n} x_n$ and $y_{n+1} = V_{r_{n+1}} x_{n+1}$, we have

$$\phi(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \ge 0, \quad \forall \eta \in C \tag{3.14}$$

and

$$\phi(y_{n+1}, \eta) + \frac{1}{r_{n+1}}\langle \eta - y_{n+1}, y_{n+1} - x_{n+1}\rangle \ge 0, \forall \eta \in C. \tag{3.15}$$

Putting $\eta = y_{n+1}$ in (3.14) and $\eta = y_n$ in (3.15), we have

$$\phi(y_n, y_{n+1}) + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \ge 0$$

and

$$\phi(y_{n+1}, y_n) + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \ge 0.$$

Adding up these two inequalities and then using the condition (A_2) to simplify, we have

$$\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\rangle \ge 0.$$

That is,

$$\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\rangle \ge 0.$$

By condition (C_4), without loss of generality, we can assume that there exists a real number m such that $\gamma_n > m > 0$ for all n, and so

$$\|y_{n+1} - y_n\|^2 \le \|y_{n+1} - y_n\|(\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|(\|y_{n+1} - x_{n+1}\|).$$

Simplifying it, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|(\|y_{n+1} - x_{n+1}\|)) \\ &\le \|x_{n+1} - x_n\| + \frac{M_1}{m}|r_{n+1} - r_n|), \end{aligned} \tag{3.16}$$

where $M_1 = \sup_{n\geq 1} \|y_n - x_n\|$. Because S_i and $U_{n,i}$ are nonexpansive, from (2.1), we have

$$\begin{aligned}\|W_{n+1}\rho_n - W_n\rho_n\| &= \|\lambda_1 S_1 U_{n+1,2}\rho_n + (1-\lambda_1)\rho_n - (\lambda_1 S_1 U_{n,2}\rho_n + (1-\lambda_1)\rho_n)\| \\ &\leq \lambda_1 \|U_{n+1,2}\rho_n - U_{n,2}\rho_n\| \\ &\leq \lambda_1 \|\lambda_2 S_2 U_{n+1,3}\rho_n + (1-\lambda_2)\rho_n - (\lambda_2 S_2 U_{n,3}\rho_n + (1-\lambda_2)\rho_n)\| \\ &\leq \lambda_1\lambda_2 \|U_{n+1,3}\rho_n - U_{n,3}\rho_n\| \\ &\;\;\vdots \\ &\leq (\prod_{i=1}^{n}\lambda_i)\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \\ &= (\prod_{i=1}^{n}\lambda_i)\|\lambda_{n+1}S_{n+1}U_{n+1,n+2}\rho_n + (1-\lambda_{n+1})\rho_n - \rho_n\| \\ &= (\prod_{i=1}^{n}\lambda_i)\|\lambda_{n+1}S_{n+1}\rho_n - \lambda_{n+1}\rho_n\| \\ &= (\prod_{i=1}^{n+1}\lambda_i)\|S_{n+1}\rho_n - \rho_n\| \leq L(\prod_{i=1}^{n+1}\lambda_i), \end{aligned} \tag{3.17}$$

where $L = \sup_{n\geq 1} \|S_{n+1}\rho_n - \rho_n\|$. Substituting (3.16) and (3.17) into (3.13) yields that

$$\begin{aligned}&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ \leq\; &\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\| + \|AW_{n+1}\rho_{n+1}\|) + \frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\| + \|AW_n\rho_n\|) \\ &+L(\prod_{i=1}^{n+1}\lambda_i) + \frac{M_1}{m}|r_{n+1} - r_n|. \end{aligned} \tag{3.18}$$

Because $0 < \lambda_i \leq b < 1, \forall i > 1$, from (3.18) and conditions $(C_1)-(C_3)$,

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.6, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.$$

Hence, from (3.12), (3.16) and condition (C_3), we have $\lim_{n\to\infty} \|\rho_{n+1} - \rho_n\| = 0$.

Step 3 Next, we prove that

$$\lim_{n\to\infty} \|x_n - W_n\rho_n\| = 0. \tag{3.19}$$

Because

$$\begin{aligned}\|x_n - W_n\rho_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n\rho_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - AW_n\rho_n\| + \beta_n\|x_n - W_n\rho_n\|,\end{aligned}$$

simplifying it, we have

$$\|x_n - W_n\rho_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - AW_n\rho_n\|.$$

As $\alpha_n \to 0, \|x_{n+1} - x_n\| \to 0$, and $\{\gamma f(x_n) - AW_n\rho_n\}$ is bounded, from condition (C_4), we have $\|x_n - W_n\rho_n\| \to 0$.

Step 4 Next, we prove that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.20}$$

In fact, for any given $p \in G$, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|V_{r_n}x_n - V_{r_n}p\|^2 \le \langle V_{r_n}x_n - V_{r_n}p, x_n - p\rangle \\ &= \langle y_n - p, x_n - p\rangle = \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2),\end{aligned}$$

and so

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2. \tag{3.21}$$

From (3.6) and (3.9), we have

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ = {}& \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n - p\|^2 \\ = {}& \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - W_n\rho_n) + (I - \alpha_n A)(W_n\rho_n - p)\|^2 \\ \le {}& \|(I - \alpha_n A)(W_n\rho_n - p) + \beta_n(x_n - W_n\rho_n)\|^2 + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1} - p\rangle \\ \le {}& [\|(I - \alpha_n A)(W_n\rho_n - p)\| + \beta_n\|(x_n - W_n\rho_n)\|]^2 + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1} - p\rangle \\ \le {}& [(1 - \alpha_n\bar{\gamma})\|\rho_n - p\| + \beta_n\|x_n - W_n\rho_n\|]^2 + 2\alpha_n\langle\gamma f(x_n) - Ap, x_{n+1} - p\rangle \\ = {}& (1 - \alpha_n\bar{\gamma})^2\|\rho_n - p\|^2 + \beta_n^2\|x_n - W_n\rho_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - W_n\rho_n\| \\ & +2\alpha_n\|\gamma f(x_n) - Ap\|\cdot\|x_{n+1} - p\|.\end{aligned} \tag{3.22}$$

Substituting (3.21) into (3.22) yields

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ \le {}& (1 - \alpha_n\bar{\gamma})^2\{\|x_n - p\|^2 - \|x_n - y_n\|^2\} + \beta_n^2\|x_n - W_n\rho_n\|^2 \\ & +2(1 - \alpha_n\bar{\gamma})\cdot\beta_n\|\rho_n - p\|\cdot\|x_n - W_n\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\cdot\|x_{n+1} - p\| \\ = {}& (1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2)\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})^2\|x_n - y_n\|^2 + \beta_n^2\|x_n - W_n\rho_n\|^2 \\ & +2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p)\|\cdot\|x_n - W_n\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\cdot\|x_{n+1} - p\|.\end{aligned}$$

Simplifying it, we have

$$\begin{aligned}&(1 - \alpha_n\bar{\gamma})^2\|x_n - y_n\|^2 \\ \le {}& \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 + \beta_n^2\|x_n - W_n\rho_n\|^2 \\ & +2(1 - \alpha_n\bar{\gamma})\beta_n\|\rho_n - p\|\cdot\|x_n - W_n\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\cdot\|x_{n+1} - p\|.\end{aligned}$$

As $\alpha_n \to 0, \|x_{n+1} - x_n\| \to 0, \|x_n - W_n\rho_n\| \to 0$, by condition (C_4), it yields $\|x_n - y_n\| \to 0$.

Step 5 Now, we prove that, for any given $p \in G$,

$$\lim_{n\to\infty} \|By_n - Bp\| = 0. \tag{3.23}$$

In fact, it follows from (3.9) that

$$\begin{aligned}\|\rho_n - p\|^2 \le \|\xi_n - p\|^2 &= \|J_{M,\lambda}(I-\lambda B)y_n - J_{M,\lambda}(I-\lambda B)p\|^2\\ &\le \|(I-\lambda B)y_n - (I-\lambda B)p\|^2\\ &= \|y_n - p\|^2 - 2\lambda\langle y_n - p, By_n - Bp\rangle + \lambda^2\|By_n - Bp\|^2\\ &\le \|y_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2\\ &\le \|x_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2. \end{aligned}\tag{3.24}$$

Substituting (3.24) into (3.22), we obtain

$$\begin{aligned}&\|x_{n+1} - p\|^2\\ \le\ &(1-\alpha_n\bar\gamma)^2\{\|x_n - p\|^2 + \lambda(\lambda - 2\alpha)\|By_n - Bp\|^2\} + \beta_n^2\|x_n - W_n\rho_n\|^2\\ &+2(1-\alpha_n\bar\gamma)\beta_n\|\rho_n - p\|\cdot\|x_n - W_n\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\cdot\|x_{n+1} - p\|.\end{aligned}$$

Simplifying it, we have

$$\begin{aligned}&(1-\alpha_n\bar\gamma)^2\lambda(2\alpha - \lambda)\|By_n - Bp\|^2\\ \le\ &\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\bar\gamma^2\|x_n - p\|^2 + \beta_n^2\|x_n - W_n\rho_n\|^2\\ &+2(1-\alpha_n\bar\gamma)\beta_n\|\rho_n - p\|\cdot\|x_n - W_n\rho_n\| + 2\alpha_n\|\gamma f(x_n) - Ap\|\cdot\|x_{n+1} - p\|.\end{aligned}$$

As $\alpha_n \to 0, 0 < \liminf\limits_{n\to\infty}\beta_n \le \limsup\limits_{n\to\infty}\beta_n < 1$, $\|x_{n+1} - x_n\| \to 0, \|x_n - W_n\rho_n\| \to 0$, and $\{\gamma f(x_n) - Ap\}, \{x_n\}$ are bounded, these imply that $\|By_n - Bp\| \to 0$ $(n\to\infty)$.

Step 6 Next, we prove that

$$\lim_{n\to\infty}\|y_n - \rho_n\| = 0. \tag{3.25}$$

In fact, as

$$\|y_n - \rho_n\| \le \|y_n - \xi_n\| + \|\xi_n - \rho_n\|,$$

for the purpose, it is sufficient to prove $\|y_n - \xi_n\| \to 0$ and $\|\xi_n - \rho_n\| \to 0$.

(a) First, we prove that $\|y_n - \xi_n\| \to 0$. In fact, as

$$\begin{aligned}&\|\xi_n - p\|^2\\ =\ &\|J_{M,\lambda}(I-\lambda B)y_n - J_{M,\lambda}(I-\lambda B)p\|^2\\ \le\ &\langle y_n - \lambda By_n - (p - \lambda Bp), \xi_n - p\rangle\\ =\ &\frac12\{\|y_n - \lambda By_n - (p-\lambda Bp)\|^2 + \|\xi_n - p\|^2 - \|y_n - \lambda By_n - (p - \lambda Bp) - (\xi_n - p)\|^2\}\\ \le\ &\frac12\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n - \lambda(By_n - Bp)\|^2\}\\ \le\ &\frac12\{\|y_n - p\|^2 + \|\xi_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, By_n - Bp\rangle - \lambda^2\|By_n - Bp\|^2\},\end{aligned}$$

we have

$$\|\xi_n - p\|^2 \le \|y_n - p\|^2 - \|y_n - \xi_n\|^2 + 2\lambda\langle y_n - \xi_n, By_n - Bp\rangle - \lambda^2\|By_n - Bp\|^2. \tag{3.26}$$

Substituting (3.26) into (3.22) yields

$$\begin{aligned}\|x_{n+1}-p\|^2 \le{}& (1-\alpha_n\bar{\gamma})^2\{\|y_n-p\|^2-\|y_n-\xi_n\|^2\\ &+2\lambda\langle y_n-\xi_n, By_n-Bp\rangle-\lambda^2\|By_n-Bp\|^2\}+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}$$

Simplifying it, we have

$$\begin{aligned}&(1-\alpha_n\bar{\gamma})^2\|y_n-\xi_n\|^2\\ \le{}& (\|x_n-x_{n+1}\|)\cdot(\|x_n-p\|+\|x_{n+1}-p\|)+\alpha_n\bar{\gamma}^2\|x_n-p\|^2\\ &+2(1-\alpha_n\bar{\gamma}^2)\lambda\langle y_n-\xi_n, By_n-Bp\rangle-(1-\alpha_n\bar{\gamma})^2\lambda^2\|By_n-Bp\|^2+\beta_n^2\|x_n-W_n\rho_n\|^2\\ &+2(1-\alpha_n\bar{\gamma})\beta_n\|\rho_n-p\|\cdot\|x_n-W_n\rho_n\|+2\alpha_n\|\gamma f(x_n)-Ap\|\cdot\|x_{n+1}-p\|.\end{aligned}$$

As $\alpha_n\to 0, 0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, $\|x_n-W_n\rho_n\|\to 0$, $\|By_n-Bp\|\to 0(n\to\infty)$, $\|x_{n+1}-x_n\|\to 0$, and $\{\gamma f(x_n)-Ap\},\{x_n\},\{\rho_n\}$ are bounded, these imply that $\|y_n-\xi_n\|\to 0(n\to\infty)$.

(b) Next, we prove that

$$\lim_{n\to\infty}\|\xi_n-\rho_n\|=0. \tag{3.27}$$

In fact, as $\|\xi_n-\rho_n\|=\|J_{M,\lambda}(I-\lambda B)y_n-J_{M,\lambda}(I-\lambda B)\xi_n\|\le\|y_n-\xi_n\|\to 0$, and so $\|y_n-\rho_n\|=\|y_n-\xi_n+\xi_n-\rho_n\|\le\|y_n-\xi_n\|+\|\xi_n-\rho_n\|\to 0$.

Step 7 Next, we prove that

$$\lim_{n\to\infty}\|y_n-W_ny_n\|=0. \tag{3.28}$$

In fact, because

$$\begin{aligned}\|y_n-W_ny_n\|&\le\|W_ny_n-W_n\rho_n\|+\|W_n\rho_n-x_n\|+\|x_n-y_n\|\\ &\le\|y_n-\rho_n\|+\|W_n\rho_n-x_n\|+\|x_n-y_n\|,\end{aligned}$$

form (3.19), (3.20), and (3.25), we have $\|y_n-W_ny_n\|\to 0$.

Step 8 Next, we prove that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Aq, x_n-q\rangle\le 0, \quad \text{where} \quad q=P_G(\gamma f+(I-A))(q). \tag{3.29}$$

For the purpose, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, such that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Aq, x_n-q\rangle=\lim_{i\to\infty}\langle\gamma f(q)-Aq, x_{n_i}-q\rangle.$$

Because $\{y_{n_i}\}$ is bounded, there exists a subsequence of $\{y_{n_i}\}$, still denoted by $\{y_{n_i}\}$, such that $y_{n_i}\rightharpoonup w$.

Now, we show that $w\in G$.

(a) First, we prove that $w\in EP(\phi)$.

In fact, as $y_n=V_{r_n}x_n$, we have

$$\phi(y_n,\eta)+\frac{1}{r_n}\langle\eta-y_n, y_n-x_n\rangle\ge 0, \quad \forall\eta\in C.$$

By condition (A_2),

$$\langle \eta - y_n, \frac{y_n - x_n}{r_n}\rangle \geq \phi(\eta, y_n).$$

Hence, we have

$$\langle \eta - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq \phi(\eta, y_{n_i}).$$

As $\frac{y_{n_i}-x_{n_i}}{r_{n_i}} \to 0, y_{n_i} \rightharpoonup w$, by condition (A_4), we have $\phi(\eta, w) \leq 0$ for all $\eta \in C$. For any t with $0 < t \leq 1$ and $\eta \in C$, let $\eta_t = t\eta + (1-t)w$. Because $\eta \in C$ and hence $\phi(\eta_t, w) \leq 0$, from conditions (A_1) and (A_4), we have

$$0 = \phi(\eta_t, \eta_t) \leq t\phi(\eta_t, \eta) + (1-t)\phi(\eta_t, w) \leq t\phi(\eta_t, \eta).$$

This implies that $\phi(\eta_t, \eta) \geq 0$. Hence, from condition (A_3), we have $\phi(w, \eta) \geq 0$ for all $\eta \in C$, and hence $w \in EP(\phi)$.

(b) Now, we prove that $w \in F(W)$.

If not, we have $w \neq Ww$. From Opial's lemma [13], we have

$$\begin{aligned}\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Ww\| \\ &\leq \liminf_{i\to\infty}\{\|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Ww\|\} \\ &\leq \liminf_{i\to\infty}\{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\}. \end{aligned} \tag{3.30}$$

By Lemma 2.5 and (3.28),

$$\begin{aligned}\lim_{i\to\infty} \|Wy_{n_i} - y_{n_i}\| &\leq \lim_{i\to\infty}\{\|Wy_{n_i} - W_{n_i}y_{n_i}\| + \|W_{n_i}y_{n_i} - y_{n_i}\|\} \\ &\leq \lim_{i\to\infty}\{\sup_{x\in C}\|Wx - W_{n_i}x\|\} + \lim_{i\to\infty}\|W_{n_i}y_{n_i} - y_{n_i}\| = 0, \end{aligned} \tag{3.31}$$

therefore, we have

$$\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - w\|.$$

This is a contradiction. So, we have $w \in F(W)$.

(c) Now, we prove that $w \in VI(H, B, M)$.

In fact, as B is α-inverse-strongly monotone, it follows from Proposition 1.1 that B is a $\frac{1}{\alpha}$-Lipschitz continuous monotone mapping and $D(B) = H$ (where $D(B)$ is the domain of B). From Lemma 2.1, $M + B$ is maximal monotone. Let $(\nu, g) \in \text{Graph}(M + B)$, that is, $g - B\nu \in M(\nu)$. Because $\rho_n = J_{M,\lambda}(I - \lambda B)\xi_n$, $\|y_n - \rho_n\| \to 0, y_{n_i} \rightharpoonup w$, we can prove that $\rho_{n_i} \rightharpoonup w$. Again as $\rho_{n_i} = J_{M,\lambda}(I - \lambda B)\xi_{n_i}$, we have

$$\xi_{n_i} - \lambda B\xi_{n_i} \in (I + \lambda M)\rho_{n_i}, \quad \text{that is,} \quad \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i}) \in M(\rho_{n_i}).$$

By virtue of the maximal monotonicity of $M + B$, we have

$$\langle \nu - \rho_{n_i}, g - B\nu - \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i})\rangle \geq 0,$$

and so

$$\begin{aligned}\langle \nu - \rho_{n_i}, g\rangle &\geq \langle \nu - \rho_{n_i}, B\nu + \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i} - \lambda B\xi_{n_i})\rangle \\ &= \langle \nu - \rho_{n_i}, B\nu - B\rho_{n_i} + B\rho_{n_i} - B\xi_{n_i} + \frac{1}{\lambda}(\xi_{n_i} - \rho_{n_i})\rangle \\ &\geq 0 + \langle \nu - \rho_{n_i}, B\rho_{n_i} - Bk_{n_i}\rangle + \langle \nu - \rho_{n_i}, \frac{1}{\lambda}(k_{n_i} - \rho_{n_i})\rangle. \end{aligned}$$

Because $\|\xi_n - \rho_n\| \to 0, \|B\xi_n - B\rho_n\| \to 0$, and $\rho_{n_i} \rightharpoonup w$, we have

$$\lim_{n_i \to \infty} \langle \nu - \rho_{n_i}, g\rangle = \langle \nu - w, g\rangle \geq 0.$$

As $M + B$ is maximal monotone, this implies that $\theta \in (M + B)(w)$, that is, $w \in VI(H, B, M)$, and so $w \in G$. Because $q = P_G(\gamma f + (I - A))(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle &= \lim_{i\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q\rangle \\ &= \lim_{i\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - y_{n_i} + y_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, w - q\rangle \leq 0.\end{aligned}$$

Step 9 Finally, we prove that

$$x_n \to q = P_G(\gamma f + (I - A))(q).$$

Indeed, from (3.6) and (3.9), we have

$$\begin{aligned}&\|x_{n+1} - q\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Aq) + \beta_n(x_n - q) + ((1 - \beta_n)I - \alpha_n A)(W_n\rho_n - q)\|^2 \\ &\leq \|\beta_n(x_n - q) + ((1 - \beta_n)I - \alpha_n A)(W_n\rho_n - q)\|^2 + 2\alpha_n\langle \gamma f(x_n) - Aq, x_{n+1} - q\rangle \\ &\leq [\|((1 - \beta_n)I - \alpha_n A)(W_n\rho_n - q)\| + \beta_n\|x_n - q\|]^2 + 2\alpha_n\gamma\langle f(x_n) - f(q), x_{n+1} - q\rangle \\ &\quad +2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q\rangle \\ &\leq [\|(1 - \beta_n - \alpha_n\bar{\gamma})\|\rho_n - q\| + \beta_n\|x_n - q\|]^2 + 2\alpha_n\gamma h\|x_n - q\| \cdot \|x_{n+1} - q\| \\ &\quad +2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q\rangle \\ &\leq (1 - \alpha_n\bar{\gamma})^2\|x_n - q\|^2 + \alpha_n\gamma h\{\|x_n - q\|^2 + \|x_{n+1} - q\|^2\} \\ &\quad +2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - q\|^2 &\leq \frac{(1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma h}{1 - \alpha_n\gamma h}\|x_n - q\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma h}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle \\ &= \left[1 - \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}\right]\|x_n - q\|^2 + \frac{(\alpha_n\bar{\gamma})^2}{1 - \alpha_n\gamma h}\|x_n - q\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma h}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle \\ &\leq \left[1 - \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}\right]\|x_n - q\|^2 + \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}\left\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma h)}\|x_n - q\|^2\right. \\ &\quad \left. + \frac{1}{\bar{\gamma} - \gamma h}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle\right\} \\ &= (1 - l_n)\|x_n - q\|^2 + \delta_n,\end{aligned}$$

where

$$l_n = \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}$$

and

$$\delta_n = \frac{2(\bar{\gamma} - \gamma h)\alpha_n}{1 - \alpha_n\gamma h}\left\{\frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \gamma h)}\|x_n - q\|^2 + \frac{1}{\bar{\gamma} - \gamma h}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle\right\}.$$

It is seen that $l_n \to 0$, $\sum_{n=1}^{\infty} l_n = \infty$, and $\limsup_{n\to\infty} \frac{\delta_n}{l_n} \le 0$. Hence, by Lemma 2.7, the sequence $\{x_n\}$ converges strongly to q.

Theorem 3.2 Let C be a nonempty closed convex subset of a Hilbert space H, B and D be two different α-inverse-strongly monotone mapping from C into H. Let F be a bifunction from $C \times C \to R$ which satisfies the conditions $(A_1)-(A_4)$. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of infinite k_n-strict pseudo-contractive mappings with $0 \le k_n < 1$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, b]$ for some $b \in (0, 1)$, $\{S_n\}_{n=1}^{\infty} : C \to C$ be a family of infinitely nonexpansive mappings defined by (3.4) such that $G = F(W) \bigcap VI(C, B) \bigcap EP \neq \emptyset$, where $F(W) := \bigcap_{n=1}^{\infty} F(S_n)$. Let A be a strongly positive linear bounded operator with coefficient $\bar\gamma > 0$, $f : H \to H$ be a contraction with a contraction constant $h \in (0, 1)$, and $0 < \gamma < \frac{\bar\gamma}{h}$. Let $\{x_n\}$, $\{\rho_n\}$, $\{\xi_n\}$, and $\{y_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{cases} F(y_n,\eta) + \langle Dy_n, \eta - y_n\rangle + \dfrac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \ge 0, \quad \forall \eta \in C, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n\rho_n, \\ \rho_n = P_C(I-\lambda B)\xi_n, \\ \xi_n = P_C(I-\lambda B)y_n, \end{cases} \qquad \forall n \ge 1, \tag{3.32}$$

where $\{W_n : C \to C\}$, $\lambda \in (0, 2\alpha]$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ are the same sequences as given in Theorem 3.1, then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in G$, where $q = P_G(\gamma f + (I - A))(q)$.

Proof Take $M = \partial\delta_C : H \to 2^H$ in Theorem 3.1, where $\delta_C : H \to [0, \infty)$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C, \end{cases}$$

then, the variational inclusion problem (1.6) is equivalent to the variational inequality (1.7), that is, to find $u \in C$ such that

$$\langle B(u), v - u\rangle \ge 0, \quad \forall v \in C.$$

Again, as $M = \partial\delta_C$, the restriction of $J_{M,\lambda}$ on C is an identity mapping, that is, $J_{M,\lambda}|_C = I$ and so we have

$$P_C(I-\lambda B)k_n = J_{M,\lambda}(P_C(I-\lambda B)k_n); \quad P_C(I-\lambda B)y_n = J_{M,\lambda}(P_C(I-\lambda B)y_n).$$

Hence, the conclusion of Theorem 3.2 can be obtained from Theorem 3.1 immediately.

Theorem 3.3 Let H be a real Hilbert space, $B : H \to H$ be a α-inverse strongly monotone mapping, and $M : H \to 2^H$ be a maximal monotone mapping and let $\{T_n\}_{n=1}^{\infty} : H \to H$ be a family of infinite k_n-strict pseudo-contractive mappings with $0 \le k_n < 1$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0, b]$ for some $b \in (0, 1)$, $\{S_n\}_{n=1}^{\infty} : H \to H$ be a family of infinite nonexpansive mappings defined by (3.4) such that $G = F(W) \bigcap VI(H, B, M) \neq \emptyset$, where $F(W) := \bigcap_{n=1}^{\infty} F(S_n)$, $f : H \to H$ be a contractive mapping with a contraction constant

$h \in (0,1)$. For any given $x_1 \in H$, let $\{x_n\}$, $\{\rho_n\}$, and $\{\xi_n\}$ be the iterative sequences defined by

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)W_n\rho_n, \\ \rho_n = J_{M,\lambda}(I-\lambda B)\xi_n, \\ \xi_n = J_{M,\lambda}(I-\lambda B)x_n, \end{cases} \quad \forall n \geq 1, \tag{3.33}$$

where $\lambda \in (0, 2\alpha]$, $\{W_n : H \to H\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are the same as given in Theorem 3.1, then, $\{x_n\}$ cnoverges strongly to $p \in F(W) \bigcap VI(H,B,M)$, where $p = P_{F(W)\bigcap VI(H,B,M)} f(p)$.

Proof In Theorem 3.1, taking $F(x,y) = 0, D = 0$ for all $x, y \in H$, $\gamma = 1, r_n = 1$, and $A = I$, we get $y_n = x_n$. Therefore, the conclusion of Theorem 3.3 can be obtained from Theorem 3.1.

References

[1] Mann W R. Mean value methods in iteration. Proc Amer Math Soc, 1953, **4**: 506–510

[2] Acedo G L, Xu H. Iterative methods for strict pseudo-contractions in Hilbert space. Nonlinear Anal, 2007, **67**: 2258–2271

[3] Verma R U. Approximation solvability of a class of monotone set-valued variational inclusions involving (A, η)-monotone mappings. J Math Anal Appl, 2008, **337**: 969–975

[4] Zhang S. Set-valued variational inclusions in Banach space. J Math Anal Appl, 2000, **248**: 438–454

[5] Zhang S. Existence and approximation of solutions of Set-valued variational inclusions in Banach space. Nonlinear Anal, 2001, **47**(1): 583–594

[6] Demyanov V F, Stavroulakis G E, Polyakova L N, et al. Quasi-differentiability and nonsmooth modeling in mechanics, engineering and economics. Holland: Kluwer Academic, 1996

[7] Lions J L, Stampacchia G. Variational inequalities. Comm Pure Appl Math, 1967, **20**: 493–517

[8] Chang S S, Joseph Lee J H, Chan C K. Algorithms of common solutions for quasi-variational inclusion and fixed point problems. Appl Math Mech, 2008, **29**: 1–11

[9] Takahashi W, Toyoda M. Weak convergence theorems for nonexpansive mapping and monotone mapings. J Optim Theory Appl, 1998, **118**: 417–428

[10] Takahashi S, Takahshi W. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert space. J Math Anal Appl, 2007, **331**: 506–515

[11] Chang S S, Joseph Lee H W, Chan C. A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. Nonlinear Analysis, 2009, **70**(9): 3307–3319

[12] Takahashi S, Takahashi W. Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. Nonlinear Analysis, 2008, **69**(3): 1025–1033

[13] Opial Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull Amer Math Soc, 1967, **73**: 591–597

[14] Pascali Dan. Nolinear Mappings of Monotone Type. The Netherlands: Sijthoff and Noordhoff International Publishers, 1978

[15] Shimoji K, Takahashi W. Strong convergence to common fixed ponts of infinite nonexpansive mappings and applications. Taiwanese J Math, 2001, **5**: 387–404

[16] Suzuki T. Strong convergence of Krasnoselskii and Mann type sequence for one parameter nonexpansive semigroups without Bochner integrals. J Math Anal Appl, 2005, **305**: 227–239

[17] Zhang S. Some problems and results in the study of nonlinear analysis. Nonlinear Anal, 1997, **30**: 4197–4208

[18] Blum E, Oettli W. From optimization and varitional inequalities to equilibrium problems. Math Student, 1994, **63**: 123–145

[19] Combettes P L, Hirstoaga S A. Equilibrium problems in Hilbert space. J Nonlinear Convex Anal, 2005, **6**: 117–136