Milnor K-theory of smooth varieties

Reza Akhtar

Abstract

Let k be a field and X a smooth projective variety of dimension d over k. Generalizing a construction of Kato and Somekawa, we define a Milnor-type group $K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$ which is isomorphic to the ordinary Milnor K-group $K_s^m(k)$ in the case $X = \text{Spec } k$. We prove that $K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$ is isomorphic to both the higher Chow group $CH^{d+s}(X,s)$ and the Zariski cohomology group $H^d_{Zar}(X,\mathcal{K}^M_{d+s})$.

1 Introduction

Let k be a field. Among the algebraic invariants associated to k are the *Milnor K-groups*, one for each integer $n \geq 0$. These abelian groups were first defined (but not so named) by Milnor in the context of quadratic forms [Mi]. The definition is completely algebraic; nevertheless, a beautiful geometric connection with Bloch's higher Chow groups was discovered by Nesterenko-Suslin [NS] and Totaro [To]; precisely, there is a natural map inducing an isomorphism

$$K_s^M(k) \xrightarrow{\cong} CH^s(k,s)$$

From the definition of the higher Chow groups [Bl1], one sees immediately that $CH^s(k,s)$ is a group of *zero-dimensional* cycles. In 1990, Somekawa [So] studied a generalized Milnor K-group associated to a family $K(k;G_1,\ldots,G_s)$ of semi-abelian varieties and proved that in the case $G_1 = \ldots = G_s = \mathbf{G}_m$ (the multiplicative group scheme), there is a natural map inducing an isomorphism

$$K_s^M(k) \xrightarrow{\cong} K(k;\underbrace{\mathbf{G}_m,\ldots,\mathbf{G}_m}_{s})$$

In this paper we prove a generalization of these two theorems which in effect extends the setting of Milnor K-theory from that of fields to that of smooth projective varieties. Given a field k and a geometrically integral quasiprojective variety X which is smooth of dimension d over k, we define a Somekawa-type group

$K(k;\mathcal{CH}_0(X), G_1, \ldots, G_s)$ and prove that when X is projective and $G_1 = \ldots = G_s = \mathbf{G}_m$, this is isomorphic to the higher Chow group $CH^{d+s}(X,s)$ of zero-dimensional cycles. Furthermore, for any smooth quasiprojective X, there is an isomorphism $H^d_{Zar}(X, \mathcal{K}^M_{s+d}) \cong CH^{d+s}(X,s)$ (here $\mathcal{K}^M_{s+d}$ is a sheafified version of the Milnor K-group). The method of proof followed here is somewhat different from that developed in the author's Ph.D. thesis [A]: instead of recovering the Nesterenko-Suslin / Totaro Theorem as a special case of our result, we use their theorem to obtain a more concise proof of our result. This approach also circumvents some of the annoying technicalities encountered in [A] and avoids the introduction of further cumbersome notation. We conclude with an application to the (integral) calculation of various higher Chow groups of zero-cycles.

When referring to an *algebraic scheme* over a field k, we mean a separated scheme of finite type over Spec k. A *variety* over k will be taken to mean an integral algebraic scheme over k. We say that a variety X is *defined over* k if it is geometrically irreducible. The codimension i points of an equidimensional scheme X will be denoted X^i. For any point $x \in X$, we use $k(x)$ to indicate the residue field of x on X. If X is irreducible, the notation $k(X)$ denotes the function field of X, i.e. the residue field of the generic point of X. When A is a k-algebra, we use the notation $X \times_k A$, or simply X_A if the ground field is clear, as shorthand for $X \times_{Spec\ k} A$. If L/k is a field extension and $f : \text{Spec } L \longrightarrow Y$ is a morphism of schemes over Spec k, we use the notation f_L to denote the morphism $f \times id : \text{Spec } L \longrightarrow Y \times_k L$.

The results of this paper formed part of my Ph.D. thesis at Brown University. I would like to thank my advisor, Stephen Lichtenbaum, for his support and guidance; I am also indebted to Wayne Raskind for providing some useful references and hints. I also acknowledge helpful discussions and correspondence with Andrew Kresch, Niranjan Ramachandran, and Wayne Raskind, and Michael Spiess. Finally, I acknowledge the Natural Sciences and Engineering Research Council of Canada for their generous financial support during my years at Brown and the referee for several helpful suggestions.

2 Milnor K-groups

Here we provide a quick survey of some relevant definitions and facts concerning Milnor K-groups. More complete treatments of the material may be found in [Mi], [BT], or [Ke].

2.1 Definitions

Let k be a field and r an integer.

Definition 2.1. *The* Milnor K-groups $K_r^M(k)$ *are defined as follows:*

$$K_r^M(k) = 0 \text{ for } r < 0,\ K_0^M(k) = \mathbf{Z},\ K_1^M(k) = k^*$$

and for $r \geq 2$

$$K_r^M(k) = \frac{(k^*)^{\otimes r}}{I_r}$$

where $I_r \subseteq (k^*)^{\otimes r}$ *is the subgroup generated by elements of the form* $a_1 \otimes \ldots \otimes a_r$ *such that* $a_i + a_j = 1$ *for some* $1 \leq i < j \leq r$. *The class of* $a_1 \otimes \ldots \otimes a_r$ *in* $K_r^M(k)$ *is typically denoted* $\{a_1, \ldots, a_r\}$.

We write $K_*^M(k) = \oplus_{n \in \mathbf{Z}} K_n^M(k)$ and $\cdot$ for the product on this (noncommutative) ring. For arbitrary field extensions L/k, the covariant "restriction" maps will be denoted $\mathrm{res}_{L/k}^M$ and for finite field extensions, the contravariant "norm" maps will be denoted $N_{L/K}^M$. When there is no danger of confusion, we simply write $\mathrm{res}_{L/k}$ and $N_{L/k}$.

Let K be a field equipped with a discrete valuation v, and $k(v)$ the corresponding residue field. Then there is a natural homomorphism $\partial_v^M : K_*^M(K) \longrightarrow K_{*-1}^M(k(v))$ of graded groups of degree -1 (cf. [BT], I.4); in degree 1, the map $\partial_v^M : K^* = K_1^M(K) \longrightarrow K_0^M(k(v)) = \mathbf{Z}$ coincides with the valuative map ord_v, and in degree 2, the map $\partial_v^M : K_2^M(K) \longrightarrow K_1^M(k(v)) = k(v)^*$ is given on generators by $\partial_v^M(\{f, g\}) = T_v(f, g)$, where $T_v(f, g)$ is the tame symbol associated to v.

We conclude this section by listing two fundamental results involving the maps defined above:

Proposition 2.2. *(Bass-Tate, [BT], I.5.4)*
(Projection Formula)
Let $i : k \hookrightarrow L$ *be a finite extension of fields,* $x \in K_*^M(k)$, *and* $y \in K_*^M(L)$. *Then*

$$N_{L/k}(res_{L/k} x \cdot y) = x \cdot N_{L/k} y$$

Proposition 2.3. *(Suslin, [Su])*
(Reciprocity Law)
Let k *be a field and* K *an extension field, finitely generated of transcendence degree* 1 *over* k. *Then*

$$\sum_v N_{k(v)/k} \circ \partial_v^M = 0$$

It is worth noting that in degree 1, the reciprocity law simply reduces to the well-known formula $\sum_v [k(v):k]\mathrm{ord}_v(x) = 0$ for $x \in K^*$.

3 Mixed K-groups

3.1 Definitions

In this section we introduce mixed K-groups, a generalization of Milnor K-groups. The adjective 'mixed' refers to the fact that these groups are defined as a class incorporating both the groups studied by Somekawa in [So] and those studied by Raskind-Spiess in [RS]. The former correspond to the case $r = 0$ and the latter to the case $s = 0$ below.

Let k be a field, and X a smooth quasiprojective variety defined over k. We use the notation $CH_0(X)$ to denote the group of zero-cycles on X modulo rational equivalence. If G is a group scheme defined over k and A is a k-algebra, we use the notation $G(A)$ for the group of A-rational points, i.e. morphisms Spec $A \longrightarrow G$ which commute with the structure maps.

Now suppose $r \geq 0$ and $s \geq 0$ are integers. Let $X_1, \ldots, X_r$ be smooth quasiprojective varieties defined over k and $G_1, \ldots, G_s$ semi-abelian varieties defined over k. Set

$$T = \bigoplus_{E/k \text{ finite}} CH_0((X_1)_E) \otimes \ldots \otimes CH_0((X_r)_E) \otimes G_1(E) \otimes \ldots \otimes G_s(E)$$

We use the notation $(a_1 \otimes \ldots \otimes a_r \otimes b_1 \otimes \ldots \otimes b_s)_E$ to refer to a homogeneous element living in the direct summand of T corresponding to the field E.

As we work towards our definition, we need to identify particular elements of T, which we classify as type **M1** or type **M2**:

- **M1.** For convenience of notation, set $H_i(E) = CH_0((X_i)_E)$ for $i = 1, \ldots, r$ and $H_j(E) = G_{j-r}(E)$ for $j = r+1, \ldots, r+s$.

  For every diagram $k \hookrightarrow E_1 \overset{\phi}{\hookrightarrow} E_2$ of finite extensions of k, all choices $i_0 \in \{1, \ldots, r+s\}$ and all choices $h_{i_0} \in H_{i_0}(E_2)$ and $h_i \in H_i(E_1)$ for $i \neq i_0$, define the element $R_1(E_1; E_2; i_0; h_1, \ldots, h_{r+s})$ to be:

$$(\phi^*(h_1) \otimes \ldots h_{i_0} \otimes \ldots \otimes \phi^*(h_{r+s}))_{E_2} - (h_1 \otimes \ldots \otimes \phi_*(h_{i_0}) \otimes \ldots \otimes h_{r+s})_{E_1}$$

Here we have used the notation ϕ^* (ϕ_*) to denote the pullback (pushforward) map for the Chow group structure on H_i (if $1 \leq i \leq r$) or the group scheme structure on H_i (if $s \leq i \leq r+s$).

- **M2.** For every K finitely generated of transcendence degree 1 over k, all choices of $h \in K^*$, $f_i \in CH_0((X_i)_K)$ for $i = 1, \ldots, r$ and $g_j \in G_j(K)$ for $j = 1, \ldots, s$ such that for every place v of K such that $v(k) = 0$, there exists $j_0(v)$ such that $g_j \in G_j(O_v)$ for all $j \neq j_0(v)$, define the element $R_2(K; h; f_1, \ldots, f_r; g_1, \ldots, g_s)$ by the following rule. If $s > 0$, it is defined by:

$$\sum_v (s_v(f_1) \otimes \ldots \otimes s_v(f_r) \otimes g_1(v) \otimes \ldots \otimes \tilde{T}_v(g_{j_0(v)}, h) \otimes \ldots \otimes g_s(v))_{k(v)}$$

Here O_v is the valuation ring of v, $s_v : CH_0((X_i)_K) \longrightarrow CH_0((X_i)_{k(v)})$ the specialization map for Chow groups (cf. [F], 20.3), and $g_i(v) \in G_i(k(v))$ the reduction of $g_i \in G_i(O_v)$. The notation $\tilde{T}_v$ refers to the "extended tame symbol" as defined in [So]; in the case $G_1 = \ldots = G_s = \mathbf{G}_m$, which is our only concern in this paper, this coincides with the (ordinary) tame v-adic symbol T_v defined in Section 2.1.

If $s = 0$, the element $R_2(K; h; f_1, \ldots, f_r)$ is defined by:

$$\sum_v \mathrm{ord}_v(h)(s_v(f_1) \otimes \ldots \otimes s_v(f_r))_{k(v)}$$

Now define $R \subseteq T$ to be the subgroup generated by all elements of types **M1 M2**; define the *mixed K-group* $K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r); G_1, \ldots, G_s)$ as the quotient T/R, at least when $r + s > 0$. We denote the class of a generator $(h_1 \otimes \ldots \otimes h_{r+s})_E$ by $\{h_1, \ldots, h_{r+s}\}_{E/k}$. We refer to the classes of elements of the form **M1** and **M2** as *relations* of the mixed K-group. If $r = s = 0$, we simply define our group to be $\mathbf{Z}$.

Remark.
Note the similarity between the relations **M1** and the projection formula and between the relations **M2** and the reciprocity law.

We will mostly be interested in the case $G_1 = \ldots = G_s = \mathbf{G}_m$; hence we use $K_s(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_{0(}X_r); \mathbf{G}_m)$ as shorthand for

$K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_{0(}X_r);\underbrace{\mathbf{G}_m,\ldots,\mathbf{G}_m}_{s})$. We also adopt the practice of omitting superfluous semicolons; for example, if $r=0$, we simply write $K(k;G_1,\ldots,G_s)$ for the group above.

3.2 Some results on mixed K-groups

There are various interesting functorial properties possessed by mixed K-groups; we only state those of relevance to us here. For a more complete list, together with proofs, we refer the reader to [A].

Proposition 3.1.

- *(Covariant functoriality) Let k be a field, $X_1,\ldots,X_r$ smooth quasiprojective varieties defined over k and $G_1,\ldots,G_s$ semi-abelian varieties defined over k. Let $i:k\hookrightarrow k'$ be any field extension. Then there is a naturally induced map on mixed K-groups:*

$$res^{MM}_{k'/k}:K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_r);G_1,\ldots,G_s)\longrightarrow$$

$$K(k';\mathcal{CH}_0((X_1)_{k'}),\ldots,\mathcal{CH}_0((X_r)_{k'});(G_1)_{k'},\ldots,(G_s)_{k'})$$

- *(Contravariant functoriality) Suppose $j:k\hookrightarrow L$ is a finite extension of fields, $X_1,\ldots,X_r$ smooth quasiprojective varieties defined over k and $G_1,\ldots,G_s$ semi-abelian varieties defined over k. Then there is a naturally induced map:*

$$N^{MM}_{L/k}:K(L;\mathcal{CH}_0((X_1)_L),\ldots,\mathcal{CH}_0((X_r)_L);(G_1)_L,\ldots,(G_s)_L)\longrightarrow$$
$$K(k;\mathcal{CH}_0(X_1),\ldots,\mathcal{CH}_0(X_r);G_1,\ldots,G_s)$$

Furthermore, $N^{MM}_{L/k}$ is independent of the embedding of k in L.

It seems likely that the original motivation behind Kato's definition of these groups was to obtain the following theorem, corresponding to the case $r=0$ in our notation:

Theorem 3.2. *(Somekawa, [So], Theorem 1.4)*
Let k be a field and $s\geq 0$ an integer. Then the map

$$\gamma=\gamma_k:K^M_s(k)\longrightarrow K_s(k;\mathbf{G}_m)$$

given by $\gamma(\{a_1,\ldots,a_s\})=\{a_1,\ldots,a_s\}_{k/k}$ is a (well-defined) isomorphism.

The construction of the map γ leads directly to the following:

Proposition 3.3. *The covariant and contravariant functorial maps for Milnor K-groups are compatible via the isomorphism γ with the corresponding maps for mixed K-groups. Specifically, using the above notation, we have*

$$\gamma_{k'} \circ res^{M}_{k'/k} = res^{MM}_{k'/k} \circ \gamma_k$$

and

$$\gamma_k \circ N^{M}_{L/k} = N^{MM}_{L/k} \circ \gamma_L$$

At the other extreme, when $s = 0$, we have the following:

Theorem 3.4. *(Raskind-Spiess, [RS], Theorem 2.4)*
Let k be a field, $r \geq 0$ an integer, and $X_1, \ldots, X_r$ smooth projective varieties defined over k. Then there exist natural isomorphisms

$$CH_0(X_1 \times_k \ldots \times_k X_r) \cong K(k; \mathcal{CH}_0(X_1), \ldots, \mathcal{CH}_0(X_r)) \cong K(k; \mathcal{CH}_0(X_1 \times_k \ldots \times_k X_r))$$

We conclude this section with a useful lemma:

Lemma 3.5. *Let S be a smooth quasiprojective variety defined over a field k, and suppose $Q \in CH_0(S)$.*

For every finite extension F/k, there exists a well-defined homomorphism $j_F(Q) : K_s(F; \mathbf{G}_m) \longrightarrow K_s(F; \mathcal{CH}_0(X_F); \mathbf{G}_m)$ taking a generator $\{b_1, \ldots, b_s\}_{M/F}$ to the element $\{[Q_M], b_1, \ldots, b_s\}_{M/F}$.

Let $k \hookrightarrow F_1 \hookrightarrow F_2$ be a diagram of finite extensions of k.
Then the following diagram commutes:

$$\begin{array}{ccccc}
K^M_S(F_2) & \xrightarrow{\gamma_{F_2}} & K_S(F_2; \mathbf{G}_m) & \xrightarrow{j_{F_2}(Q)} & K_s(F_2; \mathcal{CH}_0(X_{F_2}); \mathbf{G}_m) \\
\Big\downarrow N^M_{F_2/F_1} & & \Big\downarrow N^{MM}_{F_2/F_1} & & \Big\downarrow N^{MM}_{F_2/F_1} \\
K^M_s(F_1) & \xrightarrow{\gamma_{F_1}} & K_s(F_1; \mathbf{G}_m) & \xrightarrow{j_{F_1}(Q)} & K_s(F_1; \mathcal{CH}_0(X_{F_1}); \mathbf{G}_m)
\end{array}$$

The proof of the first statement of the above lemma is routine; in the second statement, commutativity of the left square follows from the construction of the maps γ, cf. [So], Theorem 1.4. Commutativity of the right square is immediate from the definitions once we observe that $(\phi_{F_2/F_1})^*[Q_{F_1}] = [Q_{F_2}]$.

4 Higher Chow groups

4.1 Definitions

The original definition of the higher Chow groups for schemes of finite type over a field goes back to Bloch [Bl1], who defined them as the homotopy groups of a certain simplicial abelian group; by the Dold-Kan correspondence, these coincide with the homology groups of a related chain complex. Not long afterward, Levine defined a complex, quasi-isomorphic to Bloch's complex using "cubical" instead of simplicial constructions; following Totaro [To] it is this definition which we employ. As observed by Totaro, the cubical construction makes possible an explicit description of the product structure on higher Chow groups. Throughout this section we fix a base field k.

Let $n \geq 0$ be an integer. Following Totaro [To], we define the n-cube $\square_k^n$ (or simply $\square^n$ if there is no danger of ambiguity) to be $(\mathbf{P}_k^1 - \{1\})^n$. We will use coordinates $t_1, \ldots, t_n : \square_k^n \longrightarrow (\mathbf{P}_k^1 - \{1\})^n$ to describe (closed) points on $\square^n$; in terms of these,

$$\square_k^n \cong \text{Spec } k[\frac{1}{1-t_1}, \ldots, \frac{1}{1-t_n}]$$

If we change to coordinates $u_i = \dfrac{1}{1-t_i}$ for each $i = 1, \ldots, n$, we obtain a natural identification of $\square^n$ with $\mathbf{A}_k^n$. (The reason for the choice of the apparently less natural t_i in the definition will become evident in later proofs) The subscheme defined by the equations $t_{i_1} = \ldots = t_{i_r} = 0$, $t_{j_1} = \ldots = t_{j_s} = \infty$ is called a (codimension $r+s$) *face* of $\square^n$.

Given a strictly increasing map $\rho : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ and elements $\varepsilon_i \in \{0, \infty\}$ for all $i \notin \text{Im } (\rho)$, we define the *face map* $\tilde{\rho}^\varepsilon : \square^m \longrightarrow \square^n$ by

$$(\tilde{\rho}^\varepsilon)^* t_i = \begin{cases} t_j & \text{if } i = \rho(j) \text{ for some } j \\ \varepsilon_i & \text{if } i \notin \rho(\{1.\ldots, m\}) \end{cases}$$

Now let Y be an equidimensional algebraic scheme. Define $c^*(Y, p)$ to be the free abelian group, graded by codimension, on the set of subvarieties of $Y \times_k \square_k^p$ which intersect all faces of the cube properly, i.e. in the expected dimension; if $p < 0$, declare this group to be 0. We call such cycles *well-positioned.* One shows (cf. [F], 6.6) that if ρ is a codimension m face, the operation of intersection induces a (well-defined) pullback map $\rho^* : c^*(Y, p) \longrightarrow c^*(Y, p-m)$. Define $\rho_i^0 : \{1, \ldots, n-1\} \longrightarrow \{1, \ldots, n\}$ by

$$\rho_i^0(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

and $\varepsilon_i = 0$; we define ρ_i^∞ in the analogous manner. Assembling these maps together, we obtain a diagram of maps denoted $c^*(Y, \cdot)$:

$$\ldots \longrightarrow c^*(Y, 2) \xrightarrow{d_2} c^*(Y, 1) \xrightarrow{d_1} c^*(Y, 0)$$

The maps d_n are defined by the formula

$$d_n = \sum_{i=1}^{n} (-1)^i (\rho_i^0 - \rho_i^\infty)$$

It follows readily that $c^*(Y, \cdot)$ is a chain complex. Now consider the subcomplex $d^*(Y, \cdot) \subseteq c^*(Y, \cdot)$ generated by "degenerate cycles"; that is, $d^i(X, n)$ is the subgroup of $c^i(X, n)$ generated by those cycles on $Y \times_k \square_k^n$ pulled back from cycles on $Y \times_k \square_k^{n-1}$ by a projection $Y \times_k \square_k^n \longrightarrow Y \times_k \square_k^{n-1}$ of the form $(y, a_1, \ldots, a_n) \mapsto (y, a_1, \ldots, \widehat{a_k}, \ldots, a_n)$ for some k, $1 \leq k \leq n$. Let $z^*(Y, \cdot)$ be the quotient complex $c^*(Y, \cdot)/d^*(Y, \cdot)$; we then define the *higher Chow groups* $CH^*(Y, \cdot)$ as the homology groups of the complex $z^*(Y, \cdot)$. Precisely, $CH^i(Y, n)$ is the codimension i graded piece of the nth homology group of $z^*(Y, \cdot)$.

Some basic properties of the higher Chow groups, along with a few elementary calculations, may be found in the next section; a more complete treatment is given in [Bl1]. The case of immediate interest to us is the group $CH^s(\text{Spec } k, s)$, where $s \geq 0$ is an integer; we abbreviate this group as $CH^s(k, s)$ to ease notation. Tracing through the definition, one finds that the generators of $z^s(k, s-1)$ correspond to codimension s subvarieties of $\square_k^{s-1}$, of which there are obviously none. Hence $z^s(k, s-1) = 0$ and $CH^s(k, s)$ is isomorphic to the cokernel of the map $d_{s+1} : z^s(k, s+1) \longrightarrow z^s(k, s)$. In particular, the elements of $CH^s(k, s)$ are classes of elements of $z^s(k, s)$, and the latter are zero-dimensional cycles on the scheme $\square_k^s$.

The main result of relevance here is:

Theorem 4.1. *(Nesterenko-Suslin, [NS]; Totaro, [To])*
Let k be a field and $s \geq 0$ an integer. Then there is a natural map inducing an isomorphism:

$$T_k = T : K_s^M(k) \xrightarrow{\cong} CH^s(k, s)$$

4.2 Functoriality and Products

The higher Chow groups $CH^*(X, \cdot)$ of an equidimensional algebraic scheme X over a field k have a number of useful functorial properties as well as a product structure:

- Given a proper morphism $\phi : X \longrightarrow Y$ over k, there exists a morphism $\phi_* : CH^*(X, \cdot) \longrightarrow CH^{*+\dim Y - \dim X}(Y, \cdot)$ induced by push-forward of cycles.

- Given a flat morphism $\psi : X \longrightarrow Y$ over k, there exists a morphism $\psi^* : CH^*(Y, \cdot) \longrightarrow CH^*(X, \cdot)$ induced by pull-back of cycles. A deeper result states that ψ^* exists even if we assume only that Y is smooth over k.

- Given two equidimensional algebraic schemes X, Y over k, one may construct an "external product"

$$\times : z^*(X, \cdot) \otimes z^*(Y, \cdot) \longrightarrow z^*(X \times_k Y, \cdot)$$

  defined on homogeneous generators as follows: if $V \subseteq X \times_k \square^n$ and $W \subseteq Y \times_k \square^m$ are well-positioned subvarieties, we define $[V] \times [W]$ to be the cycle associated to the scheme $V \times_k W$, which is isomorphic in a natural way to a well-positioned subscheme of $X \times_k Y \times_k \square^{n+m}$.

  (This is the point at which the choice of the cubical structure in the definition of the higher Chow groups is most expedient; in [Bl1], Bloch needs to work harder to achieve the construction simplicially).

- If X is smooth, we may compose the external product map (at the level of higher Chow groups) $\times : CH^*(X, \cdot) \otimes CH^*(X, \cdot) \longrightarrow CH^*(X \times_k X, \cdot)$ with pullback along the diagonal $\Delta_X \hookrightarrow X \times_k X$ to obtain the *intersection product*:

$$\cdot : CH^*(X, \cdot) \otimes CH^*(X, \cdot) \longrightarrow CH^*(X, \cdot)$$

The map T is compatible with the norm maps on Milnor K-theory and the covariant maps on higher Chow groups in the following sense:

Lemma 4.2. *Let E/k be a finite field extension. Then the norm maps on Milnor K-theory are compatible with the isomorphism of Theorem 4.1, i.e.*

$$(\phi_{E/k})_* \circ T_E = T_k \circ N^M_{E/k}$$

We conclude this section with three further facts about the higher Chow groups. For a more comprehensive list, we refer the reader to [Bl1].

- Let X be any equidimensional algebraic scheme defined over k. Then for any i there is a natural isomorphism $CH^i(X, 0) \xrightarrow{\cong} CH^i(X)$, where the group on the right is the ordinary Chow group of codimension i cycles on X.

- If X is furthermore smooth, there is an isomorphism $CH^1(X,1) \cong \Gamma(X, \mathcal{O}_X^*)$; the term on the right is the group of invertible global sections of the structure sheaf on X.

- (Projection formula) Let $f : X \longrightarrow Y$ be a finite morphism of equidimensional algebraic schemes over k, with Y smooth. Then for any $x \in CH^*(X, \cdot)$ and $y \in CH^*(Y, \cdot)$,
$$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$$

4.3 Localization

Suppose $i : Z \hookrightarrow X$ is a closed subscheme of pure codimension d in X; let $j : U = X - Z \hookrightarrow X$ be its complement. Noting that i is proper and j flat, we clearly have an exact sequence of complexes:

$$0 \longrightarrow z^{*-d}(Z, \cdot) \xrightarrow{i_*} z^*(X, \cdot) \xrightarrow{j^*} z^*(U, \cdot)$$

where $d = \dim X - \dim Z$.

The map j^* is not in general surjective; however, there is an important and highly nontrivial *localization* theorem of Bloch:

Theorem 4.3. *(Bloch, [Bl2]) With notation as above, the induced map*

$$\frac{z^*(X, \cdot)}{i_*(z^{*-d}(Z, \cdot))} \overset{j^*}{\hookrightarrow} z^*(U, \cdot)$$

is a quasi-isomorphism.

We will be interested in an extension of the theorem to the following scenario: let S be any quasiprojective variety and C a smooth projective curve over k. Consider the directed system consisting of open neighborhoods $\{V_g\}$ of the generic point g of C. Applying the localization theorem with $X = S \times_k C$ and $U = S \times_k V_g$, once for each V_g, we obtain a directed system of exact sequences; we may then take limits and are led to the following:

Corollary 4.4. *With notation as above, the induced map*

$$\frac{z^*(S \times_k C, \cdot)}{\bigoplus_{c \in C^1} (i_c)_* z^{*-1}(S \times_k k(c), \cdot)} \overset{j^*}{\hookrightarrow} z^*(S \times_k k(C), \cdot)$$

is a quasi-isomorphism.

Our interest in this form of the localization theorem is the following corollary:

Theorem 4.5. *(Reciprocity law for Higher Chow Groups) Let k be a field, S a quasiprojective variety over k and C a smooth quasiprojective curve over k with function field $K = k(C)$. Let $m, n \geq 0$ be integers. Let $\partial = (\partial_c) : CH^{m+1}(S \times_k K, n+1) \longrightarrow \bigoplus_{c \in C^1} CH^m(S \times_k k(c), n)$ denote the connecting homomorphism associated to the map of complexes in 4.4 and for each $c \in C^1$, let $(f_c)_* : CH^m(S \times_k k(c), n) \longrightarrow CH^m(S, n)$ be the covariant map induced functorially by the canonical morphism $f_c : S \times_k k(c) \longrightarrow S$. Let $\sigma : C \longrightarrow Spec\ k$ denote the structure morphism. Then*

$$\sum_{c \in C^1} (f_c)_* \circ \partial_c = 0$$

Proof.
The proof follows that of the Reciprocity Law of [Gi], p.275. We are indebted to Wayne Raskind for bringing this result to our attention. Consider the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{c \in C^1} z^m(S \times_k k(c), \cdot) & \xrightarrow{(i_c)_*} & z^{m+1}(S \times_k C, \cdot) & \xrightarrow{j^*} & \frac{z^{m+1}(S \times_k C, \cdot)}{\bigoplus_{c \in C^1} z^m(S \times_k k(c), \cdot)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sum (f_c)_*} & & \downarrow{\scriptstyle \sigma_*} & & \downarrow & & \\
0 & \longrightarrow & z^m(S, \cdot) & \xrightarrow{id} & z^m(S, \cdot) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

If we examine the long exact homology sequences associated to the top and bottom short exact sequences, we obtain a commutative diagram, taking into account Theorem 4.4, as follows:

$$\begin{array}{ccccc}
CH^{m+1}(S \times_k K, n+1) & \xrightarrow{\partial} & \bigoplus_{c \in C^1} CH^m(S \times_k k(c), n) & \longrightarrow & CH^{m+1}(S \times_k C, n) \\
\downarrow & & \downarrow{\scriptstyle \sum_c N_{k(c)/k}} & & \downarrow \\
0 & \xrightarrow{\partial} & CH^m(S, n) & \xrightarrow{id} & CH^m(S, n)
\end{array}$$

By commutativity of the left square, it is clear that

$$\sum_{c \in C^1} N_{k(c)/k} \circ \partial_c = 0$$

5 The Main Theorem: Part I

Before stating our main theorem, we need one more definition:

Definition 5.1. *Let X be a scheme and $s \geq 0$ an integer. The* Milnor K-sheaf *on (the Zariski site of) X, denoted $\mathcal{K}_s^M$ is the sheaf associated to the presheaf $U \mapsto K_s^M(\mathcal{O}_X(U))$. If X is connected and A is an abelian group, we (abusively) use the notation A to denote the constant (Zariski) sheaf on X with value A.*

These constant sheaves are interesting primarily for the following reason: suppose $y \in X$ is a point corresponding to some closed subvariety $Y \subseteq X$ and z is a point corresponding to a codimension 1 subvariety $Z \subseteq Y$. If Y is normal, then z corresponds to some discrete valuation on its function field $k(y)$; hence, there is a boundary homomorphism

$$\partial_z^M : K_*^M(k(y)) \longrightarrow K_{*-1}^M(k(z))$$

If Y is not normal, we let $\pi : \tilde{Y} \longrightarrow Y$ denote its normalization and define, for each such z, $\partial_z^M = \sum_{\tilde{z} \in \pi^{-1}(z)} N_{k(\tilde{z})/k(z)}^M \circ \partial_{\tilde{z}}^M$.

Now let X be an algebraic scheme over some field k; set $d = \dim X$, and let $s \geq 0$ be an integer. We may assemble the various boundary maps above into a sequence:

$$\bigoplus_{x \in X^0} (i_x)_* K_s^M(k(x)) \xrightarrow{\partial^M} \bigoplus_{x \in X^1} (i_x)_* K_{s-1}^M(k(x)) \xrightarrow{\partial^M} \dots \xrightarrow{\partial^M} \bigoplus_{x \in X^d} (i_x)_* K_{s-d}^M(k(x)) \longrightarrow 0$$

where for each $x \in X$, $i_x : \operatorname{Spec} k(x) \longrightarrow X$ is the inclusion map.

The main result of value to us the following:

Theorem 5.2. *(Kato, [Ka2] 4. Theorem 3) With notation as above,*

$$H_{Zar}^d(X, \mathcal{K}_{d+s}^M) \cong \mathit{Coker}\ \Big(\bigoplus_{x \in X^{d-1}} K_{s+1}^M(k(x)) \xrightarrow{\partial^M} \bigoplus_{x \in X^d} K_s^M(k(x)) \Big)$$

We also need the following version of the "Gersten conjecture" for higher Chow groups. Letting $\mathcal{CH}^r(s)$ denote the sheaf associated to the presheaf $U \mapsto CH^r(U, s)$, we have:

Theorem 5.3. *(Bloch, [Bl1], 10) Let X be a smooth quasiprojective variety of dimension d over a field k. There is an exact sequence of flasque Zariski sheaves on X:*

$$0 \longrightarrow \mathcal{CH}^r(s) \longrightarrow \bigoplus_{x \in X^0} (i_x)_* CH^r(k(x), s) \xrightarrow{\partial} \bigoplus_{x \in X^1} (i_x)_* CH^{r-1}(k(x), s-1) \xrightarrow{\partial} \dots$$

$$\cdots \xrightarrow{\partial} \bigoplus_{x\in X^d} (i_x)_* CH^{r-d}(k(x), s-d) \longrightarrow 0$$

The maps ∂ are defined as limits of the boundary homomorphisms arising from the various localization sequences; for details, we refer the reader to [Bl1], Section 10.

The following lemma will be useful:

Lemma 5.4. *(Geisser-Levine [GL], Lemma 3.2)*
Let F be a field and K be a finitely generated field of transcendence degree 1 over F. Let v be a discrete valuation on K such that $v(K) = 0$ and $F(v)$ the residue field. Then the following diagram commutes:

$$\begin{array}{ccc} K^M_{s+1}(K) & \xrightarrow{\partial^M_v} & K^M_s(F(v)) \\ \downarrow T & & \downarrow T \\ CH^{s+1}(K, s+1) & \xrightarrow{\partial} & CH^s(F(v), s) \end{array}$$

At last we come to the statement of our theorem:

Theorem 5.5. *Let X be a smooth quasiprojective variety defined over a field k; set $d = \dim X$, and let $s \geq 0$ be an integer. Then there is an isomorphism:*

$$H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \xrightarrow{\cong} CH^{d+s}(X, s)$$

Proof.
The diagram

$$\begin{array}{ccc} \bigoplus_{x\in X^{d-1}} K^M_{s+1}(k(x)) & \xrightarrow{\partial^M} & \bigoplus_{x\in X^d} K^M_s(k(x)) \\ \downarrow T & & \downarrow T \\ \bigoplus_{x\in X^{d-1}} CH^{s+1}(k(x), s+1) & \xrightarrow{\partial} & \bigoplus_{x\in X^d} CH^s(k(x), s) \end{array}$$

commutes by Lemma 3.2 of [GL], and the vertical maps are isomorphisms by Theorem 4.1. Thus, the maps T induce an isomorphism Coker $\partial^M \cong$ Coker ∂; by Theorems 5.2 and 5.3, this gives an isomorphism

$$H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \cong H^d_{Zar}(X, \mathcal{CH}^{d+s}(d+s))$$

On the other hand, the discussion of [Bl1], Section 10 asserts the existence of a spectral sequence (for every r):

$$E_2^{p,q} = H^p_{Zar}(X, \mathcal{CH}^r(-q)) \Longrightarrow CH^r(X, -p-q)$$

We will consider the case $r = d+s$ in order to study the group $CH^{d+s}(X, s)$. The relevant E_2-terms correspond to the terms $E_2^{p,-p-s} = H^p_{Zar}(X, \mathcal{CH}^{d+s}(p+s))$. Note that for $p > d$, this term is zero, as X has trivial Zariski cohomology in dimension greater than d. When $p < d$, the sheaf $\mathcal{CH}^{d+s}(p+s)$ is zero by Theorem 5.3, as all terms in the resolution are zero. Furthermore, we have $E_n^{d,-d-s} = E_2^{d,-d-s}$ for $n \geq 2$. Thus the spectral sequence degenerates, and we have an isomorphism

$$H^d_{Zar}(X, \mathcal{CH}^{d+s}(d+s)) \xrightarrow{\cong} CH^{d+s}(X,s)$$

and hence

$$H^d_{Zar}(X, \mathcal{K}^M_{d+s}) \cong CH^{d+s}(X,s)$$

6 The Main Theorem: Part II

Our goal in this section is to prove the following:

Theorem 6.1. *Let k be a field and X a smooth projective variety of dimension d defined over k, and $s \geq 0$ an integer. Then there is a natural map inducing an isomorphism:*

$$CH^{d+s}(X,s) \xrightarrow{\cong} K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$$

6.1 Some lemmas

Let S be a scheme over Spec A, where A is a finitely generated algebra over a field F. Then there is a natural identification of A^* with a subgroup of $CH^1(S,1) \cong \Gamma(S, \mathcal{O}_S^*)$. With this identification in mind, we present two technical lemmas necessary for the proof of Theorem 6.1.

The first result follows immediately from Theorem 4.1 and the relation $\{a, -a\} = 0$ in $K_2^M(F)$:

Lemma 6.2. *Let S be an equidimensional algebraic scheme over a field F. Then for any $x \in CH_0(S)$ and $a, b_1, \ldots, b_{s-2} \in F^*$ and $1 \leq i \leq j \leq s-2$, the element $x \cdot b_1 \cdot \ldots \cdot b_i \cdot a \cdot b_{i+1} \cdot \ldots \cdot b_j \cdot (-a) \cdot b_{j+1} \cdot \ldots \cdot b_{s-2} \in CH^{d+s}(S,s)$ is equal to zero.*

Lemma 6.3. *Let X be a smooth quasiprojective variety of dimension d defined over k. Let K be a finitely generated extension of transcendence degree 1 over k and C its smooth projective model over k. For any valuation v of K such that $v(k) = 0$, we denote its valuation ring by O_v and its residue field by $k(v)$. Let $s \geq 0$ be an integer* *and* $\partial_v = \partial_{v,s,X} : CH^{d+s+1}(X_K, s+1) \longrightarrow CH^{d+s}(X_{k(v)}, s)$ *the connecting homomorphism arising from the long exact sequence associated to the quasi-isomorphism of the localization theorem, Corollary 4.4. Let* $P \in X_K$ *be a closed point and* $[P]$ *its class in* $CH_0(X_K)$*. Then for every* $g_1, \ldots, g_s \in O_v^*$ *and* $h \in O_v$,

$$\partial_{v,s,X}([P] \cdot g_1 \cdot \ldots \cdot g_s \cdot h) = ord_v(h)(s_v([P]) \cdot g_1(v) \cdot \ldots \cdot g_s(v))$$

As before, s_v *is the specialization map and* $\cdot$ *the intersection product on the higher Chow groups.*

Proof.

The assertion is trivial if h or any of the g_i are equal to 1; we assume henceforth that none of them are. Furthermore, by multilinearity of the external product, we may assume $0 \le \mathrm{ord}_v(h) \le 1$.

The valuation v corresponds to some closed point on C; choose an affine neighborhood Spec B of this point such that the only zero of h lying in this neighborhood is v and such that $g_1, \ldots, g_s$ have no zeros or poles in this neighborhood. It is clear from the local nature of the problem that we may replace C by Spec B in the statement of the theorem. In fact, we may even take a limit over all such B and replace C by the local ring O_v.

Consider the diagram:

$$\begin{array}{ccccccc}
\uparrow d_s & & \uparrow d_s & & \uparrow d_s & & \uparrow d_s \\
z^{*-1}(X_{k(v)}, s) & \xrightarrow{i_v} & z^*(X_{O_v}, s) & \xrightarrow{\pi} & \frac{z^*(X_{O_v}, s)}{z^{*-1}(X_{k(v)}, s)} & \overset{\rho}{\hookrightarrow} & z^*(X_K, s) \\
\uparrow d_{s+1} & & \uparrow d_{s+1} & & \uparrow d_{s+1} & & \uparrow d_{s+1} \\
z^{*-1}(X_{k(v)}, s+1) & \xrightarrow{i_v} & z^*(X_{O_v}, s+1) & \xrightarrow{\pi} & \frac{z^*(X_{O_v}, s+1)}{z^{*-1}(X_{k(v)}, s+1)} & \overset{\rho}{\hookrightarrow} & z^*(X_K, s+1) \\
\uparrow d_{s+2} & & \uparrow d_{s+2} & & \uparrow d_{s+2} & & \uparrow d_{s+2}
\end{array}$$

Let $c = [P] \cdot g_1 \cdot \ldots \cdot g_s \cdot h$. The computation of $\partial_{v,s,X}(c)$ may be described by performing the chase defining $\partial_{v,s,X}$ around the above diagram. First, since ρ is a quasi-isomorphism, pick an element $z(c) \in z^*(X_K, s+1) \in \mathrm{Im}\ (\rho)$ whose class in $CH^*(X_K, s+1)$ is c; then choose some cycle $y(c) \in z^*(X_{O_v}, s+1)$ such that $z(c) = \rho(\pi(y(c)))$. Next, observe that commutativity of the diagram implies that $\rho(\pi(d_{s+1}(y(c)))) = d_{s+1}(\rho(\pi(y(c)))) = d_{s+1}(z(c)) = 0$. Since ρ is injective, it follows that $\pi(d_{s+1}(y(c))) = 0$, and hence that $x_c = d_{s+1}(y(c))$ is in the image of the map i_v.

Letting $w(c) \in z^{*-1}(X_{k(v)}, s)$ denote the pullback of the cycle $x(c)$ via the natural map $X_{k(v)} \hookrightarrow X_{O_v}$, we finally obtain $\partial_{v,s,X}(c)$ by taking the class of $w(c)$ in $CH^{d+s}(X_{k(v)}, s)$.

Lemma 6.4. *With notation as above,*

$$\partial_{v,s,X}([P] \cdot g_1 \cdot \ldots \cdot g_s \cdot h) = s_v[P] \cdot \partial_{v,s,k}(g_1 \cdot \ldots \cdot g_s \cdot h)$$

Proof.
Following the procedure described above, we may take

$$z(c) = P \times_K g_1 \times_K \ldots g_s \times_K h \in z^*(X_K, s+1).$$

Let Q be the closure of the image of P under the natural map $X_K \longrightarrow X_{O_v}$. Then we may take

$$y(c) = Q \times_{O_v} g_1 \times_{O_v} \ldots \times_{O_v} g_s \times_{O_v} h$$

and hence

$$x(c) = Q \times_{O_v} d_{s+1}(g_1 \times_{O_v} \ldots \times_{O_v} g_s \times_{O_v} h).$$

Last, we have

$$w(c) = (Q \times_{O_v} k(v)) \times_{k(v)} (d_{s+1}((g_1 \times_{O_v} \ldots \times_{O_v} g_s \times_{O_v} h) \times_{O_v} k(v)))$$

Finally, the class in $CH^d(X_{k(v)})$ of the cycle defined by $Q \times_{O_v} k(v)$ is exactly the specialization $s_v[P]$ as described in [F], Section 20.3 and the class of $d_{s+1}(g_1 \times_{O_v} \ldots \times_{O_v} g_s \times_{O_v} h) \times_{O_v} k(v))$ in $CH^s(k(v), s)$ is simply $\partial_{v,s,k}(g_1 \cdot \ldots \cdot g_s \cdot h)$. Thus

$$c = s_v[P] \cdot \partial_{v,s,k}(g_1 \cdot \ldots \cdot g_s \cdot h)$$

To conclude the proof of Lemma 6.3, we simply observe that $\partial_{v,s,k}(g_1, \cdot \ldots \cdot g_s \cdot h) = \mathrm{ord}_v(h)(g_1(v) \cdot \ldots \cdot g_s(v))$ by Lemma 5.4.

6.2 Proof of the Main Theorem, Part II

Recall that X is now assumed to be a smooth *projective* variety. We begin by constructing a map $\beta : K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \longrightarrow CH^{d+s}(X, s)$. We propose the following definition on generators of $K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$, extending in the obvious manner by linearity. In the following, $\phi_{E/k}$: Spec $E \longrightarrow$ Spec k is the canonical map; x is interpreted as being in $CH_0(X_E) \cong CH^d(X_E, 0)$; $a_1, \ldots, a_s$ are interpreted as being in $E^* \cong CH^1(X_E, 1)$, and $\cdot$ represents the product structure on the higher Chow groups.

$$\beta(\{x, a_1, \ldots, a_s\}_{E/k}) = (\phi_{E/k})_*(x \cdot a_1 \cdot \ldots \cdot a_s)$$

Towards a verification that β is well-defined, we note that the assertion that β kills relations of type **M1** follows immediately from the projection formula for higher Chow groups. Now suppose $R = R_2(K; h; y; g_1, \ldots, g_s)$ is a relation of type **M2**.
Then

$$\beta(R) = \sum_v ((\phi_{E/k})_{k(v)})_*(s_v(y) \cdot g_1(v) \cdot \ldots \cdot T_v(g_{j_0(v)}, h) \cdot \ldots \cdot g_s(v))$$

Lemma 6.5. *For each v,*

$$s_v(y) \cdot g_1(v) \cdot \ldots \cdot T_v(g_{j_0(v)}, h) \cdot \ldots \cdot g_s(v) = \partial_v(y \cdot g_1 \cdot \ldots \cdot g_s \cdot h)$$

Proof.
Fix a uniformizer π_v for v and write $g_{j_0}(v) = u_1\pi_v^a$, $h = u_2\pi_v^b$, where $u_1, u_2 \in O_v^*$ and $a, b \in \mathbf{Z}$. Consider $g_{j_0(v)} \cdot h \in CH^2(K, 2)$; by Lemma 6.2 and multilinearity of the intersection product we have

$$g_{j_0(v)} \cdot h = u_1 \cdot \pi_v^b + \pi_v^a \cdot u_2 + u_1 \cdot u_2 + ab(\pi_v \cdot \pi_v)$$

$$= u_1 \cdot \pi_v^b + u_2 \cdot \pi_v^{-a} + u_1 \cdot u_2 + ab((-1) \cdot \pi_v)$$

Furthermore, it is clear from the properties of the tame symbol (cf. [Se], III.1) that

$$T_v(g_{j_0(v)}, h) = T_v(u_1\pi_v^a, u_2\pi_v^b) = T_v(u_1, \pi_v^b)T_v(\pi_v^a, u_2)T_v(\pi_v^a, \pi_v^b)$$

$$= T_v(u_1, \pi_v^b)T_v(u_2, \pi_v^{-a})T_v(-1, \pi_v)^{ab}$$

Since each side of the equation in the statement of the Lemma is (separately) linear in the terms $g_{j_0}(v)$ and h, it suffices, in light of the above reasoning, to prove the lemma when $g_{j_0}(v) \in O_v^*$. This is exactly the conclusion of Lemma 6.3 and thus concludes the proof of Lemma 6.5.

Continuing the calculation leading towards a proof of Theorem 6.1, we obtain

$$\beta(R) = \sum_v (\phi_{k(v)/k})_*(s_v(y) \cdot g_1(v) \cdot \ldots \cdot T_v(g_{j_0(v)}, h) \cdot \ldots \cdot g_s(v))$$

$$= \sum_v (\phi_{k(v)/k})_*(\partial_v(y \cdot g_1 \cdot \ldots \cdot g_s \cdot h))$$

By the Reciprocity Law (Theorem 4.5), this sum is zero, thus completing the verification that β is well-defined.

We continue by constructing a map $\alpha : CH^{d+s}(X,s) \longrightarrow K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$. Since the group $CH^{d+s}(X,s)$ consists of zero-cycles, it is generated by classes $[P]$, where $P : \mathrm{Spec}\ k(P) \longrightarrow X \times_k \square_k^s$ is a closed point. By definition of fibered product, we see that P is determined by the data $x : \mathrm{Spec}\ k(P) \longrightarrow X$ and $a_1,\dots,a_s \in k(P)^*$. Now define

$$\alpha : CH^{d+s}(X,s) \xrightarrow{\alpha} K_s(k;\mathcal{CH}_0(X);\mathbf{G}_m)$$

by

$$[P] \mapsto \{[x_{k(P)}], a_1,\dots,a_s\}_{k(P)/k}$$

In terms of our previous notation, we have

$$\alpha[P] = N^{MM}_{k(P)/k}(j_{k(P)}[Q](\gamma_{k(P)}(\{a_1,\dots,a_s\})))$$

To check that this rule is well-defined, we must show that elements of $d_{s+1}(z^{d+s}(X,s+1))$ are killed by α.

The generators of the group $z^{d+s}(X,s+1)$ correspond to well-positioned dimension 1 subvarieties $C \subseteq X \times_k \square_k^{s+1}$. This means that for any face F of $\square_k^{s+1}$, C intersects $X \times_k F$ in dimension 0 (i.e. in points) if F has codimension 1 and does not meet $X \times_k F$ at all if F has codimension ≥ 2. Let $\nu : \tilde{C} \longrightarrow C$ be the normalization of C.

The inclusion $i : C \subseteq X \times_k \square_k^{s+1}$ is defined by a collection of $s+2$ morphisms: $f : C \longrightarrow X$, $g_i : C \longrightarrow \square_k^1 \hookrightarrow \mathbf{P}_k^1$, $i = 1 \dots s+1$. Likewise, the composition $\tilde{i} = \nu \circ i : \tilde{C} \longrightarrow X$ is defined by $s+2$ morphisms: $\tilde{f} : \tilde{C} \longrightarrow X$, $\tilde{g}_i : \tilde{C} \longrightarrow \square_k^1 \hookrightarrow \mathbf{P}_k^1$.

The assumption that C meets the codimension 1 faces of the cube in points translates into the fact that none of the g_i are identically 0 or ∞. The condition that C does not meet the codimension ≥ 2 faces of the cube at all implies that given any $x \in C^1$, at most of one of the g_i assumes a value of 0 or ∞ at x. In particular, for every $v \in \tilde{C}^1$, at most one of the $\tilde{g}_i$ assumes a value of 0 or ∞ at v.

For $x \in C^1$, define $j(x)$ to be the index j such that $g_j(x) \in \{0,\infty\}$ if such an index exists, or 1 if not such index exists. Likewise, for $v \in \tilde{C}^1$, define $\tilde{j}(v)$ to be the index $\tilde{j}$ such that $\tilde{g}_{\tilde{j}}(v) \in \{0,\infty\}$ if such an index exists, or 1 if no such index exists.

By definition of the boundary map d_{s+1}, we have:

$$d_{s+1}(C) = \sum_{x \in C^1} (-1)^{j(x)} \mathrm{ord}_x(g_{j(x)})(f(x) \times g_1(x) \times \dots \times \widehat{g_{j(x)}(x)} \times \dots \times g_{s+1}(x))$$

Now let D denote the proper smooth model for $k(C)$ over k. Since $\tilde{C}$ is normal, it may be identified with a subset of D. The functions $\tilde{g}_i : \tilde{C} \longrightarrow \mathbf{P}^1_k$ may thus be considered *rational* functions on D, which of course extend to morphisms which we denote (somewhat abusively) by $\tilde{g}_i : D \longrightarrow \mathbf{P}^1_k$. Furthermore, since X is projective, $\tilde{f} : \tilde{C} \longrightarrow X$ extends to a morphism $\tilde{f} : D \longrightarrow X$.

Lemma 6.6. *Suppose $v \in D - \tilde{C}$. Then there exists $i(v) \in \{1, \ldots, s+1\}$ such that $\tilde{g}_{i(v)}(v) = 1$.*

Proof.
Let K denote the function field $k(C) = k(\tilde{C}) = k(D)$. If $\tilde{g}_i(v) \neq 1$ for all $i = 1, \ldots, s+1$, then all the $\tilde{g}_i$ are regular at $v \in D$, and therefore (by completeness of X) we have a morphism Spec $O_v \xrightarrow{\phi_v} X \times_k \square_k^{s+1}$. Putting all of this data into one diagram, we have:

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & \tilde{C} \\ \downarrow & & \downarrow {\scriptstyle \tilde{f} \times \tilde{g}_1 \times \ldots \times \tilde{g}_{s+1}} \\ \text{Spec } O_v & \xrightarrow{\phi_v} & X \times_k \square_k^{s+1} \end{array}$$

Now the right vertical map is proper, so the valuative criterion for properness gives a (unique) map Spec $O_v \longrightarrow \tilde{C}$ making the resulting diagram commutative. This is a contradiction, because v was chosen to be in $D - \tilde{C}$.

Returning to our calculation, we note:

$$\alpha(d_{s+1}(C)) = \alpha(\sum_{x \in C^1} (-1)^{j(x)} \text{ord}_x(g_{j(x)})(f(x) \times g_1(x) \times \ldots \times \widehat{g_{j(x)}(x)} \times \ldots \times g_{s+1}(x))$$

$$= \sum_{x \in C^1} (-1)^{j(x)} \text{ord}_x(g_{j(x)}) \{[f(x)_{k(x)}], g_1(x), \ldots, \widehat{g_{j(x)}(x)}, \ldots, g_{s+1}(x))\}_{k(x)/k}$$

By [F], ex. 1.2.3, we have $\text{ord}_x(g_{j(x)}) = \sum_{v : \nu(v) = x} [k(v) : k(x)] \text{ord}_v(\tilde{g}_{\tilde{j}(v)})$. (Of course, $g_{j(x)} = \tilde{g}_{\tilde{j}(v)}$ as elements of $k(C) = k(\tilde{C})$) Thus, the above expression equals:

$$= \sum_{v \in \tilde{C}^1} (-1)^{\tilde{j}(v)} [k(v) : k(\nu(v))] \text{ord}_v(\tilde{g}_{\tilde{j}(v)}) \{[\tilde{f}(v)_{k(v)}], \tilde{g_1}(v), \ldots, \widehat{\tilde{g}_{\tilde{j}(v)}(v)}, \ldots, \tilde{g}_{s+1}(v))\}_{k(\nu(v))/k}$$

Using a relation of type **M1**, this may be identified with:

$$= \sum_{v \in \tilde{C}^1} (-1)^{\tilde{j}(v)} \text{ord}_v(\tilde{g}_{\tilde{j}(v)}) \{[\tilde{f}(v)_{k(v)}], \tilde{g_1}(v), \ldots, \widehat{\tilde{g}_{\tilde{j}(v)}(v)}, \ldots, \tilde{g}_{s+1}(v))\}_{k(v)/k}$$

By Lemma 6.6, this is the same as the above sum taken over *all* valuations of $k(C)$ fixing k:

$$= \sum_{v \in D} (-1)^{\tilde{j}(v)} \mathrm{ord}_v(\tilde{g}_{\tilde{j}(v)}) \{[\tilde{f}(v)_{k(v)}], \tilde{g_1}(v), \ldots, \widehat{\tilde{g}_{\tilde{j}(v)}(v)}, \ldots, \tilde{g}_{s+1}(v))\}_{k(v)/k}$$

Letting $\phi : \mathrm{Spec}\ k(C) \longrightarrow X$ denote the map naturally induced by f, we see at last that this is a relation $R_2(k(C); g_{s+1}; \phi_{k(C)}; g_1, \ldots, g_s)$ of the group $K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$. This concludes the proof that α is well-defined.

It remains to check that the compositions $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity maps on the appropriate groups. We begin with the former.

It suffices to verify the assertion on generators $[P]$ of $CH^{d+s}(X, s)$ corresponding to closed points $P : \mathrm{Spec}\ k(P) \longrightarrow X \times_k \square_k^s$. As in the proof of the well-definedness of α, we observe that P is determined by $y : \mathrm{Spec}\ k(P) \longrightarrow X$ and elements $a_1, \ldots, a_s \in k(P)^*$.

By the product structure on the cubical complex, we have:

$$[P_{k(P)}] = [y_{k(P)}] \cdot [a_1] \cdot \ldots \cdot [a_s]$$

viewing $[y_{k(P)}]$ as an element of $CH^d(X_{k(P)}, 0)$ and $[a_1], \ldots, [a_s]$ as elements of $CH^1(k(P), 1)$.
Then, computing directly from the definitions,

$$\beta(\alpha([P])) = \beta(\{[y_{k(P)}], a_1, \ldots, a_s\}_{k(P)/k})$$

$$= (\phi_{k(P)/k})_*([y_{k(P)}] \cdot [a_1] \cdot \ldots \cdot [a_s])$$

$$= (\phi_{k(P)/k})_*([P_{k(P)}])$$

$$= [P]$$

To show that $\alpha \circ \beta = id$, it suffices to show that α is surjective. To this end, fix a generator $\{[P], a_1, \ldots, a_s\}_{E/k} \in K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m)$, where $P \in X_E$ and $a_1, \ldots, a_s \in E^*$. Let $F = k(P)$, and $\phi_{F/E} : E \hookrightarrow F$ the inclusion map. Then $(\phi_{F/E})_*[P_F] = [P]$, so by a relation of type **M1**, we have $\{[P], a_1, \ldots, a_s\}_{E/k} = \{[P_F], \phi_{F/E}(a_1), \ldots, \phi_{F/E}(a_s)\}_{F/k}$. This calculation shows that we may assume without loss of generality that P is an E-rational point of X.

The data $P, a_1, \dots, a_s$ define a closed point x_E of $X_E \times_E \square_E^s$; consider its image x under the natural map $X_E \times_E \square_E^s \longrightarrow X \times_k \square_k^s$. Evidently, x is a closed point Spec $L \longrightarrow X \times_k \square_k^s$; let Q : Spec $L \longrightarrow X$ be defined by projection of x onto the first factor.

We argue now that the element $y = [P] \cdot a_1 \cdot \ldots \cdot a_s \in CH^{d+s}(X_E, s)$ has the same image under both compositions in the diagram:

$$\begin{array}{ccc} CH^{d+s}(X_E, s) & \xrightarrow{\alpha_E} & K_s(E; \mathcal{CH}_0(X_E); \mathbf{G}_m) \\ \Big\downarrow{\scriptstyle (\phi_{E/L})_*} & & \Big\downarrow{\scriptstyle N^{MM}_{E/L}} \\ CH^{d+s}(X_L, s) & \xrightarrow{\alpha_L} & K_s(L; \mathcal{CH}_0(X_L); \mathbf{G}_m) \end{array}$$

and that $N^{CH}_{E/L}(y) \in CH^{d+s}(X_L, s)$ has the same image under both compositions in the diagram:

$$\begin{array}{ccc} CH^{d+s}(X_L, s) & \xrightarrow{\alpha_L} & K_s(L; \mathcal{CH}_0(X_L); \mathbf{G}_m) \\ \Big\downarrow{\scriptstyle (\phi_{L/k})_*} & & \Big\downarrow{\scriptstyle N^{MM}_{L/k}} \\ CH^{d+s}(X, s) & \xrightarrow{\alpha} & K_s(k; \mathcal{CH}_0(X); \mathbf{G}_m) \end{array}$$

This will suffice to prove our assertion.

For the first diagram,

$$N^{MM}_{E/L}(\alpha_E([P] \cdot a_1 \cdot \ldots \cdot a_s))$$

$$= \{[P], a_1, \dots, a_s\}_{E/L}$$

Also,

$$\alpha_L((\phi_{E/L})_*([P] \cdot a_1 \cdot \ldots \cdot a_s)) = \alpha_L((\phi_{E/L})_*(\phi^*_{E/L}[Q_L] \cdot a_1 \cdot \ldots \cdot a_s))$$

By the projection formula for higher Chow groups,

$$= \alpha_L([Q_L] \cdot (\phi_{E/L})_*(a_1 \cdot \ldots \cdot a_s))$$

By Lemma 4.2,

$$= \alpha_L([Q_L] \cdot T_k(N^M_{E/L}\{a_1, \dots, a_s\}))$$

From the definition of α_L,

$$= j_L[Q](\gamma_L(N^M_{E/L}\{a_1, \dots, a_s\}))$$

By Lemma 3.5,

$$= N^{MM}_{E/L}(j_E[Q](\gamma_E\{a_1, \dots, a_s\}))$$

$$= N^{MM}_{E/L}(\{[Q_E], a_1, \dots, a_s\}_{E/E})$$

$$= \{[Q_E], a_1, \dots, a_s\}_{E/L} = \{[P], a_1, \dots, a_s\}_{E/L}$$

which completes the verification for the first diagram.

For the second diagram,

$$\alpha((\phi_{L/k})_*((\phi_{E/L})_*(y))) = \alpha((\phi_{L/k})_*([Q_L] \cdot (\phi_{E/L})_*(a_1 \cdot \dots \cdot a_s)))$$

$$= \alpha([Q] \cdot (\phi_{E/k})_*(a_1 \dots a_s))$$

$$= \alpha([Q] \cdot T_k(N^M_{E/k}(\{a_1, \dots, a_s\})))$$

$$N^{MM}_{L/k}(j_L[Q](\gamma_L(N^M_{E/L}(\{a_1, \dots, a_s\}))))$$

Also,

$$N^{MM}_{L/k}(\alpha_L((\phi_{E/L})_*(y))) = N^{MM}_{L/k}(\alpha_L([Q_L] \cdot N^{CH}_{E/L}(a_1, \dots, a_s)))$$

$$= N^{MM}_{L/k}(\alpha_L([Q_L] \cdot T_k(N^M_{E/L}\{a_1, \dots, a_s\})))$$

By the definition of α_L,

$$= N^{MM}_{L/k}(j_L(\gamma_L(N^M_{E/L}\{a_1, \dots, a_s\})))$$

By Lemma 3.5,

$$= N^{MM}_{L/k}(N^{MM}_{E/L}(j_E(\gamma_E\{a_1, \dots, a_s\})))$$

$$= N^{MM}_{E/k}(j_E(\gamma_E\{a_1, \dots, a_s\})) = \{[Q_E], a_1, \dots, a_s\}_{E/k}$$

7 Some calculations

In situations where we have some knowledge of the Milnor K-groups of the base field and its algebraic extensions, the combined strength of Theorems 5.2 and 5.5 sometimes enables us to carry out explicit computations, as in the following:

Corollary 7.1. *Let k be a finite field, and let X be a smooth quasiprojective variety defined over k. Let $d = \dim X$. Then for $s \geq 2$, $CH^{d+s}(X, s) = 0$.*

Proof.
By Theorem 5.5, there is an isomorphism $CH^{d+s}(X, s) \cong H^d_{Zar}(X, \mathcal{K}^M_{d+s})$, and by Theorem 5.2, the latter group is a quotient of $\bigoplus_{x \in X^d} K^M_s(k(x))$. Since each of the fields $k(x)$ is finite and $s \geq 2$, Steinberg's computations show that the terms $K^M_s(k(x))$ are zero, whence our result.

We remark that by taking direct limits, one can prove that Corollary 7.1 also holds when k is an arbitrary algebraic extension of a finite field.

Using the calculation of the Milnor K-theory of global fields from [BT], the same reasoning yields:

Corollary 7.2. *Let k be a global field (or an algebraic extension thereof) and X a smooth quasiprojective variety of dimension d defined over k. Let $s \geq 2$ be an integer. Then $CH^{d+s}(X, s)$ is a torsion group when $s = 2$, a 2-torsion group when $s \geq 3$ and char $k = 0$, and 0 when $s \geq 3$ and char $k > 0$.*

References

[A] R. Akhtar, *Milnor K-theory and zero-cycles on algebraic varieties*, Ph.D. Thesis, Brown University, 2000.

[BT] H. Bass and J. Tate, *The Milnor ring of a global field*, Lecture Notes in Math. **342** (1973), 349–446.

[Bl1] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. **61** (1986), 267–304.

[Bl2] S. Bloch, *The moving lemma for higher Chow groups*, J. Algebraic Geom. **3** (1994), no. 3, 537–568.

[F] W. Fulton, *Intersection Theory*, Springer-Verlag, 1984.

[Ga] O. Gabber, Letter to Bruno Kahn, February 1998.

[GL] T. Geisser and M. Levine, *The K-theory of fields in characteristic p*, Invent. Math. **139** (2000), 459-493.

[Gi] H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Adv. Math. **40** (1981), no. 3, 203–289.

[Ka1] K. Kato, *A generalization of local class field theory by using K-group II*, J. Fac. Sci. Univ. Tokyo, Sec. IA **27** (1983), 603–683.

[Ka2] K. Kato, *Milnor K-theory and the Chow group of zero cycles*, Contemp. Math. **55** (1986), 241–253.

[Ke] I. Kersten, *Brauergruppen von Köpern*, Vieweg, 1990.

[Mi] J. Milnor, *Algebraic K-theory and quadratic forms*, Invent. Math. **9** (1969-1970), 318–344.

[MSEV] S. Müller-Stach and P. Elbaz-Vincent, *Milnor K-theory, Higher Chow groups and Applications*, Invent. Math. **148** (2002), no. 1, 177–206.

[NS] Y. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring and Milnor's K-theory.* Izv. Math. **53** (1989), no. 1, 121–146.

[Qu] D. Quillen, *Higher Algebraic K-theory I* Lecture Notes in Math. **341** (1973), 85–147.

[RS] W. Raskind and M. Spiess: Milnor K-groups and zero-cycles on products of curves over p-adic fields. *Compositio Math.* textbf121 (2000), no.1, 1-33.

[Ro] M. Rost, *Chow groups with Coefficients*, Doc. Math. **1** (1996), no. 16, 319–393.

[Se] J-P. Serre, *Algebraic Groups and Class Fields*, Springer, 1997.

[So] M. Somekawa, *On Milnor K-groups attached to semiabelian varieties*, K-theory **4** (1990), 105–119.

[Su] A. A. Suslin, *Reciprocity laws and the stable rank of rings of polynomials*, Izv. Math. **43** (1979), no. 6, 1394–1429.

[To] B. Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-theory **6** (1992), 177-189.

Reza Akhtar
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056
reza@calico.mth.muohio.edu