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Zeros and special values of Witten zeta functions and Witten L-functions

Jeongwon Min

Department of Mathematics, Tokyo Institute of Technology, Oo-okayama, Meguro-ku, Tokyo 152-8551, Japan

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ABSTRACT

We calculate special values of Witten L-functions for $SU(2)$ and $SU(3)$. We prove some algebraicity for such special values. Moreover, we prove that such values turn out to be zero in many cases. These are generalizations of classical results due to Euler for the Riemann zeta function.

1. Introduction

Witten [5] introduced the Witten zeta function for a compact topological group G:

$$\zeta_G^W(s) \stackrel{\text{def}}{=} \sum_{\rho \in \widehat{G}} \deg(\rho)^{-s}$$

where $\widehat{G}$ is the unitary dual of G, that is a set of equivalence classes of the irreducible unitary representations.

E-mail address: min.j.aa@m.titech.ac.jp.

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For example, the Riemann zeta function $\zeta(s)$ is identified as the Witten zeta function for $SU(2)$,

$$\zeta(s) = \zeta^W_{SU(2)}(s)$$

since $\widehat{SU(2)} = \{Sym^m \mid m = 0, 1, 2, \ldots\}$, $\deg(Sym^m) = m + 1$. Witten [5] and Zagier [6] proved similar results to Euler's one, which is that $\zeta^W_{SU(2)}(2m) = \zeta(2m) \in \pi^{2m}\mathbb{Q}$ for $m = 1, 2, 3, \ldots$, for special values of some Witten zeta functions at positive even integers. In addition, Kurokawa and Ochiai [2] observed the special values of opposite site, in other words, special values of $s = -2m$ $(m = 1, 2, 3, \ldots)$.

In [2], Kurokawa and Ochiai introduced the Witten L-function which is defined as follows:

$$\zeta^W_G(s; g_1, \ldots, g_r) \stackrel{\text{def}}{=} \sum_{\rho \in \widehat{G}} \frac{\operatorname{trace}(\rho(g_1))}{\deg(\rho)} \cdots \frac{\operatorname{trace}(\rho(g_r))}{\deg(\rho)} \deg(\rho)^{-s}$$

for a compact topological group G and elements $g_1, \ldots, g_r \in G$.

When $r = 1$, $\zeta^W_G(s; \mathbb{1}) = \zeta^W_G(s)$ ($\mathbb{1} \in G$ is the identity element of G), so the Witten L-function includes the Witten zeta function. We remark that

$$\zeta^W_G(-2; g) = \begin{cases} |G| & \text{if } g = \mathbb{1}, \\ 0 & \text{otherwise} \end{cases}$$

for each finite group G. In [2], Kurokawa and Ochiai made the following conjecture:

$$\zeta^W_G(-2, g) = 0$$

for all infinite groups G.

In this paper, we study the generalization of the problem for $\zeta^W_G(-2; g_1, \ldots, g_r)$ for $G = SU(2)$ and $SU(3)$. In these cases, we express $\zeta^W_G(s; g_1, \ldots, g_r)$ via a generalized polylogarithm function, and we obtain the analyticity and the algebraicity.

Theorem 1.

(1) *We have*

$$\zeta^W_{SU(2)}(s; g_1, \ldots, g_r) = \sum_{n \geqslant 1} \frac{\sin(n\theta_1) \cdots \sin(n\theta_r)}{n^r \sin\theta_1 \cdots \sin\theta_r} n^{-s},$$

in $\operatorname{Re}(s) > 1$ *when* $g_j \in SU(2)$ *is conjugate to* $\begin{pmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{pmatrix}$, $0 \leqslant \theta \leqslant \pi$, $1 \leqslant j \leqslant r$ *where*

$$\frac{\sin(n\theta)}{\sin\theta} = \begin{cases} 1, & \theta = 0, \\ (-1)^{n-1}, & \theta = \pi. \end{cases}$$

(2) *We have*

$$\zeta^W_{SU(3)}(s;g)=\sum_{\rho\in\widehat{SU(3)}}\frac{\operatorname{trace}(\rho(g))}{\deg(\rho)}\deg(\rho)^{-s}$$

where

$$\operatorname{trace}\bigl(\rho(g)\bigr)=\frac{1}{\prod_{1\leqslant j<k\leqslant 3}(e^{i\theta_j}-e^{i\theta_k})}\cdot\det\begin{bmatrix} e^{i\theta_1(n_1+n_2)} & e^{i\theta_2(n_1+n_2)} & e^{i\theta_3(n_1+n_2)}\\ e^{i\theta_1 n_2} & e^{i\theta_2 n_2} & e^{i\theta_3 n_3}\\ 1 & 1 & 1\end{bmatrix}$$

$$\bigl(\textit{when } g\sim\operatorname{diag}\bigl(e^{i\theta_1},e^{i\theta_2},e^{i\theta_3}\bigr),\ \theta_1+\theta_2+\theta_3=0\bigr),$$

$$\deg(\rho)=\frac{n_1n_2(n_1+n_2)}{2},\quad n_1,\ n_2\ \textit{integers with } n_1,n_2\geqslant 1.$$

Theorem 2. *The function*

$$\zeta^W_{SU(2)}(s,g_1,\ldots,g_r)=\sum_{n\geqslant 1}\frac{\sin(n\theta_1)\cdots\sin(n\theta_r)}{n^r\sin\theta_1\cdots\sin\theta_r}n^{-s}$$

is continued analytically for $s\in\mathbb{C}$ to a meromorphic function.

Theorem 3.

(1) *It holds that*

$$\zeta^W_{SU(2)}(m;g_1,\ldots,g_r)=0$$

for every negative integer $m\leqslant -r$.

(2) *When $r=3$, we have*

$$\zeta^W_{SU(2)}(-2,g_1,g_2,g_3) = \begin{cases}\frac{\pi}{4\sin\theta_1\sin\theta_2\sin\theta_3}, & (\theta_1,\theta_2,\theta_3)\in\operatorname{Int}(V),\\ \frac{\pi}{8\sin\theta_1\sin\theta_2\sin\theta_3}, & (\theta_1,\theta_2,\theta_3)\in\operatorname{Int}(S_1\cap V)\cup\cdots\cup\operatorname{Int}(S_4\cap V),\\ 0, & \textit{otherwise}\end{cases}$$

where

$$\begin{aligned} V&=\bigl\{(\theta_1,\theta_2,\theta_3)\bigm|\theta_1+\theta_2+\theta_3\leqslant 2\pi\bigr\}\\ &\quad\cap\bigl\{(\theta_1,\theta_2,\theta_3)\bigm|\theta_1+\theta_2-\theta_3\geqslant 0\bigr\}\\ &\quad\cap\bigl\{(\theta_1,\theta_2,\theta_3)\bigm|\theta_1-\theta_2-\theta_3\leqslant 0\bigr\}\\ &\quad\cap\bigl\{(\theta_1,\theta_2,\theta_3)\bigm|\theta_1-\theta_2+\theta_3\geqslant 0\bigr\} \end{aligned}\tag{1.1}$$

and

$$
\begin{aligned}
S_1 &= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1+\theta_2-\theta_3=0\},\\
S_2 &= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1-\theta_2-\theta_3=0\},\\
S_3 &= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1-\theta_2+\theta_3=0\},\\
S_4 &= \{(\theta_1,\theta_2,\theta_3) \mid \theta_1+\theta_2+\theta_3=2\pi\}. \qquad (1.2)
\end{aligned}
$$

(3) *When $r=4$, it holds that $\zeta^W_{SU(2)}(-2;g_1,g_2,g_3,g_4)=0$ except for the following cases.*
(3-1) *If $0<\theta_1,\theta_2,\theta_3,\theta_4<\pi$ satisfy*

$$
\begin{aligned}
2\pi &\leqslant \theta_1+\theta_2+\theta_3+\theta_4 < 4\pi,\\
0 &\leqslant \theta_1-\theta_2+\theta_3+\theta_4 \leqslant 2\pi,\\
0 &\leqslant \theta_1+\theta_2-\theta_3+\theta_4 \leqslant 2\pi,\\
0 &\leqslant \theta_1+\theta_2+\theta_3-\theta_4 \leqslant 2\pi,\\
-2\pi &\leqslant \theta_1-\theta_2-\theta_3-\theta_4 \leqslant 0, \qquad (1.3)
\end{aligned}
$$

it holds that

$$\zeta^W_{SU(2)}(-2;g_1,g_2,g_3,g_4)=\frac{\pi^2}{4\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4}.$$

(3-2) *If one of $\theta_j=\pi$ $(j=1,2,3,4)$; say $\theta_4=\pi$ and $(\theta_1,\theta_2,\theta_3,\theta_4)$ satisfy*

$$
\begin{aligned}
\pi &< \theta_1+\theta_2+\theta_3 < 3\pi,\\
-\pi &< \theta_1-\theta_2+\theta_3 < \pi,\\
-\pi &< \theta_1+\theta_2-\theta_3 < \pi,\\
-\pi &< \theta_1-\theta_2-\theta_3 < \pi, \qquad (1.4)
\end{aligned}
$$

it holds that

$$\zeta^W_{SU(2)}(-2;g_1,g_2,g_3,g_4)=\frac{\pi}{4\sin\theta_1\sin\theta_2\sin\theta_3}.$$

(3-3) *If one of $\theta_4=0$, it returns to the case of $r=3$.*

Furthermore, if s is non-negative even integer, the special values of the Witten L-function satisfy the following:

Theorem 4. *The special values at even integers m satisfy the following algebraicity when $\theta_1,\dots,\theta_r\in\pi\mathbb{Q}$, where $0<\theta_1,\dots,\theta_r<\pi$:*

$$\zeta^W_{SU(2)}(m;g_1,\dots,g_r)\in\pi^{m+r}\overline{\mathbb{Q}}.$$

Theorem 5. *We have*

$$\int\limits_{SU(2)^r} \zeta^W_{SU(2)}(-2; g_1, \ldots, g_r)\, dg_1 \ldots dg_r = \begin{cases} 0, & r = 1, 2, \\ 1, & r = 3 \end{cases}$$

where dg is the normalized Haar measure.

Theorem 6.

(1) *The function $\zeta^W_{SU(3)}(s, g)$ is continued analytically for $s \in \mathbb{C}$ to a meromorphic function.*

(2) *We have*

$$\zeta^W_{SU(3)}(m; g) = 0$$

for all negative integers m.

2. Proof of Theorem 1

(1) We have

$$\widehat{SU(2)} = \{Sym^m \mid m = 0, 1, 2, \ldots\}$$

where

$$Sym^m : SU(2) \to SU(m+1)$$

is the symmetric tensor representation.

Hence

$$\begin{aligned}
\zeta^W_{SU(2)}(s, g_1, \ldots, g_r) &= \sum_{m=0}^{\infty} \frac{\operatorname{tr}(Sym^m(g_1))}{\deg(Sym^m)} \cdots \frac{\operatorname{tr}(Sym^m(g_r))}{\deg(Sym^m)} \deg(Sym^m)^{-s} \\
&= \sum_{m=0}^{\infty} \frac{\sin((m+1)\theta_1)}{(m+1)\sin\theta_1} \cdots \frac{\sin((m+1)\theta_r)}{(m+1)\sin\theta_r}(m+1)^{-s},
\end{aligned}$$

where $g_j \sim \begin{pmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{pmatrix}$.

Thus we have (1).

(2) The character formula for $SU(3)$ (see Hall [1] for example) gives (2) exactly as in the above (1). □

3. The case of $SU(2)$

3.1. Analytic continuation: Proof of Theorem 2

By Theorem 1(1) we have the following expression for $\mathrm{Re}(s) > 1$:

$$\begin{aligned}
&\zeta^W_{SU(2)}(s; g_1, \dots, g_r) \\
&= \sum_{n=1}^{\infty} \frac{\sin(n\theta_1)\cdots\sin(n\theta_r)}{n^{s+r}\sin\theta_1\cdots\sin\theta_r} \\
&= \frac{1}{\prod_{j=1}^{r}(e^{i\theta_j} - e^{-i\theta_j})} \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{r}(e^{i\theta_j n} - e^{-i\theta_j n})}{n^{s+r}} \\
&= \frac{1}{\prod_{j=1}^{r}(e^{i\theta_j} - e^{-i\theta_j})} \Biggl[\sum_{n=1}^{\infty} \frac{1}{n^{s+r}} \sum_{(\alpha_1,\dots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} \\
&\quad \times \Biggl\{\exp\Biggl\{in\Biggl(\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j\Biggr)\Biggr\} + (-1)^r \exp\Biggl\{-in\Biggl(\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j\Biggr)\Biggr\}\Biggr\}\Biggr] \\
&= \frac{1}{\prod_{j=1}^{r}(e^{i\theta_j} - e^{-i\theta_j})} \sum_{(\alpha_1,\dots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\sum \alpha_{j-1}} \\
&\quad \times \Biggl\{Z\Biggl(s+r; \theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j\Biggr) + (-1)^r Z\Biggl(s+r; -\Biggl(\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j\Biggr)\Biggr)\Biggr\} \\
&= \begin{cases} \frac{(2i)^r}{\prod_{j=1}^{r}(e^{i\theta_j}-e^{-i\theta_j})}[\sum_{n=1}^{\infty}\frac{1}{n^{s+r}}\sum_{(\alpha_1,\dots,\alpha_{r-1})\in\{0,1\}^{r-1}}(-1)^{\sum\alpha_{j-1}} \\ \quad\times \sin\{n(\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j)\}] \quad (r\text{: even}), \\ \frac{(2i)^r}{\prod_{j=1}^{r}(e^{i\theta_j}-e^{-i\theta_j})}[\sum_{n=1}^{\infty}\frac{1}{n^{s+r}}\sum_{(\alpha_1,\dots,\alpha_{r-1})\in\{0,1\}^{r-1}}(-1)^{\sum\alpha_{j-1}} \\ \quad\times \cos\{n(\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j)\}] \quad (r\text{: odd}). \end{cases}
\end{aligned} \tag{3.1}$$

Here $Z(s;\theta) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^s}$ for $\theta \in \mathbb{R}$ is the polylogarithm function, which is extended as a meromorphic function for $s \in \mathbb{C}$ in various ways: see Milnor [3], for example. We notice a simple proof here for reader's convenience.

For $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned}
Z(s;\theta) &= \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^s} = e^{i\theta} + \frac{e^{2i\theta}}{2^s} + \sum_{n=3}^{\infty} \frac{e^{in\theta}}{n^s} \\
&= e^{i\theta} + \frac{e^{2i\theta}}{2^s} + \sum_{n=2}^{\infty} \frac{e^{i(n+1)\theta}}{(n+1)^s} \\
&= e^{i\theta} + \frac{e^{2i\theta}}{2^s} + \sum_{n=2}^{\infty} e^{i(n+1)\theta} n^{-s}\bigl(1+n^{-1}\bigr)^{-s}
\end{aligned}$$

$$= e^{i\theta} + \frac{e^{2i\theta}}{2^s} + \sum_{n=2}^{\infty} e^{i(n+1)\theta} n^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} n^{-k}$$

$$= e^{i\theta} + \frac{e^{2i\theta}}{2^s} + e^{i\theta} \sum_{k=0}^{\infty} \binom{-s}{k} \big(Z(s+k;\theta) - e^{i\theta}\big)$$

$$= e^{i\theta} + \frac{e^{2i\theta}}{2^s} + e^{i\theta}\big(Z(s;\theta) - e^{i\theta}\big) + e^{i\theta} \sum_{k=1}^{\infty} \binom{-s}{k} \big(Z(s+k;\theta) - e^{i\theta}\big).$$

It follows

$$\big(1 - e^{i\theta}\big) Z(s;\theta) = e^{i\theta} + e^{2i\theta}\big(2^{-s} - 1\big) + e^{i\theta} \sum_{k=1}^{\infty} \binom{-s}{k} \big(Z(s+k;\theta) - e^{i\theta}\big) \tag{3.2}$$

and the right-hand side converges absolutely in $\operatorname{Re}(s) > 0$. We obtain the analytic continuation to whole $s \in \mathbb{C}$ repeating this argument.

Thus we have the analytic continuation of the Witten L-function. □

3.2. Zeros: Proof of Theorem 3(1) and (2)

(1) First, we notice the following equality

$$Z(m;\theta) + (-1)^m Z(m;-\theta) = 0 \quad \text{for } m \in \mathbb{Z}_{<0}, \tag{3.3}$$

which is deduced from the functional equation for $Z(s,\theta)$:

$$e^{-\pi i s/2} Z(s;\theta) + e^{\pi i s/2} Z(s;-\theta) = \frac{(2\pi)^s}{\Gamma(s)} \zeta\left(1 - s, \frac{\theta}{2\pi}\right), \tag{3.4}$$

where the right-hand side is the Hurwitz zeta function.

Concerning this functional equation, we refer to Milnor [3]. By applying (3.3) to (3.1) we see that $\zeta^W_{SU(2)}(m; g_1, \ldots, g_r) = 0$ for all negative even integers $m < -r$.

When $m = -r$, use the following:

$$Z(0,\theta) + Z(0,-\theta) = -1 \tag{3.5}$$

since $Z(0,\theta) = \frac{e^{i\theta}}{1-e^{i\theta}}$ if $\theta \neq 0$ and $Z(0,\theta) = \zeta(0) = -\frac{1}{2}$ if $\theta = 0$.

We see $\zeta^W_{SU(2)}(m; g_1, \ldots, g_r) = 0$ at $m = -r$ by applying (3.5) to (3.1).

(2) The case of $r = 3$:

First, by (3.1),

$$\zeta^W_{SU(2)}(s; g_1, g_2, g_3) = \sum_{n=1}^{\infty} \frac{\sin(n\theta_1)\sin(n\theta_2)\sin(n\theta_3)}{n^3 \sin\theta_1 \sin\theta_2 \sin\theta_3} n^{-s}$$

$$= \frac{1}{\prod_{j=1}^{3}(e^{i\theta_j} - e^{-i\theta_j})}$$

$$\times \Big[\big\{ Z(s+3;\theta_1+\theta_2+\theta_3) - Z(s+3;-\theta_1-\theta_2-\theta_3) \big\} - \big\{ Z(s+3;\theta_1+\theta_2-\theta_3) - Z(s+3;-\theta_1-\theta_2+\theta_3) \big\} + \big\{ Z(s+3;\theta_1-\theta_2-\theta_3) - Z(s+3;-\theta_1+\theta_2+\theta_3) \big\} - \big\{ Z(s+3;\theta_1-\theta_2+\theta_3) - Z(s+3;-\theta_1+\theta_2-\theta_3) \big\} \Big] \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^s} = 2i\big\{ Z(s;\theta) - Z(s;-\theta) \big\} \tag{3.7}$$

for $\mathrm{Re}(s) > 1$. The left-hand side of (3.7) be extended to a meromorphic function for $s \in \mathbb{C}$ because of Theorem 2. Thus, we obtain the expression

$$\zeta_{SU(2)}^{W}(s;g_1,g_2,g_3) = \frac{1}{4\sin\theta_1 \sin\theta_2 \sin\theta_3} \times \left[-\sum_{n=1}^{\infty} \frac{\sin(\theta_1+\theta_2+\theta_3)n}{n^{s+3}} + \sum_{n=1}^{\infty} \frac{\sin(\theta_1+\theta_2-\theta_3)n}{n^{s+3}} - \sum_{n=1}^{\infty} \frac{\sin(\theta_1-\theta_2-\theta_3)n}{n^{s+3}} + \sum_{n=1}^{\infty} \frac{\sin(\theta_1-\theta_2+\theta_3)n}{n^{s+3}} \right]. \tag{3.8}$$

Note that

$$\frac{\pi-\theta}{2} = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} \tag{3.9}$$

at $0 < \theta < 2\pi$ (Fourier expansion of Bernoulli polynomial).

First, we consider the case of $0 < \theta_1, \theta_2, \theta_3 < \pi$. If $(\theta_1, \theta_2, \theta_3) \in \mathrm{Int}(V)$, $(\theta_1, \theta_2, \theta_3)$ satisfies the following:

$$\begin{gathered} 0 < \theta_1+\theta_2+\theta_3 < 2\pi, \\ 0 < \theta_1+\theta_2-\theta_3 < 2\pi, \\ 0 < -\theta_1+\theta_2+\theta_3 < 2\pi, \\ 0 < \theta_1-\theta_2+\theta_3 < 2\pi \end{gathered} \tag{3.10}$$

hence, we get the following result by applying (3.9) to (3.8):

$$\zeta_{SU(2)}^{W}(-2;g_1,g_2,g_3) = \frac{1}{4\sin\theta_1 \sin\theta_2 \sin\theta_3} \times \left[-\frac{\pi-(\theta_1+\theta_2+\theta_3)}{2} + \frac{\pi-(\theta_1+\theta_2-\theta_3)}{2} \right.$$

$$+\frac{\pi-(-\theta_1+\theta_2+\theta_3)}{2}+\frac{\pi-(\theta_1-\theta_2+\theta_3)}{2}\Bigg]$$
$$=\frac{\pi}{4\sin\theta_1\sin\theta_2\sin\theta_3}(\neq 0). \tag{3.11}$$

If $(\theta_1,\theta_2,\theta_3)\in V^c$ and $0<\theta_1,\theta_2,\theta_3<\pi$, we also get $\zeta^W_{SU(2)}(-2;g_1,g_2,g_3)=0$ in the same way as above.

Next, look at the case of $(\theta_1,\theta_2,\theta_3)\in \operatorname{Int}(S_1\cap V)$:

$$\begin{aligned}
&0<\theta_1+\theta_2+\theta_3<2\pi,\\
&\theta_1+\theta_2-\theta_3=0,\\
&0<-\theta_1+\theta_2+\theta_3<2\pi,\\
&0<\theta_1-\theta_2+\theta_3<2\pi.
\end{aligned} \tag{3.12}$$

Then

$$\begin{aligned}
\zeta^W_{SU(2)}(-2;g_1,g_2,g_3)&=\frac{1}{4\sin\theta_1\sin\theta_2\sin\theta_3}\Bigg[-\frac{\pi-(\theta_1+\theta_2+\theta_3)}{2}\\
&\quad+\frac{\pi-(\theta_1-\theta_2+\theta_3)}{2}+\frac{\pi-(-\theta_1+\theta_2+\theta_3)}{2}\Bigg]\\
&=\frac{\pi}{8\sin\theta_1\sin\theta_2\sin\theta_3}(\neq 0)
\end{aligned} \tag{3.13}$$

in the same way as above. We also see this in the case of $(\theta_1,\theta_2,\theta_3)\in S_2,S_3$ or S_4. If one of $g_j=1$ $(j=1,2,3)$, it returns to the case of $r=2$, and if $(\theta_1,\theta_2,\theta_3)=(\pi,\pi,\pi)$

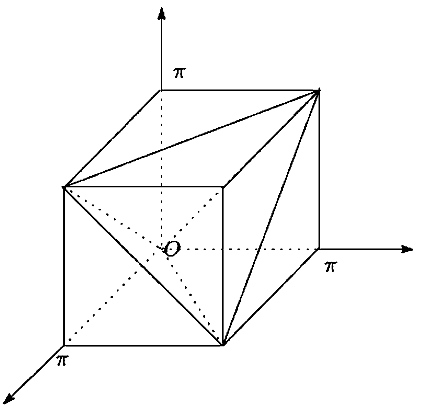
$$\zeta^W_{SU(2)}(s;g_1,g_2,g_3)=-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=-\big(1-2^{1-s}\big)\zeta(s). \tag{3.14}$$

Finally, when one of $\theta_j=\pi$,

$$\begin{aligned}
&\zeta^W_{SU(2)}(s;g_1,g_2,g_3)\\
&\quad=\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\theta_1\sin n\theta_2}{n^{s+2}\sin\theta_1\sin\theta_2}\\
&\quad=\frac{1}{(e^{i\theta_1}-e^{-i\theta_1})(e^{i\theta_2}-e^{-i\theta_2})}\sum_{n=1}^{\infty}\frac{e^{in\pi}(e^{in\theta_1}-e^{-in\theta_1})(e^{in\theta_2}-e^{-in\theta_2})}{n^{s+2}}\\
&\quad=\frac{1}{(e^{i\theta_1}-e^{-i\theta_1})(e^{i\theta_2}-e^{-i\theta_2})}\big[\big\{Z(s+2;\theta_1+\theta_2-\pi)+Z(s+2;-\theta_1-\theta_2+\pi)\big\}\\
&\qquad-\big\{Z(s+2;\theta_1-\theta_2+\pi)+Z(s+2;-\theta_1+\theta_2-\pi)\big\}\big].
\end{aligned}$$

Thus, it returns to the case of $r = 2$; we see easily that when two of θ_j's are π, it returns to the case of $r = 1$.

Therefore, we prove by using (3.8) and (3.9) in the same way as the other cases.



We illustrate above:

We notice that $s = -2$ is not zero in the interior of the tetrahedron having O, $(0, \pi, \pi)$, $(\pi, 0, \pi)$, $(\pi, \pi, 0)$ as its vertices and the interior of its faces.

(3) The case of $r = 4$:

We prove this case in more general situation in Section 3.3 below. □

3.3. *Special values*

3.3.1. *Proof of Theorem 4*

We divides into 3 cases (A), (B) and (C).

(A) For even $m \geqslant 0$:

When

$$g_1 \sim \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix}, \quad \ldots, \quad g_r \sim \begin{pmatrix} e^{i\theta_r} & 0 \\ 0 & e^{-i\theta_r} \end{pmatrix}, \quad \theta_1, \ldots, \theta_r \in \pi\mathbb{Q},$$

we express the θ's as follows:

$$\theta_1 = \frac{M_1}{N}\pi, \quad \ldots, \quad \theta_r = \frac{M_r}{N}\pi, \quad M_1, \ldots, M_r = 1, \ldots, N-1$$

then, for every positive even integer m,

$$\zeta^W_{SU(2)}(m; g_1, \ldots, g_r)$$
$$= \frac{1}{(\sin \frac{M_1}{N}\pi) \cdots (\sin \frac{M_r}{N}\pi)} \sum_{n=1}^{\infty} \frac{\sin(\frac{M_1}{N}\pi n) \cdots \sin(\frac{M_r}{N}\pi n)}{n^{m+r}}$$

$$= \frac{1}{(\sin\frac{M_1}{N}\pi)\cdots(\sin\frac{M_r}{N}\pi)}\sum_{l=0}^{\infty}\sum_{k=1}^{2N}\left\{\sin\left(\frac{(2Nl+k)M_1}{N}\pi\right)\cdots\sin\left(\frac{(2Nl+k)M_r}{N}\pi\right)\right.$$
$$\left.\times(2Nl+k)^{-m-r}\right\}$$
$$= \frac{1}{(\sin\frac{M_1}{N}\pi)\cdots(\sin\frac{M_r}{N}\pi)}\left\{\sum_{k=1}^{2N}\sin\left(\frac{kM_1}{N}\pi\right)\cdots\sin\left(\frac{kM_r}{N}\pi\right)\zeta\left(m+1,\frac{k}{2N}\right)\right\}$$
$$= \frac{(2N)^{-m-r}}{(\sin\frac{M_1}{N}\pi)\cdots(\sin\frac{M_r}{N}\pi)}\left\{\sum_{k=1}^{2N}\sin\left(\frac{kM_1}{N}\pi\right)\cdots\sin\left(\frac{kM_r}{N}\pi\right)\right.$$
$$\left.\times\left(\zeta\left(m+r,\frac{k}{2N}\right)+(-1)^r\zeta\left(m+r,1-\frac{k}{2N}\right)\right)\right\}. \tag{3.15}$$

Here, we know that

$$\zeta(k,\alpha)+(-1)^r\zeta(k,1-\alpha)\in\pi^k\overline{\mathbb{Q}} \tag{3.16}$$

for all rational numbers $0<\alpha<1$ and all positive integers $k\geqslant 2$. In fact it is shown as follows. It holds that

$$\begin{aligned}\zeta(k,\alpha)+(-1)^r\zeta(k,1-\alpha) &= \sum_{n=0}^{\infty}(n+\alpha)^{-k}+(-1)^r\sum_{n=0}^{\infty}(n+1-\alpha)^{-k}\\ &= \sum_{n=0}^{\infty}(n+\alpha)^{-k}+\sum_{m=-\infty}^{-1}(m+\alpha)^{-k}\\ &= \sum_{m=-\infty}^{\infty}(m+\alpha)^{-k}\end{aligned} \tag{3.17}$$

and

$$\sum_{m=-\infty}^{\infty}(m+x)^{-2}=\frac{\pi^2}{\sin^2\pi x}=\pi^2\left(\cot^2\pi x+1\right). \tag{3.18}$$

In addition,

$$\sum_{m=-\infty}^{\infty}(m+x)^{-k}\in\pi^k\mathbb{Q}[\cot\pi x] \tag{3.19}$$

by the termwise differentiation of (3.18). Then, we obtain (3.16) for $x=\alpha\in\mathbb{Q}$. Moreover, when $k=1$, we see (3.16) by interpreting $\sum_{m=-\infty}^{\infty}(m+\alpha)^{-1}$ as

$$\alpha^{-1}+\sum_{m=1}^{\infty}\left((m+\alpha)^{-1}+(-m+\alpha)^{-1}\right).$$

(B) When m is an even negative integer such that $m \leqslant -r$, $\zeta^W_{SU(2)}(s; g_1, \ldots, g_r) = 0 \in \pi^{m+r}\overline{\mathbb{Q}}$ by Theorem 3(1).

(C) For even m such that $-r < m < 0$, we prove the statement as follows. First, we denote by $B_k(x)$ the Bernoulli polynomial of order k which is defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}. \tag{3.20}$$

It is known that

$$B_k\bigl(x - [x]\bigr) = -\frac{k!}{(2\pi i)^k} \lim_{N\to\infty} \sum_{n=-N,\, n\neq 0}^{N} \frac{e^{2\pi inx}}{n^k} \tag{3.21}$$

for all $x \in \mathbb{R}$ when $k \geqslant 2$; in the case of $k = 1$, this holds for $x \in \mathbb{R} \setminus \mathbb{Z}$. Note that

$$\sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} \left(\theta_1 + \sum_{j=2}^{r} (-1)^{\alpha_{j-1}}\theta_j\right)^k = 0 \tag{3.22}$$

for a positive integer $k < r - 1$. Let

$$\ell_{(\alpha_1,\ldots,\alpha_{r-1})} = \frac{\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j}{2\pi} - \left[\frac{\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j}{2\pi}\right]. \tag{3.23}$$

Then, when $0 < k < r - 1$ is even,

$$\begin{aligned} &\sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} \sum_{n=1}^{\infty} \frac{\cos(n(\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j))}{n^k} \\ &\quad = \frac{(2\pi)^k}{(-1)^{\frac{k}{2}-1}2k!} \left\{ \sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} B_k(\ell_{(\alpha_1,\ldots,\alpha_{r-1})}) \right\}. \end{aligned} \tag{3.24}$$

In addition, when $1 < k < r-1$ is odd or when $k = 1$ and $\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j \neq 2l\pi$ for all $(\alpha_1, \ldots, \alpha_{r-1}) \in \{0,1\}^{r-1}$,

$$\begin{aligned} &\sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} \sum_{n=1}^{\infty} \frac{\sin(n(\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j))}{n} \\ &\quad = \frac{(2\pi)^k}{(-1)^{\frac{k+1}{2}}2k!} \left\{ \sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} B_k(\ell_{(\alpha_1,\ldots,\alpha_{r-1})}) \right\}. \end{aligned} \tag{3.25}$$

Moreover, when $k = 1$ and one of $\theta_1 + \sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j = 2l\pi$ with an integer l (say $(\beta_1, \ldots, \beta_{r-1})$ is an element of $\{0,1\}^{r-1}$ such that $\ell_{(\beta_1,\ldots,\beta_{r-1})} = 0$),

$$\sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} \sum_{n=1}^{\infty} \frac{\sin(n(\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j))}{n}$$
$$= -\pi(-1)^{\beta_1+\cdots+\beta_{r-1}}\left(\frac{\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j}{2\pi}-l\right)$$
$$+ \sum_{(\beta_\nu)\neq(\alpha_\nu)\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} \sum_{n=1}^{\infty} \frac{\sin(n(\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j))}{n}$$
$$= -\pi\left(-l+\frac{1}{2}(-1)^{\beta_1+\cdots+\beta_{r-1}} - \sum_{(\alpha_\nu)\neq(\beta_\nu)}\left[\frac{\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j}{2\pi}\right]\right)$$

by (3.21) and (3.22). Hence, $-r < m < 0$, r being even,

$$\zeta^W_{SU(2)}(m; g_1,\ldots,g_r) = \frac{1}{2^{r-1}\sin\theta_1\cdots\sin\theta_r}\cdot\frac{(2\pi)^{m+r}}{(-1)^{\frac{m+r}{2}-1}2(m+r)!}$$
$$\times\left\{\sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}} (-1)^{\alpha_1+\cdots+\alpha_{r-1}} B_{m+r}(\ell_{(\alpha_1,\ldots,\alpha_{r-1})})\right\}$$
$$\in \pi^{m+r}\overline{\mathbb{Q}}. \tag{3.26}$$

In addition, when $-r < m < 0$, r being odd,

$$\zeta^W_{SU(2)}(m; g_1,\ldots,g_r)$$
$$= \begin{cases} \frac{1}{2^{r-1}\sin\theta_1\cdots\sin\theta_r}\cdot\frac{(2\pi)^{m+r}}{(-1)^{\frac{m+r+1}{2}}2(m+r)!} \\ \quad\times\{\sum_{(\alpha_1,\ldots,\alpha_{r-1})\in\{0,1\}^{r-1}}(-1)^{\alpha_1+\cdots+\alpha_{r-1}} \\ \quad\times B_{m+r}(\ell_{(\alpha_1,\ldots,\alpha_{r-1})})\} \quad \text{if } m+r\geqslant 2, \\ \frac{-\pi}{2^{r-1}\sin\theta_1\cdots\sin\theta_r}(-l+\frac{1}{2}(-1)^{\beta_1+\cdots+\beta_{r-1}} \\ \quad -\sum_{(\alpha_\nu)\neq(\beta_\nu)}[\frac{\theta_1+\sum_{j=2}^{r}(-1)^{\alpha_{j-1}}\theta_j}{2\pi}]) \\ \in\pi^{m+r}\overline{\mathbb{Q}} \quad \text{if } m+r=1. \end{cases} \quad\square \tag{3.27}$$

3.3.2. Proof of Theorem 3(3)

If $m = -2$, when $0 < \theta_1, \theta_2, \theta_3, \theta_4 < \pi$,

$$\sum_{(\alpha_1,\alpha_2,\alpha_3)\in\{0,1\}^3} (-1)^{\alpha_1+\alpha_2+\alpha_3} B_2(\ell_{(\alpha_1,\alpha_2,\alpha_3)})$$
$$= \begin{cases} 2, & (\theta_1,\theta_2,\theta_3,\theta_4) \text{ satisfies (1.3)}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.28}$$

So, we see the statement by applying (3.28) to (3.26). We see the other cases in the same way. □

3.4. An average over the group: Proof of Theorem 5

When $g \in SU(2)$ is conjugate to $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ (where $0 \leqslant \theta \leqslant \pi$), we obtain

$$\int_{SU(2)} f(g)\,dg = \frac{2}{\pi}\int_0^{\pi} f\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right)\sin^2\theta\,d\theta$$

for a class function f.

Here, we consider the average of Witten L-function over $SU(2)$ at $s=-2$, i.e.

$$\begin{aligned}&\int_{SU(2)^r} \zeta^W_{SU(2)}(-2; g_1,\dots,g_r)\,dg_1\dots dg_r \\ &= \left(\frac{2}{\pi}\right)^r \int_0^{\pi}\cdots\int_0^{\pi} \zeta^W_{SU(2)}\left(-2; \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix},\dots,\begin{pmatrix} e^{i\theta_r} & 0 \\ 0 & e^{-i\theta_r} \end{pmatrix}\right) \\ &\quad\times \sin^2\theta_1\cdots\sin^2\theta_r\,d\theta_1\dots d\theta_r,\end{aligned}$$

where $g_j \sim \begin{pmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{pmatrix}$.

When $r=1$ or 2, $\zeta^W_{SU(2)}(-2;\theta_1,\dots,\theta_r)=0$ because of Theorem 3, so the integration becomes 0.

Let us see the case $r=3$. It is sufficient to compute the integral in the interior of the tetrahedron V in (1.1).

Since $\zeta^W_{SU(2)}(-2; g_1,g_2,g_3) = \frac{\pi}{4\sin\theta_1\sin\theta_2\sin\theta_3}$, the integral is

$$\begin{aligned}&\frac{8}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \zeta^W_{SU(2)}\left(-2; \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix},\dots,\begin{pmatrix} e^{i\theta_3} & 0 \\ 0 & e^{-i\theta_3} \end{pmatrix}\right) \\ &\quad\times \sin^2\theta_1\sin^2\theta_2\sin^3\theta_3\,d\theta_1\,d\theta_2\,d\theta_3 \\ &= \frac{2}{\pi^2}\int_{\mathrm{Int}(V)} \sin\theta_1\sin\theta_2\sin\theta_3\,d\theta_1\,d\theta_2\,d\theta_3.\end{aligned}$$

We put

$$\begin{aligned} u &= \theta_1+\theta_2+\theta_3, \\ v &= \theta_1+\theta_2-\theta_3, \\ w &= \theta_1-\theta_3+\theta_3. \end{aligned}$$

Then, $-\theta_1 + \theta_2 + \theta_3 = u - v - w$, so the integral becomes

$$\frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi - w} \int_{v+w}^{2\pi} \sin\frac{v+w}{2} \sin\frac{u-w}{2} \sin\frac{u-v}{2}\, du\, dv\, dw \tag{3.29}$$

and it turns out to be 1 by direct calculation. □

4. The case of $SU(3)$

4.1. Proof of Theorem 6

First, we have

$$\zeta^W_{SU(3)}(s;g) = \frac{2}{\prod_{1 \leqslant j < k \leqslant 3}(e^{i\theta_j} - e^{i\theta_k})} \sum_{m,n=1}^{\infty} m^{-(s+1)} n^{-(s+1)} (m+n)^{-(s+1)} \times \det \begin{bmatrix} (e^{i\theta_1})^{m+n} & (e^{i\theta_2})^{m+n} & (e^{i\theta_3})^{m+n} \\ (e^{i\theta_1})^{n} & (e^{i\theta_2})^{n} & (e^{i\theta_3})^{n} \\ 1 & 1 & 1 \end{bmatrix} \tag{4.1}$$

by Theorem 1(2).

Now, we introduce a generalized polylogarithm function

$$Z(s;\alpha,\beta) \stackrel{\text{def}}{=} \sum_{m,n=1}^{\infty} \frac{e^{im\alpha} e^{in\beta}}{m^s n^s (m+n)^s} \tag{4.2}$$

for $\alpha, \beta \in \mathbb{R}$. We have the following expression when $\zeta^W_{SU(3)}(s;g)$ converges absolutely:

$$\begin{aligned} \zeta^W_{SU(3)}(s;g) = {} & \frac{2}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)} \\ & \times \big[Z(s+1;\theta_2,-\theta_1) - Z(s+1;-\theta_1,-\theta_1-\theta_2) \\ & - Z(s+1;\theta_1,-\theta_2) + Z(s+1;-\theta_1-\theta_2,-\theta_2) \\ & + Z(s+1;\theta_1,\theta_1+\theta_2) - Z(s+1;\theta_2,\theta_1+\theta_2)\big]. \end{aligned} \tag{4.3}$$

We denote $x_j = e^{i\theta_j}$ $(j = 1,2,3)$ for simplicity. $Z(s;\alpha,\beta)$ be extended to a meromorphic function on $s \in \mathbb{C}$ (see Nakamura [4]), so $\zeta^W_{SU(3)}(s;g)$ is continued to a meromorphic function on $s \in \mathbb{C}$.

We prepare two lemmas.

Lemma 1. *For all $k \in \mathbb{N}$,*

$$Z(-k;-\alpha,-\beta) = (-1)^k Z(-k;\alpha,\beta). \tag{4.4}$$

Proof. First, for $\mathrm{Re}(s) > 2$,

$$Z(s-1;\alpha,\beta) = i\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)Z(s;\alpha,\beta) \tag{4.5}$$

and

$$Z(s-1;-\alpha,-\beta) = -i\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)Z(s;-\alpha,-\beta). \tag{4.6}$$

Thus, we see the following by repeating k times above:

$$Z(s-k;\alpha,\beta) = i^k\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)^k\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)^k Z(s;\alpha,\beta) \tag{4.7}$$

and

$$Z(s-k;-\alpha,-\beta) = (-i)^k\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)^k\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)^k Z(s;-\alpha,-\beta). \tag{4.8}$$

In addition, we see the following by analytic continuation: First,

$$\begin{aligned} Z(0;\alpha,\beta) &= \sum_{m,n\geqslant 1} e^{im\alpha}e^{in\beta} \\ &= \frac{e^{i\alpha}e^{i\beta}}{(1-e^{i\alpha})(1-e^{i\beta})} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} Z(0;-\alpha,-\beta) &= \sum_{m,n\geqslant 1} e^{-im\alpha}e^{-im\beta} \\ &= \frac{1}{(1-e^{i\alpha})(1-e^{i\beta})} = \left(\frac{e^{i\alpha}}{1-e^{i\alpha}}+1\right)\left(\frac{e^{i\beta}}{1-e^{i\beta}}+1\right). \end{aligned} \tag{4.10}$$

Thus, it be deduced from above 2 equations:

$$\begin{aligned} Z(-1;\alpha,\beta) &= i\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)Z(0;\alpha,\beta) \\ &= i\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)\frac{e^{i\alpha}e^{i\beta}}{(1-e^{i\alpha})(1-e^{i\beta})} \\ &= i\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)\frac{1}{(1-e^{i\alpha})(1-e^{i\beta})} \\ &= i\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)Z(0;-\alpha,-\beta) \\ &= -Z(-1;-\alpha,-\beta). \end{aligned} \tag{4.11}$$

Then, our statement is proved by putting $s = 0$ for (4.7) and (4.8) and applying (4.11). □

Lemma 2. *For all* $k \in 2\mathbb{N}$,

$$Z(-k;\alpha,\alpha+\beta)+Z(-k;\beta,-\alpha)+Z(-k;-\alpha-\beta,-\beta)=0. \tag{4.12}$$

Proof. We see the following by the same way as Lemma 1.

$$\begin{aligned} Z(-k;\alpha,\alpha+\beta) &= i^k\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right)^k\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)^k Z(0;\alpha,\alpha+\beta),\\ Z(-k;\beta,-\alpha) &= i^k\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right)^k\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)^k Z(0;\beta,-\alpha),\\ Z(-k;-\alpha-\beta,-\beta) &= i^k\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right)^k\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)^k Z(0;-\alpha-\beta,-\beta). \end{aligned} \tag{4.13}$$

Thus,

$$\begin{aligned} &Z(-k;\alpha,\alpha+\beta)+Z(-k;\beta,-\alpha)+Z(-k;-\alpha-\beta,-\beta)\\ &\quad= i^{k-1}\left(\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\beta}\right)^{k-1}\left(\frac{\partial^2}{\partial\alpha\partial\beta}\right)^{k-1}\\ &\qquad\times\left\{Z(-1;\beta,-\alpha)+Z(-1,-\alpha-\beta,-\beta)+Z(-1;\alpha,\alpha+\beta)\right\}\\ &\quad= 0. \qquad \square \end{aligned} \tag{4.14}$$

Here, we prove Theorem 6 by using above two lemmas. First, for all odd integers $m < -1$, it be proved that

$$\zeta^W_{SU(3)}(m;g)=0$$

by Lemma 1 and (4.3). In addition, it follows

$$\begin{aligned} \zeta^W_{SU(3)}(m;g) = {}& \frac{2}{(e^{i\alpha}-e^{i\beta})(e^{i\beta}-e^{-i(\alpha+\beta)})(e^{-i(\alpha+\beta)}-e^{i\alpha})}\\ &\times 2\big\{Z(m+1;\alpha,\alpha+\beta)+Z(m+1,\beta,-\alpha)\\ &+Z(m+1;-\alpha-\beta,-\beta)\big\} \end{aligned} \tag{4.15}$$

for even integer $m+1<0$ by (4.3) and Lemma 1. Also,

$$\zeta^W_{SU(3)}(-1;g)=0 \tag{4.16}$$

because

$$Z(0;\alpha,\beta)=\frac{e^{i\alpha}e^{i\beta}}{(1-e^{i\alpha})(1-e^{i\beta})}$$

and (4.3). Thus, we see that $\zeta^W_{SU(3)}(m;g) = 0$ for all negative integers by using Lemma 2. □

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