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Milnor K-theory of rings, higher Chow groups and applications

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Abstract. If R is a smooth semi-local algebra of geometric type over an infinite field, we prove that the Milnor K-group $\mathrm{K}_n^M(R)$ surjects onto the higher Chow group $CH^n(R, n)$ for all $n \geqslant 0$. Our proof shows moreover that there is an algorithmic way to represent any admissible cycle in $CH^n(R, n)$ modulo equivalence as a linear combination of "symbolic elements" defined as graphs of units in R. As a byproduct we get a new and entirely geometric proof of results of Gabber, Kato and Rost, related to the Gersten resolution for the Milnor K-sheaf. Furthermore it is also shown that in the semi-local PID case we have, under some mild assumptions, an isomorphism. Some applications are also given.

Introduction

Our motivation for this work was the wish for a better understanding of the motivic cohomology of rings (i.e. affine schemes), where we take Bloch's higher Chow groups as definition for motivic cohomology. Motivic (co)homology turned out to be a new and important subject during the past years and several results are known in the case of fields. One of the most interesting theorems is the isomorphism of Nesterenko/Suslin [36] and Totaro [45] relating the Milnor K-groups $\mathrm{K}_n^M(F)$, of a field F, with the higher Chow groups $CH^n(F, n)$. As a consequence of this result we obtain a presentation of $CH^n(F, n)$ in terms of generators and relations. The following leads us to conjecture that the isomorphism of Nesterenko/Suslin/Totaro

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continues to hold if R is a smooth semi-local k-algebra of geometric type with infinite residue fields. Indeed, by [27] (Corollary 8.2, p. 171), if X is a smooth quasi-projective variety over a field, then

$$gr_\gamma^p \mathrm{K}_q(X) \otimes \mathbb{Z}\left[\frac{1}{(d+q-1)!}\right] \cong CH^p(X\,,\ q) \otimes \mathbb{Z}\left[\frac{1}{(d+q-1)!}\right],$$

where $d = dim(X)$. Moreover, combining the results above and those of [11] (Prop. 2.5) (see also [40]), we can see that if R is a smooth, H1 k-algebra of geometric type (see [21] for the definition of H1 rings), for instance a smooth semi-local k-algebra of geometric type with infinite residue fields, we have

$$\mathrm{K}_n^M(R) \otimes \mathbb{Z}\left[\frac{1}{(d+n-1)!}\right] \cong CH^n(R\,,\ n) \otimes \mathbb{Z}\left[\frac{1}{(d+n-1)!}\right],$$

with $d = dim(R)$, and where $\mathrm{K}_n^M(R)$ is defined by generators and relations [21].

The present article is organized as follows:

In the first section we recall basic definitions and results related to Milnor K-theory and higher Chow groups. In the second section we construct a graph map

$$\varphi_n : \mathrm{K}_n^M(R) \longrightarrow CH^n(R\,,\ n),$$

by associating to a symbol $\{a_1, \ldots, a_n\}$ in $\mathrm{K}_n^M(R)$ the graph of the n-tuple of rational functions $a_1, \ldots, a_n$ on $\mathrm{Spec}(R)$. We then show in Sect. 3

Theorem 3.4. *If R is a smooth (semi)local k-algebra of geometric type with k infinite, then the map $\varphi_n : \mathrm{K}_n^M(R) \longrightarrow CH^n(R\,,\ n)$ is surjective.*

This implies that one may apply "symbolic" computations to cycles in $CH^n(R\,,\ n)$. Our proof uses two essentially new methods: first we adapt a moving technique due to Bloch and Levine, see Lemma 3.11, to show that $CH^n(R\,,\ n)$ is generated by irreducible cycles that are smooth over k and finite, surjective over $\mathrm{Spec}(R)$ and have a specific "triangular" shape of equations. The second method is a perturbation technique, see Lemma 3.14, which guarantees that the polynomials involved in the equations have sufficiently generic coefficients. The rest of the proof proceeds along the lines of [45].

As a consequence of Theorem 3.4, we get a new, geometric and conceptually simpler proof of a combination of the following results of Gabber [16] (who proved the exactness at "$codim = 0$"), Kato [25] and Rost [38] (who showed the exactness at all places "$codim \geqslant 1$") in Sect. 4, namely

Proposition 4.3 (Gersten quasi-resolution for Milnor K-theory). *If R is a smooth (semi)local k-algebra of geometric type with k infinite, then the following sequence is universally exact:*

$$\mathrm{K}_n^M(R) \longrightarrow \oplus_{codim(x)=0} \mathrm{K}_n^M(k(x)) \longrightarrow \oplus_{codim(x)=1} \mathrm{K}_{n-1}^M(k(x)) \longrightarrow \cdots .$$

In Sect. 5, if R is in a particular class of rings (e.g. semi-local PID), we show that the morphism $\mathrm{K}_n^M(R) \longrightarrow \mathrm{K}_n^M(Frac(R))$ is injective. This gives partial answer to the following conjectures:

Conjecture 0.1 (Gersten resolution for Milnor K-theory). If R is a smooth noetherian semi-local k-algebra with infinite residue fields, then the following sequence is (universally) exact:

$$0 \longrightarrow \mathrm{K}_n^M(R) \longrightarrow \oplus_{codim(x)=0} \mathrm{K}_n^M(k(x)) \longrightarrow \oplus_{codim(x)=1} \mathrm{K}_{n-1}^M(k(x)) \longrightarrow \cdots .$$

Conjecture 0.2 (Bloch/Ogus formula for Milnor K-Theory). Let X/k be a smooth quasi-projective variety (over an infinite k). For all $n \geqslant 0$, there is a natural isomorphism

$$H^n(X, \mathcal{K}_n^M) \cong CH^n(X) .$$

Here $\mathcal{K}_n^M$ denotes the Zariski sheaf associated to Milnor K-theory.

Remark 0.3. 1. We know that particular cases of the last statement are true (see [25]).

2. If R is semi-local, essentially smooth over a field k, we know by [37] (Theorem 5.11, p. 133) or [6] (Théorème 4.1, p. 111), that the “Gersten conjecture” holds for K-theory. Combining this fact with the results of Soulé [40] (Théorème 5, p. 526) and Bloch [1] on “Gersten's conjecture” for higher Chow groups and the isomorphism of Levine for fields, we get some improvements on the torsion. Namely, if R is semi-local of geometric type and essentially smooth over a field k, we have

$$gr_\gamma^p \mathrm{K}_q(R) \otimes \mathbb{Z}\left[\frac{1}{(q-1)!}\right] \cong CH^p(R\,,\,q) \otimes \mathbb{Z}\left[\frac{1}{(q-1)!}\right] .$$

And moreover, if the residue fields are infinite, we have

$$\mathrm{K}_n^M(R) \otimes \mathbb{Z}\left[\frac{1}{(n-1)!}\right] \cong CH^n(R\,,\,n) \otimes \mathbb{Z}\left[\frac{1}{(n-1)!}\right] .$$

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Added for the revised version: Since the submission of this paper V. Voevodsky [46] (see [15] for a precursor of this result) has proved in all characteristics that higher Chow groups and two other versions of motivic cohomology are isomorphic. This result justifies our definition of higher Chow groups as motivic cohomology.

1. Definitions and basic constructions

Milnor K-theory.

Definition 1.1. [21] *Let R be a ring, we define $\mathrm{K}_*^M(R)$ as the graded ring generated by the symbols $\ell(a)$, $a \in R^\times$ in degree one and with the following relations:*

1. *For all a, b in $R^\times$, we have $\ell(ab) = \ell(a) + \ell(b)$*
2. *For all a, b in $R^\times$ such that $a + b = 0, 1$, we have $\ell(a)\ell(b) = 0$.*

It follows that the abelian group $\mathrm{K}_n^M(R)$ is generated by symbols $\ell(a_1) \cdots \ell(a_n)$, with $a_1, \dots, a_n \in R^\times$. We will denote the symbol $\ell(a_1) \cdots \ell(a_n)$ by $\{a_1, \dots, a_n\}$. We have a canonical surjective morphism

$$\begin{aligned} \ell : R^\times &\longrightarrow \mathrm{K}_1^M(R) \\ a &\mapsto \ell(a). \end{aligned}$$

In particular we have $\mathrm{K}_0^M(R) = \mathbb{Z}$ and $\mathrm{K}_1^M(R) = R^\times/[R^\times, R^\times]$. Moreover, we know that if R is a commutative ring with "enough" units, the Milnor K_2 is isomorphic to the Quillen K_2 (see [24] ((8.5) p. 512)). The following proposition is straightforward

Proposition 1.2. 1. *For all $a, b \in R^\times$, we have $\ell(a)\ell(b) + \ell(b)\ell(a) = 0$.*
2. *If $\xi \in \mathrm{K}_n^M(R)$ and $\eta \in \mathrm{K}_m^M(R)$ then $\xi\eta = (-1)^{nm}\eta\xi$.*
3. *If $\sigma \in \mathfrak{S}_n$, $\{a_{\sigma(1)}, \dots, a_{\sigma(n)}\} = \varepsilon(\sigma)\{a_1, \dots, a_n\}$.*
4. *For all $a \in R^\times$, we have $\ell(a)^2 = \ell(a)\ell(-1) = \ell(-1)\ell(a)$, and $\ell(-a^{-1})\ell(a) = 0$.*

Higher Chow groups. Higher Chow groups were invented by Spencer Bloch in [1]. We fix a ground field k. Define the "simplex" Δ^n by $\mathrm{Spec}\left(k[T_0, \dots, T_n]/\left(\sum T_i - 1\right)\right)$. By setting the coordinates $t_i = 0$, one obtains $(n+1)$ linear hypersurfaces in Δ^n called codimension one faces of the simplex Δ^n. By iterating this, one gets codimension $(n-m)$-faces isomorphic to Δ^m for every $m < n$ inside of Δ^n. The higher Chow groups of an equidimensional scheme X over k are defined as the homology groups of a chain complex. Let $Z^p(X, n) \subset Z^p(X \times \Delta^n)$ be the subset of cycles of codimension p such that every irreducible component of Z meets all faces $X \times \Delta^m$ again in codimension p for $m < n$. Let $\partial_i : Z^p(X, n) \longrightarrow Z^p(X, n-1)$ be the restriction map to the i-th codimension one face for $i = 0, \dots, n$ and let $\partial = \sum (-1)^i \partial_i$, then the homology of the complex

$$\cdots \longrightarrow Z^p(X, n+1) \xrightarrow{\partial} Z^p(X, n) \xrightarrow{\partial} Z^p(X, n-1) \longrightarrow \cdots$$

at the position n is denoted by $CH^p(X\,,\,n)$, see [1]. These groups have all the functorial properties of a Borel-Moore homology theory, and we will briefly discuss certain ones which are most relevant for us.

1. **Cubical versions of higher Chow complexes**: Here let $\square^n_k = (\mathbb{P}^1_k - \{1\})^n$ with coordinates t_i and codimension one faces obtained by setting $t_i = 0, \infty$. The rest of the definition is completely analogous except that one has to divide out degenerate cycles (see [45] (pp. 178-181) for more details). It is known that the quotient complex is quasiisomorphic to the simplicial one (see [28], [2] and [45]).
2. **Functoriality** [1]: The groups $CH^*(X, *)$ are covariant for proper maps and contravariant for flat maps. They are contravariant for all maps between smooth affine schemes by [29] (4.5.12.).
3. **Products** [1,45]: If X is smooth, there is a product
$$CH^p(X, q) \otimes CH^r(X, s) \longrightarrow CH^{p+r}(X, q+s)\,.$$
4. **Homotopy Invariance** [1]: For any equidimensional k-scheme X, we have
$$CH^*(X, n) \cong CH^*(X \times \mathbb{A}^1_k, n).$$
5. **Localization** [1,2]: If $W \subset X$ is a closed subvariety of pure codimension r with X quasi-projective over a field, then one has localization
$$\cdots \to CH^*(W, n) \to CH^{*+r}(X, n) \to CH^{*+r}(X - W, n) \to CH^*(W, n-1) \to \cdots$$

2. The graph map

From now on $Z^p(X, *)$ will exclusively denote the cubical version of the higher Chow complex. Let R be a k-algebra of finite type. Any element $f \in R$ gives a map $ev_f : k[T] \longrightarrow R$, $P \mapsto P(f)$. This defines a k-morphism of schemes $f : \mathrm{Spec}(R) \longrightarrow \mathbb{P}^1_k$, $x \mapsto f(x)$. The restriction of the graph of f, $graph(f) \cap \mathrm{Spec}(R) \times \square^1_k \subset \mathrm{Spec}(R) \times \square^1_k$, is an algebraic cycle in $Z^1(R, 1)$, since it is defined by the algebraic equation $\{y = f(x)\}$ inside $\mathrm{Spec}(R) \times \square^1_k$.

In general, we have a map $\varphi_n : (R^\times)^n \longrightarrow CH^n(R, n)$, defined as
$$(f_1, \ldots, f_n) \mapsto graph(f_1, \ldots, f_n) \cap \mathrm{Spec}(R) \times \square^n_k.$$

We will call the cycles in the image of φ_n "*graph cycles*". By a result of Spencer Bloch [1] (Theorem 6.1, p. 287), we know that the regularity of R implies that φ_1 is an isomorphism. We want to define a map at the level of $\mathrm{K}^M_n(R)$, thus we need to prove bilinearity and the "Steinberg relation".

Lemma 2.1. *We have the following relations:*

1. *If $f, 1-f, f_i \in R^\times$, then $\varphi_n(\{f, 1-f, f_3, \ldots, f_n\}) = 0$ in $CH^n(R, n)$.*
2. *If $f, g, f_i \in R^\times$, then $\varphi_n(\{fg, f_2, \ldots, f_n\}) = \varphi_n(\{f, f_2, \ldots, f_n\}) + \varphi_n(\{g, f_2, \ldots, f_n\}) \in CH^n(R, n)$.*

3. *The cycle* $\varphi_n(\{f, g, f_3, \ldots, f_n\}) + \varphi_n(\{g, f, f_3, \ldots, f_n\})$ *is 0 in* $CH^n(R, n)$*, for all* $f, g, f_i \in R^\times$.
4. *The cycle* $\varphi_n(\{f, -f, f_3, \ldots, f_n\})$ *is 0 in* $CH^n(R, n)$*, for all* $f, f_i \in R^\times$.

Proof. 1. As in the following proofs, the goal is to produce a cycle $W \in Z^n(R, n+1)$ whose boundary is a linear combination of the cycles involved in the statement. The irreducible cycle $W \in Z^n(R, n+1)$, given as a parametrized subvariety by the image of the morphism

$$\operatorname{Spec}(R) \times \square_k^1 \longrightarrow \operatorname{Spec}(R) \times \square_k^{n+1},$$
$$(p, x) \mapsto \left(p, x, 1-x, \frac{f(p)-x}{1-x}, f_3(p), \ldots, f_n(p)\right),$$

has only one boundary $\varphi_n(\{f, 1-f, f_3, \ldots, f_n\})$ which proves the assertion.

2. The boundary of the cycle $V \in Z^n(R, n+1)$ given by the parametrization

$$\operatorname{Spec}(R) \times \square_k^1 \longrightarrow \operatorname{Spec}(R) \times \square_k^{n+1},$$
$$(p, x) \mapsto \left(p, x, \frac{f(p)\,x - f(p)g(p)}{x - f(p)g(p)}, f_2(p), \ldots, f_n(p)\right),$$

is equal to $-\varphi_n(\{f, f_2, \ldots, f_n\}) - \varphi_n(\{g, f_2, \ldots, f_n\}) + \varphi_n(\{fg, f_2, \ldots, f_n\})$.

3. It is suffient to establish the result for $n = 2$ since the remaining coordinates do not contribute to the boundaries. We compute the boundary of the parametrized cycle

$$W: \quad (p, t) \mapsto \left(p, t + u(p), t + v(p), \frac{t + s(p)}{t + r(p)}\right) \in Z^2(R, 3)$$

with some given elements $u, v, s, r \in R^\times$ such that the cycle is admissible. Such parametrized cycles will for simplicity be written using the symbolic notation

$$\left[t + u, t + v, \frac{t+s}{t+r}\right].$$

Continuing with a similar notation for cycles in $Z^2(R, 2)$ and substituting $x = u - s$, $y = v - s$, $z = u - r$ and $w = v - r$, we get the following boundary of W:

$$[x, y] - [z, w] - \left[x - y, \frac{y}{w}\right] + \left[y - x, \frac{x}{z}\right] = 0,$$

again assuming that all cycles involved are admissible. Then we add this relation to the one that we obtain by permuting the roles of x and y, as well as the roles of w and z. Thus we obtain the relation

$$[x, y] + [y, x] = [w, z] + [z, w].$$

Specializing to $w = 1$, $x = f$ and $y = g$ (the cycle is still admissible), we get the desired relation.

4. Again, we give the proof for $n = 2$ only. Given $f \in R^\times$, we compute the sum of the boundaries of the following two cycles in $Z^2(R, 3)$:

$$\partial\left[t+f, -\frac{t}{f}, \frac{t-f}{t+1}\right] = -[f, -f] + [2f, -1] - \left[f-1, \frac{1}{f}\right],$$
$$\partial\left[t-1, \frac{1}{t}, \frac{f-t}{1-t}\right] = [-1, f] + \left[f-1, \frac{1}{f}\right].$$

Therefore $[f, -f] = [2f, -1] + [-1, f] \in Z^2(R, 2)$. Now use (2) and (3) with $g = -1$ to obtain $[f, -f] = [2f, -1] - [f, -1] = [2, -1] = 0$ by 1. □

This lemma provides us with a well defined map (even when R is not smooth or semi-local),

$$\varphi_n : \mathrm{K}_n^M(R) \longrightarrow CH^n(R, n), \quad n \geqslant 1.$$

Moreover we have the following property

Lemma 2.2. *If R is smooth over k, the maps φ_n are compatible with products.*

Proof. By definition, the products in cubical higher Chow groups are obtained by taking the cartesian product of two cycles and pulling back along the diagonal embedding (for the latter step a moving lemma is needed to make the pullback well-defined), see [28] or [45]. The maps φ_n are clearly compatible with Cartesian products. Even more, the pullbacks along the diagonal are always well defined without moving since graph cycles have no boundary. □

Proposition 2.3. *Let R be a smooth k-algebra. Then we have a well defined homomorphism of graded rings $\varphi_*^R : \mathrm{K}_*^M(R) \longrightarrow CH^*(R, *)$. Moreover, if R' is any k-algebra extension of R, then we have, for all $n \geqslant 0$, the following commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_n^M(R) & \xrightarrow{\iota_{Milnor}} & \mathrm{K}_n^M(R') \\ \varphi_n^R \downarrow & & \downarrow \varphi_n^{R'} \\ CH^n(R, n) & \xrightarrow[\iota_{Chow}]{} & CH^n(R', n). \end{array}$$

Here ι_{Milnor} and ι_{Chow} are the natural maps induced by the embedding $\iota : R \hookrightarrow R'$.

Proof. This is a consequence of a diagram chase using Lemma 2.1. □

Remark 2.4. 1. *Graph map and Chern character*. The reader should be warned that φ_n is not the composition of the maps

$$\mathrm{K}_n^M(R) \xrightarrow{\text{can}} \mathrm{K}_n(R) \xrightarrow{\widetilde{c}_{n,n}} CH^n(R\,,\,n),$$

where $\widetilde{c}_{n,n}$ is the pre-Chern class [1] (7.3.2, p. 293 and Lemma (9.2), p. 297) which induces the Chern class in the isomorphism of Levine [27] and 'can' is the canonical morphism induced by the products in Milnor and Quillen K-theories. Indeed, the morphism 'can' is known to behave badly. In particular, if R is a global field, Weibel[1] (see also Shapiro [39] (Proposition 2, p. 96), and Musikhin/Suslin [33] for related questions) has shown that 'can' is zero for $n \geqslant 5$, and thus cannot induce the isomorphism of Nesterenko/Suslin/Totaro. Nevertheless, up to torsion, both maps coincide.

2. The map φ_*^R corresponds to the "cycle map" cl_X^M of Levine, with $X = \mathrm{Spec}(R)$, as defined in [29] ((1.1.8.1), p. 298).

3. Surjectivity of the graph map for smooth semi-local algebras of geometric type

Definition 3.1. *A (semi)local ring R is said of geometric type if it is a localization of an algebra of finite type over a field.*

Remark 3.2. The previous definition is a generalisation of [43] (p. 168); it is equivalent to say that R is essentially of finite type over k. Therefore, smooth of geometric type is the same as essentially smooth in the terminology of some authors.

Proposition 3.3. *If R is any smooth k-algebra of geometric type, then φ_1 is an isomorphism. If moreover R is semi-local and each residue field contains at least 6 elements, then φ_2 is an isomorphism.*

Proof. If R is any smooth k-algebra of finite type, we have $\mathrm{K}_1^M(R) \cong CH^1(R\,,\,1)$ (cf. [1] (Theorem 6.1, p. 287)). Furthermore, by using [24], if R is a smooth semi-local k-algebra of finite type and each residue field contains at least 6 elements, then $\mathrm{K}_2^M(R) \cong CH^2(R\,,\,2)$. Notice that this statement uses the fact that we have Gersten's conjecture for $K_2(R)$ and $CH^2(R\,,\,2)$ and all the terms in the Gersten resolution for both groups coincide. Hence $K_2(R)$ and $CH^2(R\,,\,2)$ are isomorphic on a smooth semi-local k-algebra of finite type. Together with van der Kallen's result (assuming each residue field contains at least 6 elements) both groups are also isomorphic to $\mathrm{K}_2^M(R)$. □

[1] Unpublished work

We now state our main result:

Theorem 3.4. *If R is a smooth (semi)local k-algebra of geometric type, with k infinite, then the map $\varphi_n : \mathrm{K}_n^M(R) \longrightarrow CH^n(R\,,\,n)$ is surjective.*

Remark 3.5. 1. In the case where R is not smooth one can use the example of Srinivas [41] to show that Theorem 3.4 does not necessarily hold, even if R is a two-dimensional normal domain. In addition, examples of smooth, but not semi-local k-algebras of geometric type, hence with *"few units"*, are known where the map φ_n is not surjective even rationally.

2. As both functors (Milnor K-groups and higher Chow groups) commute with direct limits, we see that we can extend the previous result to the case of rings (eventually not quasi-projective) which are direct limits of smooth (semi)local k-algebras of geometric type (assuming k infinite).
3. We notice that, as R is smooth, it is always UFD. Indeed, if R is local, then, by the well known result of Auslander/Buchsbaum [31] (Theorem 20.3, p. 163), the smoothness implies that R is UFD. We deduce the general semi-local case by localization using the Corollary 9.4 of [13] (p. 40).
4. The Theorem 3.4 is also true with coefficients. Indeed, we can consider higher Chow groups with coefficients $\mathbb{Z}/m\mathbb{Z}$, for an integer m, by tensoring the complex with $\mathbb{Z}/m\mathbb{Z}$, and denoting its homology by $CH^p(R\,,\,n\,;\,\mathbb{Z}/m\mathbb{Z})$. As R is smooth semi-local, $CH^n(R, n-1) = 0$ and hence, by the Bockstein sequence, we have $CH^n(R\,,\,n) \otimes \mathbb{Z}/m\mathbb{Z} = CH^n(R\,,\,n\,;\,\mathbb{Z}/m\mathbb{Z})$. Thus the induced map is still surjective.

Proof. Part of our proof is based upon the one of Burt Totaro for fields [45]. However there are specific ring theoretic difficulties which arise from general position problems. The cycles are first brought into a position where they are smooth over k, finite and surjective over $\mathrm{Spec}(R)$. Then we generalize the "Norm-Algorithm" of Totaro, while we take care that the cycles stay in such a good position that we can use symbolic arguments on the defining equations and all occuring boundaries are well defined. For simplicity we will assume that R is local, and when we speak about the closed fiber, it will be with respect to its maximal ideal. In general, in the following arguments, the maximal ideal sometimes has to be replaced by the Jacobson radical of R or taken as one respectively all of the maximal ideals of R (and in this case we will have several closed fibers). More precisely, the plan of the proof is as follows:

We first show that every cycle in $Z^n(R, n)$ is equivalent to a cycle for which each irreducible component is smooth over k, finite, surjective over $\mathrm{Spec}(R)$ and its ideal can be described precisely by a complete intersection of polynomials of a distinguished type. This is done in Lemma 3.11. Then using an explicit relative curve C over $\mathrm{Spec}(R)$, we will show in Lemma 3.18 that we can move, step by step, an arbitrary cycle in $Z^n(R, n)$ into a sum of graph-cycles. This curve is only well-defined in the corresponding higher Chow group, if Z is in even better position, a property which is achieved by three additional moving procedures in each level of induction. A central role in these movings is played by the technique of Lemma 3.14. So let us introduce several intermediary groups which will help to do the first moving.

We will implicitly use the two following algebraic lemmas:

Lemma 3.6. *Let A and B two integral domains. Then* $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ *is dominant if and only if* $A \hookrightarrow B$.

Lemma 3.7. *Let $i : R \hookrightarrow S$ be a local morphism of integral domains. Then the following statements are equivalent:*

1. $\mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R)$ *is finite and dominant.*
2. *S is finite as an R-module.*
3. *$S/\mathfrak{m}S$ is a finite $R/\mathfrak{m}$-vector space.*

Remark 3.8. For Lemma 3.6 see Exercise 2.18 p. 81 of [22], and for Lemma 3.7 see Exercise 3.7 p. 91 of [22].

Let $\mathfrak{m} \subset R$ be the maximal ideal (resp. the Jacobson radical in the semi-local case) and $i : \mathrm{Spec}(R/\mathfrak{m}) \to \mathrm{Spec}(R)$ be the inclusion of the closed point into $\mathrm{Spec}(R)$. As in [29] (Sect. 3.5, p. 94), we consider the subcomplex

$$Z^n(R, *)_i \subset Z^n(R, *),$$

of cycles, such that each irreducible component intersects the closed fiber properly. Notice that this embedding (of complexes) is natural and subsequently the homology groups $CH^n(R\,,\,n)_i = H_*(Z^n(R, *)_i)$ inherit the same functorial properties as the higher Chow groups.

Definition 3.9. *In homological degree n, we consider the subgroup*

$$Z^n_{sfs}(R, n) \subset Z^n(R, n)_i$$

of those cycles Z, for which the following properties hold:

- *Every irreducible component of Z is smooth over k and finite, surjective over* $\mathrm{Spec}(R)$ *and contained in $Z^n(R, n)_i$.*
- *For every smooth, irreducible component W of Z, the coordinate ring $\Gamma(W, \mathcal{O}_W)$ is isomorphic to $R[x_1, ..., x_n] = R[t_1, \ldots, t_n]/I(W)$ for some ideal $I(W)$ and does not meet any faces $t_i = 0, \infty$ at all. We have a nested sequence of integral extensions*

$$R \subset R[x_1] \subset R[x_1, x_2] \subset \cdots \subset R[x_1, \ldots, x_n]\,, \tag{1}$$

such that all the algebras are smooth over k and each x_k has a monic minimal polynomial $p_k \in R[x_1, ..., x_{k-1}][t]$. Moreover the ideal $I(W)$ is a complete intersection ideal and has the monic polynomials as generators for $1 \leqslant k \leqslant n$.

It is important to notice that the closure of such cycles in $\mathrm{Spec}(R) \times (\mathbb{P}^1)^n$ become also smooth (over k), finite and surjective over $\mathrm{Spec}(R)$ and have empty boundary (see Fact 3.21 below), i.e. they do not meet the faces at all. This fact enables us to work with integral extensions of rings. Note also that we do not form a subcomplex out of these cycles, since this definition only makes sense in the Milnor range of indices. Nevertheless we define the group $CH^n_{sfs}(R, n)$ as the quotient of $Z^n_{sfs}(R, n)$ by $\mathrm{Im}(\partial_{n+1}) \cap Z^n_{sfs}(R, n)$. Again this makes sense because every element of $Z^n_{sfs}(R, n)$ lies in $\mathrm{Ker}(\partial_n)$, since they have no boundary. Notice that the group $Z^n_{sfs}(R, n)$ is non-zero since it contains at least the graph cycles. We get a sequence of natural maps $CH^n_{sfs}(R, n) \to CH^n(R, n)_i \to CH^n(R, n)$.

By base change, we will have the following property:

Lemma 3.10. *The correspondence $R \mapsto CH^n_{sfs}(R, n)$ is a well defined (covariant) functor on the category of smooth algebras of geometric type over k. Furthermore this functor commutes with (direct) limits.*

Our proof now relies on the following key lemma[2] which is a refinement of the methods of M. Levine developed in [29, chap. II, Sect. 3.5]:

Lemma 3.11. (Marc Levine)
If R is smooth over an arbitrary (not necessarily infinite) field k, then both natural maps $CH^n_{sfs}(R, n) \to CH^n(R, n)_i$ and $CH^n(R, n)_i \to CH^n(R, n)$ are isomorphisms.

Proof. Suppose first that k is perfect. If k is finite, then the "pro-l-extension trick" [29] (3.5.14.1, p. 102) can be used to reduce the proof to the case of infinite perfect fields, since such extensions are again perfect. As a result, we can now assume that k is both perfect and infinite. By Levine [29] (Theorem 3.5.14), as R is smooth, the subcomplex

$$Z^n(R, *)_i \subset Z^n(R, *)$$

of cycles such that i^* is well defined on the level of cycles induces a quasi-isomorphism. As a consequence, there is a well defined map

$$i^* : CH^n(R, *) \to CH^n(R/\mathfrak{m}, *)$$

induced by the map $Z^n(R, *)_i \to Z^n(R/\mathfrak{m}, *)$.

Now we follow the steps of Sect. 3.5 in [29] (p. 94) to see that the proof of Levine in fact shows that every integral element in $Z^n(R, n)$ is equivalent in $CH^n(R, n)$ to a linear combination of elements in $Z^n_{sfs}(R, *)$.

For the following "geometric projection" argument, we replace $\mathrm{Spec}(R)$ momentarily by a d-dimensional affine variety X of finite type over k, such that $\mathrm{Spec}(R)$ arises as localization of X and is assumed to be embedded into some fixed affine space $\mathbb{A}^N_k$. Later the notation X will be replaced by R again.

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[2] we are grateful to Marc Levine for his help in the proof of this lemma

First denote by $\pi_{L,X} : X \to \mathbb{A}_k^d$ a projection given by a linear subspace $L \subset \mathbb{P}_k^N$ of dimension $N - d - 1$ with $N \geqslant d$. The set of all such L is parametrized by some Graßmann variety. If W is an integral generator of $Z^n(X, *)$, then

$$(\pi_{L,X} \times id)^*(\pi_{L,X} \times id)_*(W) = W + W',$$

where W' is the residual cycle of this cone construction. We use the notation

$$Z_m^n(R, *)_i = Z_{\mathcal{C},m}^n(R, *)$$

from [29] (Lemma 3.5.2, p. 94), denoting the subgroups generated by those integral cycles which meet the closed fiber in codimension $\geqslant m$, for $\mathcal{C} = \mathrm{Spec}(R/\mathfrak{m})$ and for positive integers $m \leqslant n$. Clearly

$$Z_{sfs}^n(R, n) \subset Z_n^n(R, n)_i.$$

The Lemma 3.5.6 in [29] (p. 97) shows that W' is contained in $Z_m^n(R, *)_i$ whenever W is contained in $Z_{m-1}^n(R, *)_i$ and if L is generic in the sense that it is contained in an open subset of the Graßmann variety, denoted by $\mathcal{U}_X$ in [29]. Note that here we use that k is infinite.

Going beyond the arguments in [29] now, we first note that we can reimbed X via a Veronese map into some bigger affine space $\mathbb{A}_k^M$, such that the codimension of X becomes arbitrarily large. In particular any linear projection $p : \mathbb{A}_k^M \to \mathbb{A}_k^d$ will have the following two properties for any W as above:

- The reduced image of W under $\pi_{L,X} \times id$ is generically smooth over k.
- The residual part W' is smooth over k.

These two properties hold after such a reimbedding since the secant varieties are then in general position: the first property is always satisfied even without reimbedding for a general projection, since this property holds for X itself and is therefore inherited by any of its generically smooth subvarieties. The second property follows from the following fact after reimbedding: for every pair a, b of points in X, there is a general projection given by a subspace L, such that one of its linear fibers contains a, b and the closed point of X, but which is still finite when restricted to X. A classical case of these arguments is contained in [22, ch. IV, Cor. 3.6, p. 310]. More generally these methods are described in [14, chap. 1]. A fortiori we will work with an ambient space with such a property.

Our first observation is that under such circumstances, if W is in $Z_{m-1}^n(R, n)_i$, then W' is in fact contained in $Z_{sfs}^n(R, n)$, if L is generic (maybe one has to choose an even smaller open subset). The reason is that, since we are working on local affine schemes over a perfect field, W' will be smooth over k and the projection $\pi_{L,X} : W' \to \mathbb{A}_k^d$ will be finite and surjective over $\mathrm{Spec}(R)$ along all components of W', if L is generic (see [29] (Sect. 3.5.3, p. 95)). This is true in any characteristic using the fact that k is perfect.

Our second observation is that the method of "generic translations" in Lemma 3.5.11 of [29] (p. 101) provides good properties for the rest of the cycle: the method consists of applying a generic translation to the cycle $(\pi_{L,X} \times id)_*(W)$ and then taking a pullback again. By Lemma 3.5.11 of [29], if we have enlarged the ambient space as described above, any cycle $V \in Z^n_m(R,n)$ will have the property that $i_1^* \circ T^*_{\phi_s} \circ p_F^*(V)$ (with the notations of [29]) is indeed contained in $Z^n_{s\,fs}(R_F,n)$ for some transcendental field extension F/k. The reason is that for every general linear projection $p : \mathbb{A}^M_k \to \mathbb{A}^d_k$ $(d = \dim(X))$, and any subvariety $W \subset X \times \mathbb{A}^n_k$, the generic translation of the projection of W onto $\mathbb{A}^d_k \times \mathbb{A}^n_k$ is finite and unramified and its image is smooth over k.

In [29] (3.5.12, p. 101) a homotopy is constructed, showing that the maps $\pi^*_{t,X} \circ i_1^* \circ T^*_{\phi_s} \circ \pi_{t,X*} \circ p_F^*$ and $\pi^*_{t,X} \circ \pi_{t,X*} \circ p_F^*$ are homotopic (via an intermediate field extension $k \subset K \subset F$) as maps

$$Z^n_{m-1}(R,*) \longrightarrow Z^n_{m-1}(R_F,*),$$

and factor through $Z^n_{sfs}(R_F,n)$. Hence we conclude, as in the proof of Lemma 3.5.13 of [29] (p. 102), that this construction gives a homotopically trivial map

$$P^* : \frac{Z^n_{m-1}(R,*)_i}{Z^n_m(R,*)_i} \to \frac{Z^n_{m-1}(R_F,*)_i}{Z^n_m(R_F,*)_i}$$

of complexes, induced by the base extension from k to F, with the additional property that in homological degree n, every integral generator of $Z^n_{m-1}(R,n)_i$ is mapped to $Z^n_{sfs}(R_F,n) = p_F^* Z^n_{sfs}(R,n)$. The specialization argument in [29] (p. 103) shows that P^* is injective. It follows by induction over m that every cycle W in $Z^n(R,n)$ with $\partial(W) = 0$ is equivalent to a cycle in $Z^n_{sfs}(R,n) \subset Z^n_n(R,n)_i$ and that all three groups are isomorphic. In general, we can reduce to the case where the base field is perfect by Quillen's method [37, proof of Theorem 5.11, p. 133], since all the above functors commute with direct limits. For the reader's convenience, we briefly recall how the trick works: there exists a subfield k' of k which is finitely generated over $\mathbb{F}_p$ and a semi-localization R' of a smooth k'-algebra of finite type such that $R = R' \otimes_{k'} k$. By running through the finitely generated subfields of k containing k' and using the fact that both functors commute with direct limit, we can assume that k is finitely generated over $\mathbb{F}_p$. Hence, as R is a localization of a smooth k-algebra of finite type, and k is finitely generated over $\mathbb{F}_p$, we get that R is a localization of a finite type smooth $\mathbb{F}_p$-algebra. Subsequently, we can assume $k = \mathbb{F}_p$. □

We can reformulate the previous lemma as follows.

Corollary 3.12. *Every cycle in* $Z^n(R,n)$ *is equivalent to a cycle in* $Z^n_{sfs}(R,n)$*, in particular every irreducible component is smooth over* k

and finite, surjective over Spec(R). *Moreover all the integral, separable extensions*

$$R \subset R[x_1] \subset R[x_1, x_2] \subset \cdots \subset R[x_1, \ldots, x_n] \tag{2}$$

are smooth over k.

In the course of our proof however a weaker complete intersection property will persist which we describe now:

Definition 3.13. *Let S_ℓ be the set of elements $Z \in Z^n(R, n)_i$ such that for every irreducible component W of Z, the coordinate ring $\Gamma(W, \mathcal{O}_W)$ is isomorphic to $R[x_1, \ldots, x_\ell] = R[t_1, ..., t_n]/I(W)$ for some ℓ and such that the variety W has the following set of equations:*

$$\begin{cases} p_1(t_1) = 0, \\ p_2(t_1, t_2) = 0, \\ \vdots \\ p_\ell(t_1, \ldots, t_\ell) = 0, \\ p_{\ell+1}(t_1, \ldots, t_\ell) - t_{\ell+1} = 0, \\ \vdots \\ p_n(t_1, \ldots, t_\ell) - t_n = 0. \end{cases}$$

We furthermore require the following properties:

1. *The first k equations $p_1, \ldots, p_k$ define a smooth complete intersection variety for all $k \leqslant \ell - 1$, and we have a sequence of integral extensions*

$$R \subset R[x_1] \subset R[x_1, x_2] \subset \cdots \subset R[x_1, \ldots, x_\ell].$$

2. *The coordinate x_ℓ is integral over $R[x_1, \ldots, x_{\ell-1}]$ and it has a monic polynomial, say $p_\ell \in R[x_1, \ldots, x_{\ell-1}][T]$, satisfying $p_\ell(x_1, \ldots, x_\ell) = 0$.*
3. *All algebras in this chain are integral domains, the first $\ell - 1$ are smooth and p_ℓ is the generator of the principal ideal of $I(W)$ in $R[x_1, \ldots, x_{\ell-1}][t_\ell]$ over $R[x_1, \ldots, x_{\ell-1}]$.*

We furthermore define $S_{\ell,d}$ to be the subset of S_ℓ consisting of those cycles such that the polynomials p_k have t_ℓ-degree $\leqslant d$ for all $\ell \leqslant k \leqslant n$ and the degree of each irreducible factor of the polynomials p_k for $\ell + 1 \leqslant k \leqslant n$, viewed as polynomial in $R[x_1, \ldots, x_{\ell-1}][t_\ell]$, is strictly smaller than d.

Note that every semi-local regular k-algebra is a UFD (see 3.5 (3)), so that the last property makes sense. It follows that S_0 is just the set of graph cycles, and S_n is a set of complete intersection cycles such that every irreducible component is a divisor in a smooth variety, finite, surjective over Spec(R) and does not meet any faces $t_i = 0, \infty$ at all. We also observe that $S_{\ell,d}$, with $\ell > 0$ and $d \leqslant 1$, can be identified with $S_{\ell-1}$. Notice that the definitions of S_ℓ and $S_{\ell,d}$ depend on R, and by enlarging R we will, in general, decrease the indices ℓ and d.

In order to manipulate such sets of equations easily, we want to have nicer properties for the polynomials p_k if $k > \ell$. In particular, if we want to get integrality conditions or to apply the division algorithm, it is desirable to have an invertible leading coefficient. The following result will show that we can always assume such a property.

Lemma 3.14. *Assume that R is a semi-local UFD with infinite residue fields. Let W be an irreducible component of a cycle in $S_{\ell,d}$, and let Λ be the finite subset of $\{\ell+1,\ldots,n\}$ containing all the indices i for which p_i (or one of its irreducible factors) is not monic in t_ℓ. Then there exists a family of reducible polynomials $\tilde{p}_i \in R[x_1,\ldots,x_{\ell-1}][T]$ for $i \in \Lambda$ of degree $\leqslant d$ such that each irreducible factor has degree $< d$, an invertible leading coefficient and satisfies $\tilde{p}_i(x_\ell) = x_i$. A fortiori we can replace p_i by $\tilde{p}_i$ for $i \in \Lambda$ in the presentation of $I(W)$.*

Before the proof of the lemma, we will show a key property, that we will use several times in the remaining parts of this paper.

Property 3.15. *Let A be a semi-local domain with infinite residue fields, and B a semi-local domain over A (assuming $A \hookrightarrow B$) and P_λ a finite family of polynomials of $B[T]$ such that the coefficients of each polynomial generate B (i.e. they are primitive). Suppose moreover that B is integral over A. Then there exists an infinite number of $a \in A^\times$ such that $P_\lambda(a) \in B^\times$ for all λ.*

Proof. Denote by $\mathfrak{m}_i$ (resp. $\mathfrak{n}_i$) the maximal ideals of A (resp. B), and by k_i the residue fields $A/\mathfrak{m}_i$. As B is integral over A, we get all the maximal ideals of A by intersecting those of B with it. Denote by Λ_i the set of indexes j, such that $B/\mathfrak{n}_j$ contains k_i for all $j \in \Lambda_i$. Hence, by the observation above, the set Λ_i is never empty. Denote by $\widetilde{P}_{\lambda,j}$ the reduction of P_λ modulo $\mathfrak{n}_j$. Thus, as the coefficients of P_λ generate the unit ideal of B, the $\widetilde{P}_{\lambda,j}$ are non-zero polynomials over $B/\mathfrak{n}_j$. Let U_i be the set of elements $\alpha \in {k_i}^\times$ such that $\widetilde{P}_{\lambda,j}(\alpha)$ is a unit in $B/\mathfrak{n}_j$ for all j. Since k_i is infinite, and the set of zeros of the $\widetilde{P}_{\lambda,j}$ is finite, the set U_i is infinite. Hence, by the general form of the Chinese Remainder Theorem, for each family (α_i), with $\alpha_i \in U_i$, using the coprimalities of the $\mathfrak{m}_i$, we can find $a \in A$ such that a is invertible modulo $\cap_i \mathfrak{m}_i$ and $a = \alpha_i$ modulo $\mathfrak{m}_i$. But $Rad(A) = \cap_i \mathfrak{m}_i$, meaning that a is invertible in A. As we have an infinite number of such families, we deduce that we have an infinite number of such invertible elements. Using the same arguments for B, we get that $P_\lambda(a)$ is invertible in B for all λ. □

Remark 3.16. In fact, using the notion of full ring in the sense of McDonald [32] (Sect. II, pp. 871–874), we can have a finer statement (depending of the degrees of the polynomials). For simplicity, assume that $B = A$. Let d be the maximum degree of the family P_λ. Then we can see all the P_λ as vectors in A^{d+1} inserting the constant term into the first component. If the coefficients of P_λ generate A, the associated vector of A^{d+1} is unimodular. Let M be the $r \times (d+1)$ matrix whose rows are the polynomials P_λ (without a special order). As the residue fields of A are infinite, A is a (m_1, m_2)-full ring in the sense of McDonald (op. cit.) for all $m_1 \geqslant 1$ and $m_2 \geqslant 2$. We deduce that there exist $a \in A$ and $v_1, \dots, v_r \in A^\times$, such that

$$M(1, a, \dots, a^d)^t = (v_1, \dots, v_r)^t,$$

which gives the desired property.

Proof of Lemma 3.14. Since the polynomials p_i for $i \in \Lambda$ may have degree equal to d we may first apply the division algorithm relative to p_ℓ and replace p_i for $i \in \Lambda$ by some polynomials p_i' of t_ℓ-degree $< d$. This does not affect the ideal $I(W)$. Then we can use the following argument: we apply 3.15, with $B = A = R[x_1, \dots, x_{\ell-1}]$ and take the family of polynomials $(p_i')_{i\in\Lambda}$ together with p_ℓ as P_λ. Then there exists $a \in R[x_1, \dots, x_{\ell-1}]$, such that $p_i'(a) \in R[x_1, \dots, x_{\ell-1}]^\times$ for all $i \in \Lambda \cup \{\ell\}$. We define $\tilde{p}_i$ as follows:

$$\tilde{p}_i(T) = -\frac{p_i'(a)}{p_\ell(a)} p_\ell(T) + p_i'(T).$$

Then, by the degree property of the p_i', $\deg(\tilde{p}_i) \leqslant d$ and for all $i \in \Lambda$, we have $\tilde{p}_i(x_\ell) = x_i$, the leading coefficient is invertible and $T - a$ divides $\tilde{p}_i$. In particular, every irreducible factor of $\tilde{p}_i$ for $i \in \Lambda$ is of degree $< d$ and hence the equations define again the same cycle W in $S_{\ell,d}$. □

Now, using the previous settings, we will show that we may assume that the cycles we start with are in S_n:

Lemma 3.17. *The group $Z^n_{sfs}(R, n)$ is contained in S_n.*

Proof. The fact that the rings are defined by adjoining a single equation over each other is a result of the smoothness assertion in 3.12, since any divisor on a smooth variety is Cartier. The degree properties are a consequence of [9] (Prop. 4.1, p. 117) and the division algorithm [5] (ch. IV, 1.6). As the x_i, for $i = \ell + 1, \dots, n$, are elements of $\Gamma(Z, \mathcal{O}_Z)$, we can find polynomials $p_i \in R[t_1, \dots, t_\ell]$ such that

$$p_i(x_1, \dots, x_\ell) = x_i\,.$$

Again, by the division algorithm relatively to p_ℓ, we may assume that these polynomials have the desired degree property. □

Now we will show - via an induction process over ℓ - that we can reduce an arbitrary cycle in S_n modulo equivalence in $CH^n(R, n)$ and under mild assumptions on R to a sum of graph cycles. The following “Crucial Lemma” shows that the induction process over ℓ is well-defined:

Lemma 3.18 (Crucial lemma). *Assume that R is a semi-local UFD with infinite residue fields. If $Z \in S_{\ell,d}$, then there exist cycles $Z' \in S_{\ell-1}$, $Z'' \in S_{\ell,d-1}$ and $C \in Z^n(R, n+1)$ such that*

$$Z = Z' + Z'' + \partial C.$$

Proof. Let Z be an irreducible cycle in $S_{\ell,d}$. Then $\Gamma(Z, \mathcal{O}_Z) = R[t_1, \ldots, t_\ell]/I$ for a certain ideal I. Let d be the t_ℓ-degree of the p_ℓ. If $d = 1$, then Z is already in $S_{\ell-1}$ and we are done. So we assume $d \geqslant 2$. As R fulfills the hypothesis of Lemma 3.14, we can assume that all the polynomials p_k for $k > \ell$ have an invertible leading coefficient in $R[x_1, \ldots, x_{\ell-1}]$. Now we want to use the p_i to construct C and therefore Z' and Z''.

The cycle $C \in Z^n(R, n+1)$, which we introduce next, is a deformation of the original cycle Z into a sum of other cycles which are in better position. Let $(-1)^d a_0(t_1, \ldots, t_{\ell-1})$ be the constant term of $p_\ell(t_1, \ldots, t_\ell)$ viewed as polynomial in t_ℓ. Define

$$q(t_1, \ldots, t_\ell, u) = p_\ell(t_1, \ldots, t_\ell) - (t_\ell - 1)^{d-1}(t_\ell - a_0(t_1, \ldots, t_{\ell-1}))u\,.$$

Then consider the relative curve C in $\mathrm{Spec}(R) \times \square_k^{n+1}$ defined by the following equations:

$$\begin{cases} p_1(t_1) = 0, \\ \vdots \\ p_{\ell-1}(t_1, \ldots, t_{\ell-1}) = 0, \\ q(t_1, \ldots, t_\ell, u) = 0, \\ p_{\ell+1}(t_1, \ldots, t_\ell) - t_{\ell+1} = 0, \\ \vdots \\ p_n(t_1, \ldots, t_\ell) - t_n = 0. \end{cases}$$

Unfortunately, C is in general not in good position, i.e. does not intersect the faces properly. This is mainly due to the fact that we have to deal with non-invertible elements in the ring-theoretical context. We introduce therefore a convenient definition:

Definition 3.19. *We will say that C is in* good position *if the following properties hold:*

1. *Every component of the intersection of C with every codimension 1 face is finite and surjective over* $\mathrm{Spec}(R)$.
2. *The curve C meets no codimension 2 face: this case falls into two subcases:*
   (a) *The curve C does not intersect the codimension two faces $t_i = t_j = 0$ for some $i \neq j \in \{\ell+1, \ldots, n\}$, meaning that $p_i(x_1, \ldots, x_{\ell-1}, t_\ell)$ and $p_j(x_1, \ldots, x_{\ell-1}, t_\ell)$ are coprime in $R[x_1, \ldots, x_{\ell-1}][t_\ell]$.*

(b) *The curve C does not intersect the faces $u = \infty$ and $t_i = 0$ simultaneously for some $i \in \{\ell + 1, \dots, n\}$, i.e. $p_i(x_1, \dots, x_{\ell-1}, a_0(x_1, \dots, x_{\ell-1}))$ is a unit in $R[x_1, \dots, x_{\ell-1}]$.*

3. *Invertibility conditions: the leading terms of each irreducible factor of all polynomials p_i are invertible in $R[x_1, \dots, x_{\ell-1}]$ for $i \in \{\ell + 1, \dots, n\}$. Furthermore $t_\ell - 1$, and $t_\ell - a_0(x_1, \dots, x_{\ell-1})$, are invertible in $R[x_1, \dots, x_{\ell-1}][t_\ell]/(p_i(x_1, \dots, x_{\ell-1}, t_\ell))$ for all $i \in \{\ell + 1, \dots, n\}$.*

However, if C is in good position, then our induction procedure works:

Sub-Lemma 3.20. *Suppose that C is in good position. Then the curve C intersects the codimension 1 faces as follows:*

(a) *The intersections of C with the faces $t_i = 0$ for $1 \leqslant i \leqslant \ell$ or $t_i = \infty$ for $1 \leqslant i \leqslant n$ are empty.*
(b) *The intersection of C with $u = 0$ is the given Z, and with $u = \infty$ is some cycle in $S_{\ell-1}$.*
(c) *The intersections of C with the faces $t_i = 0$ for $\ell + 1 \leqslant i \leqslant n$ give cycles in $S_{\ell,d-1}$.*

Proof. For the proof, we will often use the following fact:

Fact 3.21. *Let $W \in Z^n(R, n)$ be a cycle, finite and surjective over* Spec(R). *Let F be a face. Then if W does not intersect F on the closed fiber, i.e. the restriction to the closed fiber of W in* $\mathrm{Spec}(R) \times \square^n_k$ *is also well defined, then W does not intersect F over* Spec(R) *at all.*

Now we can deal with the different cases:

(a) First, consider the intersection with the faces $t_i = 0$ (resp. $t_i = \infty$) for $1 \leqslant i < \ell$. By noticing that the projection of C on the first $\ell - 1$ coordinates is the same as the projection of Z on the first $\ell - 1$ coordinates, and as Z has no boundary, we deduce that C intersected with the faces $t_i = 0$ (resp. $t_i = \infty$) for $1 \leqslant i < \ell$ is empty. We deduce the emptiness of the faces $t_i = \infty$ for $1 \leqslant i < \ell$, by observing that all the p_i are monic. For $t_\ell = 0$, we get the equation

$$a_0(t_1, \dots, t_{\ell-1})(1 - u) = 0.$$

But by Assumption 3.19(3), the expression $a_0(x_1, \dots, x_{\ell-1})$ is nonzero since p_ℓ is irreducible over $R[x_1, \dots, x_{\ell-1}]$, which leads, as we are working with domains, to the unique solution $u = 1$ giving an empty intersection. The intersection of C with the face $t_\ell = \infty$ gives the solution $u = 1$ and thus an empty intersection. Finally, if we intersect C with a face $t_i = \infty$ for $i \in \{\ell + 1, \dots, n\}$, this leads to the equation

$$p_i(t_1, \dots, t_\ell) = \infty,$$

which implies that some $t_k = \infty$, with $1 \leqslant k \leqslant \ell$, producing also an empty intersection.

(b) By the very definition of C, setting $u = 0$ gives exactly the cycle Z. Now, setting $u = \infty$, if $t_\ell = 1$ then the intersection is empty, otherwise the ℓ-th equation becomes

$$a_0(t_1, \dots, t_{\ell-1}) = t_\ell \,,$$

and the other polynomials $p_{\ell+1}, \dots, p_n$ give a new set of polynomials $\tilde{p}_{\ell+1}, \dots, \tilde{p}_n$ with $\ell - 1$ variables. Since Z is in S_ℓ, the subscheme defined by $p_1 = \dots = p_{\ell-1} = 0$ is smooth and hence the intersection $C \cdot \{u = \infty\}$ is an element of $S_{\ell-1}$.

(c) For $i \geqslant \ell + 1$ the polynomial p_i is monic in t_ℓ of degree $\leqslant d$ and each irreducible factor p'_i of p_i is of degree $\leqslant d - 1$. When we set $t_i = 0$ for $\ell + 1 \leqslant i \leqslant n$, we obtain the equation

$$p'_i(t_1, \dots, t_\ell) = 0$$

for every irreducible factor p'_i of p_i. Thus we have two possibilities: either the polynomial $p_i(t_1, \dots, t_\ell)$ is constant with respect to t_ℓ, and then, as

$$x_i = p_i(x_1, \dots, x_{\ell-1}, x_\ell)$$

is a unit by Assumption 3.19(2b), this will result in an empty intersection. Or the t_ℓ-degree of each factor p'_i is nonzero. Then p'_i as a polynomial in $R[x_1, \dots, x_{\ell-1}][t_\ell]$ will replace the old p_ℓ (but with a t_ℓ-degree $< d$). The new resulting cycle (which is in $Z^n(R, n)_i$) has the set of equations

$$\begin{cases} p_1(t_1) = 0, \\ \vdots \\ p_{\ell-1}(t_1, \dots, t_{\ell-1}) = 0, \\ p'_i(t_1, \dots, t_\ell) = 0, \\ p_{\ell+1}(t_1, \dots, t_\ell) - t_{\ell+1} = 0, \\ \vdots \\ p_{i-1}(t_1, \dots, t_\ell) - t_{i-1} = 0, \\ q(t_1, \dots, t_\ell, u) = 0, \\ p_{i+1}(t_1, \dots, t_\ell) - t_{i+1} = 0, \\ \vdots \\ p_n(t_1, \dots, t_\ell) - t_n = 0. \end{cases}$$

It remains to show that we can replace the equation

$$q(t_1, \dots, t_\ell, u) = 0 \,,$$

with an equivalent equation $\tilde{p}_i = u$ for some well-defined polynomial $\tilde{p}_i$. We then take u to be the new variable t_i. Doing this for each irreducible factor p_i' of p_i, exhibits the intersection of C with $\{t_i = 0\}$ as an element of $S_{\ell,d-1}$. To do this we have to solve the equation

$$p_\ell(t_1,\ldots,t_\ell) = (t_\ell - 1)^{d-1}(t_\ell - a_0(t_1,\ldots,t_{\ell-1}))u$$

in terms of u. However by Assumption 3.19(3), $t_\ell - 1$ and $t_\ell - a_0(x_1,\ldots,x_{\ell-1})$ are invertible in $R[x_1,\ldots,x_{\ell-1},t_\ell]/\big(p_i(x_1,\ldots,x_{\ell-1},t_\ell)\big)$. Thus there is a polynomial $\tilde{p}_i$ in $R[t_1,\ldots,t_\ell]$, such that

$$\tilde{p}_i(x_1,\ldots,x_{\ell-1},t_\ell) = \frac{p_\ell(x_1,\ldots,x_{\ell-1},t_\ell)}{(t_\ell-1)^{d-1}(t_\ell - a_0(x_1,\ldots,x_{\ell-1}))}.$$

Thus we may replace the equation $q = 0$ with $\tilde{p}_i = u$ and hence we are done and have obtained a cycle in $S_{\ell,d-1}$ for each irreducible factor p_i' of p_i.

Notice that all the computations of the codimension one faces give cycles with irreducible components which are empty by inspection or again complete intersections by the inductive construction of the S_ℓ. There are no codimension two faces due to the Assumptions (2a) and (2b). □

Back to the proof of Theorem 3.4: We will now give some general "moving lemmas" which will help us to correct the bad cases if they are applied in the given order. Notice again that we may assume the invertiblity condition on the leading coefficient of the p_i, with $i > \ell$, due to the Lemma 3.14. These moving lemmas will not destroy the property that each irreducible component of each cycle occuring is a complete intersection.

The case 3.19(2a) can be repaired with the following trick:

Sub-Lemma 3.22. *For every $Z \in S_\ell$ we can find an equivalent cycle $Z' \in S_\ell$ for which the case 3.19(2a) is satisfied for the corresponding relative curve C.*

Proof. In fact we only need to do the proof for the codimension 2 faces: if the cycle does not meet codimension 2 faces, it does not meet faces of higher codimension.

We will use 3.15 and the same strategy as in 3.14. Set $A = R[x_1,\ldots,x_{\ell-1}]$. Notice that A is semi-local and smooth, thus a UFD (see 3.5(3)), with infinite residue fields. Suppose that p_i and p_j are not coprime in $A[T]$. By 3.15 (with $B = A$), we know that there exists an infinite number of $a \in A^\times$, such that $p_j(a)$ and $p_\ell(a)$ are units in A. For such a, set $p_{j,a}(T) = p_j(T) - \frac{p_i(a)}{p_\ell(a)} p_\ell(T)$. As p_ℓ is irreducible, distinct from p_k for all $k > \ell$, and since the family of these polynomials is finite, there exists $a \in A^\times$, such that $p_{j,a}$ is coprime to p_i and in fact to each p_k for $k > \ell$. Replacing p_j by $p_{j,a}$ will solve the problem (and the new p_j still have an invertible leading coefficient). □

Now we want to show that we can assume simultaneously the cases 3.19(2b) and the last two subcases of case 3.19(3), by using an adequate change of coordinates. For every $\gamma \in R^\times$, consider the following relative curve C_γ in $\operatorname{Spec}(R)\times\square_k^{n+1}$

$$\begin{cases} p_1(t_1) = 0, \\ \vdots \\ p_{\ell-1}(t_1, \dots, t_{\ell-1}) = 0, \\ (t_\ell - \gamma^d) p_\ell(t_1, \dots, t_\ell) - \gamma^d (t_\ell - 1) p_\ell(t_1, \dots, t_{\ell-1}, t_\ell \gamma^{-1}) u = 0, \\ p_{\ell+1}(t_1, \dots, t_\ell) - t_{\ell+1} = 0, \\ \vdots \\ p_n(t_1, \dots, t_\ell) - t_n = 0, \end{cases}$$

where d is the degree of x_ℓ. This curve is a deformation of Z into "general position" with respect to the last coordinate t_ℓ, inspired by the idea that a replacement of x_ℓ by a multiple γx_ℓ solves part of the invertibility problems.

Sub-Lemma 3.23. *With the previous notation for C, suppose that the first subcase of 3.19(3) and the subcase 3.19(2a) are satisfied, but one (or more) of the situations below does in fact occur:*

- *For some $i \in \{\ell+1, \dots, n\}$ (or several), the element $p_i(x_1, \dots, x_{\ell-1}, a_0(x_1, \dots, x_{\ell-1}))$ is not invertible in $R[x_1, \dots, x_\ell]$.*
- *The element $t_\ell - a_0(x_1, \dots, x_{\ell-1})$ is not invertible in $R[x_1, \dots, x_{\ell-1}][t_\ell]/(p_i(x_1, \dots, x_{\ell-1}, t_\ell))$ for some $i \geqslant \ell+1$.*
- *The element $t_\ell - 1$ is not invertible in $R[x_1, \dots, x_{\ell-1}][t_\ell]/(p_i(x_1, \dots, x_{\ell-1}, t_\ell))$ for some $i \geqslant \ell+1$.*

Assume that all residue fields of R are infinite. Then for infinitely many $\gamma \in R^\times$, the relative curve C_γ is a well-defined element in $Z^n(R, n+1)$. In particular, by looking at its boundaries, we see that there exists an irreducible cycle Z_γ of type S_ℓ, equivalent to Z (modulo cycles in $S_{\ell-1}$ and $S_{\ell,d-1}$), for which the above problems do not occur for the relative curve C associated to Z_γ.

Remark 3.24. If k itself is infinite, then the choice of a general $\gamma \in k^\times \subseteq R^\times$ would be sufficient to obtain the same statement. This follows also from the method used in 3.15.

Proof. The relative curve C_γ is constructed in such a way that its intersection with the faces $t_\ell = 0$ and $t_\ell = \infty$ is empty: in both cases the equation comes down to the condition $u = 1$ which creates an empty cycle. The lower faces $t_i = 0$ and $t_i = \infty$ for $i < \ell$ are also empty, since they are empty for Z.

The intersection of C_γ with the face $u = 0$ gives the cycle Z and another smooth cycle $Z' \in S_{\ell-1}$ defined by the polynomials

$$p_1, \dots, p_{\ell-1}, t_\ell - \gamma^d, p_{\ell+1} - t_{\ell+1}, \dots, p_n - t_n.$$

The intersection of C_γ with the face $u = \infty$ results in the improved cycle Z_γ, which is a complete intersection of type S_ℓ, defined by the polynomials:

$$p_1, \dots, p_{\ell-1}, p_\ell(t_1, \dots, t_\ell \gamma^{-1}), p_{\ell+1} - t_{\ell+1}, \dots, p_n - t_n.$$

This cycle is irreducible and in S_ℓ for general γ, since this is true for $\gamma = 1$ (i.e. for Z itself) and therefore also for a general deformation of Z. The faces $t_i = 0$ and $t_i = \infty$ for $i > \ell$ all add up to some cycle $Z'' \in S_{\ell,d-1}$. We will show below that Z_γ is actually an improvement of Z in the sense that its associated relative curve C is in good position. But first we note that the codimension two faces of C_γ are shown to be empty by using the assumption that the first subcase of 3.19(3) and the subcase 3.19(2a) are satisfied (this is actually the same argument as in [45]). We therefore deduce that

$$\partial C_\gamma = Z + Z_\gamma + Z' + Z'',$$

with $Z' \in S_{\ell-1}$ and $Z'' \in S_{\ell,d-1}$ with t_ℓ-degree $\leqslant d-1$. It is very important to show now that Z' and Z'' are really in $S_{\ell-1}$ resp. in $S_{\ell,d-1}$, i.e. have the complete intersection property for a general choice of γ. But this follows as in the proof of Sublemma 3.20, cases (b) and (c), since the first $\ell - 1$ equations are untouched and form a complete intersection cycle of the required form by the fact that Z is an element of S_ℓ.

For all but finitely many choices of γ, the cycle Z_γ will have the same good properties as Z, but in addition it will satisfy 3.19(2b): we know that for $\ell + 1 \leqslant i \leqslant n$, we have

$$p_i(x_1, \dots, x_\ell) = x_i \in R[x_1, \dots, x_\ell]^\times.$$

Set $B = R[x_1, \dots, x_\ell]$ and $A = R$. Then A and B fulfill the conditions of property 3.15. We define Λ as the subset of $\{\ell+1, \dots, n\}$ containing all the indices i for which p_i does not have the property 3.19(2b). Set $Q_{i,1}(T) = p_i(x_1, \dots, x_{\ell-1}, Tx_\ell)$, $Q_{i,2}(T) = p_i(x_1, \dots, x_{\ell-1}, T^d)$, $Q_{0,1}(T) = Tx_\ell - 1$ and $Q_{0,2}(T) = Tx_\ell - T^d a_0(x_1, \dots, x_{\ell-1})$. Then we have a finite family of polynomials in $B[T]$. Since $x_i \in B^\times$ for $\ell + 1 \leqslant i \leqslant n$, for each polynomial in this family at least one of its coefficients is invertible in B. Hence we can apply property 3.15 and find an element $\gamma \in R^\times$ such that the property 3.19(2b) is satisfied for the new cycle $W = Z_\gamma$. We also notice that this process does not affect the coprimality of the p_i, neither the fact that their leading coefficients are invertible. Namely, if p_i and p_j are coprime relatively to t_ℓ, then the above change of coordinates will not affect this property.

Now it remains to show the last two cases of property 3.19(3). Assume that there is an index $i \geqslant \ell + 1$ such that either $t_\ell - 1$ or $t_\ell - a_0$ is not invertible modulo p_i (for several such indices the proof solves the problem at the same time). By the above construction of C_γ, we first get that $x_\ell - 1$ and $x_\ell - a_0(x_1, \ldots, x_{\ell-1})$ are units in $R[x_1, \ldots, x_\ell]$. This, in particular, implies that $p_\ell(x_1, \ldots, x_{\ell-1}, 1)$ is invertible in $R[x_1, \ldots, x_{\ell-1}]$. Then set $\alpha_0 = p_i(x_1, \ldots, x_{\ell-1}, a_0(x_1, \ldots, x_{\ell-1}))$, $B = R[x_1, \ldots, x_{\ell-1}]$ and

$$R' = R[x_1, \ldots, x_{\ell-1}][t_\ell]/\left(p_i(x_1, \ldots, x_{\ell-1}, t_\ell)\right).$$

We can assume that α_0 is invertible by construction of C_γ. This implies that $t_\ell - a_0(x_1, \ldots, x_{\ell-1})$ is invertible in R'. Indeed, by the division algorithm we have:

$$p_i(x_1, \ldots, x_{\ell-1}, t_\ell) = (t_\ell - a_0(x_1, \ldots, x_{\ell-1}))Q(t_\ell) + b,$$

with $b \in B$ and $Q \in B[T]$. Evaluating the above expression in $a_0(x_1, \ldots, x_{\ell-1})$ shows that $b = \alpha_0$ is invertible in R', and so is $t_\ell - a_0(x_1, \ldots, x_{\ell-1})$. It remains to satisfy the property that $t_\ell - 1$ is a unit modulo p_i. If it is not, we proceed as follows: we know that $\beta_1 = p_\ell(x_1, \ldots, x_{\ell-1}, 1)$ is invertible in B. Set $\beta_0 = p_\ell(x_1, \ldots, x_{\ell-1}, a_0(x_1, \ldots, x_{\ell-1}))$ and $\alpha_1 = p_i(x_1, \ldots, x_{\ell-1}, 1)$. We then consider the following polynomials in $B[T]$: $P_i(T) = p_i(x_1, \ldots, x_{\ell-1}, T)$, $P_\ell(T) = p_\ell(x_1, \ldots, x_{\ell-1}, T)$, $Q_0(T) = \alpha_0 P_\ell(T) - \beta_0 P_i(T)$, and $Q_1(T) = \alpha_1 P_\ell(T) - \beta_1 P_i(T)$. They are all primitive, and by applying once again 3.15 with $A = R$, we can find $a \in A^\times$ such that all the above polynomials are invertible in B when evaluated in a. Thus, replacing p_i by $p_i - \frac{P_i(a)}{P_\ell(a)} p_\ell$ will solve the problem. □

Remark 3.25. As in the proof of 3.4, we always work with an irreducible cycle Z which is smooth and proper over $\mathrm{Spec}(R)$ by 3.11. Hence, from the beginning, the cycles have empty intersection with the loci $t_i = 1$ (corresponding to infinity). Therefore $p_\ell(x_1, \ldots, x_{\ell-1}, 1)$ is invertible in $R[x_1, \ldots, x_{\ell-1}]$ (or equivalently, $x_\ell - 1$ is a unit in $R[x_1, \ldots, x_\ell]$).

Back to the proof of the "Crucial Lemma". Suppose that the cycle Z is of such a type that its associated relative curve C is not in good position. In order to achieve that C is in good position, we apply several moving lemmas on Z itself in a specific order. We first apply Lemma 3.14 to ensure the good invertibility property of the p_i with $i > \ell$ (notice that this is not, stricly speaking, a "moving lemma", but more a change of presentation). After that we apply Sub-Lemma 3.22, and, if needed, we finally apply Sub-Lemma 3.23. We can check that in this order each moving lemma preserves the effect of the previous one. Also these movings do not change the property of a cycle Z to be a complete intersection in the sense of 3.20. It remains to deal with the case 3.19(1). Thus we have finished the proof of the "Crucial Lemma" once the following result is shown.

Sub-Lemma 3.26. *Once the cases 3.19(2) and 3.19(3) are solved, the cycle C has empty codimension two faces, all codimension one faces are finite and surjective over* $\mathrm{Spec}(R)$ *and hence of the correct dimension, i.e. the case 3.19(1) are automatically satisfied.*

Proof. If the cases 3.19(2) and 3.19(3) are solved, C is a curve when restricted to the closed fiber, since then we are in the situation of fields, see [45]. On the other hand, if Z is in S_ℓ, then the polynomials

$$p_1, \dots, p_{\ell-1}, q, p_{\ell+1} - t_{\ell+1}, \dots, p_n - t_n$$

form a regular sequence in $R[x_1, \dots, x_n, u]$: by definition of S_ℓ, the p_i for $1 \leqslant i \leqslant \ell - 1$ form a regular sequence. Since q has a new variable u, it is not a zero divisor in $R[t_1, \dots, t_n, u]/(p_1, \dots, p_{\ell-1})$. The same is true for the polynomials $p_k - t_k$ in

$$R[t_1, \dots, t_n, u]/(p_1, \dots, p_{\ell-1}, q, p_{\ell+1} - t_{\ell+1}, \dots, p_{k-1} - t_{k-1})$$

for $k \geqslant \ell + 1$, since they are monic in t_k and $R[t_1, \dots, t_n, u]$ is factorial as a polynomial ring over a regular ring.

Therefore, C is always a relative curve over $\mathrm{Spec}(R)$ and then, by inspection as in the proof of 3.20, the intersections with codimension one faces are either empty or they are divisors in a smooth variety (a product of $\mathbb{A}^1_k$ with the variety defined by $p_1, \dots, p_{\ell-1}$) with a monic polynomial in t_ℓ as defining equation and hence they are finite and surjective over $\mathrm{Spec}(R)$. Consequently case 3.19(1) follows and C is in good position. □

This finishes the proof of the "crucial lemma". □

Back to the proof of Theorem 3.4. As R is smooth over an infinite field, all its residue fields are infinite. Furthermore, as R is semi-local, it is a UFD (see 3.5(3)), and the main hypothesis are satisfied. Let Z be an arbitrary cycle in $Z^n(R, n)$. Then by (3.12) and (3.17), this cycle is equivalent to a cycle in S_n, with every irreducible component smooth over k and finite, and surjective over $\mathrm{Spec}(R)$ (and in particular proper over $\mathrm{Spec}(R)$). We can then apply the inductive procedure of the "Crucial Lemma", (noting that each time the Assumptions of 3.18 are satisfied) and at the end of the induction we get

$$Z = Z_{graph} + Z_C,$$

where Z_{graph} is a sum of "graph cycles" in S_0 and Z_C is a sum of boundaries of well-defined relative curves in $Z^n(R, n+1)$. This shows that Z is in the image of φ, and finishes the proof of the main theorem. □

4. Applications

The first consequence of Theorem 3.4 is the

Corollary 4.1. *If R is a smooth (semi)local k-algebra of geometric type with k infinite, then the motivic cohomology ring $H^*_{\mathcal{M}}(R, \mathbb{Z}(*)) = \oplus_{n\in\mathbb{N}} H^n_{\mathcal{M}}(R, \mathbb{Z}(n))$ (where $H_{\mathcal{M}}$ could be taken as higher Chow groups or Suslin/Voevodsky's motivic cohomology) is generated by its elements of degree 1.*

Remark 4.2. Due to the "algorithmic" nature of the process, we can compute explicitly in the ring $H^*_{\mathcal{M}}(R, \mathbb{Z}(*))$. Notice that the above result is also true with coefficients.

One of the main applications of Theorem 3.4 is the

Proposition 4.3 (Gersten quasi-resolution for Milnor K-theory). *If R is a smooth (semi)local k-algebra of geometric type with k infinite, then the following sequence is universally exact (cf.* [25] *for the definition of the maps):*

$$\mathrm{K}^M_n(R) \longrightarrow \oplus_{codim(x)=0} \mathrm{K}^M_n(k(x)) \longrightarrow \oplus_{codim(x)=1} \mathrm{K}^M_{n-1}(k(x)) \longrightarrow \cdots$$

Observation 4.4. *If we remove the smoothness assumption, then the statement does not hold anymore. Several interesting counter-examples can be found in* [8,41].

The main ingredient in the proof of (4.3) is the following result (extended to smooth semi-local rings as in Theorem 5.11 of [37]),

Gersten resolution for higher Chow groups [1]: For X smooth there are flasque (universal) resolutions:

$$\to \mathcal{CH}^r_X(q) \to \bigoplus_{x\in X^0} i_x CH^r(k(x)\,,\, q) \to \bigoplus_{x\in X^1} i_x CH^{r-1}(k(x)\,,\, q-1) \to \cdots$$
$$\to \bigoplus_{x\in X^q} i_x CH^{r-q}(k(x)\,,\, 0) \to 0$$

where $\mathcal{CH}^r_X(q)$ is the sheaf associated to the presheaf $U \mapsto CH^r(U\,,\, q)$. In the previous, $X^n = \{x \in X$, Zariski closure of x has codim n in $X\}$, and for A an abelian group, $i_x A$ denotes the constant sheaf with stalk A on the Zariski closure $\{\bar{x}\}$. In particular we have $H^n(X, \mathcal{CH}^n_X(n)) \cong CH^n(X)$.

Remark 4.5 (On the proof of the Gersten resolution for higher Chow groups). The original proof (without the universality) is due to Bloch [1]. But we note that we can get at least three other proofs of this result. We will indicate in the following the hints and will leave the details to the interested reader.

1. *first strategy of proof (including the universality)*: by using the work of Colliot-Thélène, Hoobler and Kahn [7] (see Sects. 5, 6 and the Example 7.3.5, p. 69). Indeed, we can check that higher Chow groups fulfill the axioms developed in their work and as a consequence we get the universal exactness of the Gersten complex (for higher Chow groups).
2. *second strategy of proof (including the splitting)*: by using the work of Voevodsky [47] (Sects. 4.4 and 4.6). Moreover, in the case of particular interest (semi-local schemes), we get that the morphism $CH^p(R\,,\, n) \to CH^p(Frac(R)\,,\, n)$ is a split monomorphism.

3. *third strategy of proof*: by using the work of Colliot-Thélène and Ojanguren [6], and checking that higher Chow groups fulfill the conditions P1-P3 of Theorem 1.1 of [6] (p. 100) we can get the injectivity of the morphism $CH^p(R, n) \to CH^p(Frac(R), n)$. The step from Gersten conjecture to Gersten resolution is obtained via standard arguments.

Proof of Proposition 4.3. We give the proof in the local case only and refer to [25] for the maps involved in Milnor K-theory. By a variant of Lemma 5.2 and Proposition 2.3 for the commutativity of the first square, we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & \mathrm{K}_n^M(R) & \to & \mathrm{K}_n^M(Frac(R)) & \to & \oplus_{codim(x)=1} \mathrm{K}_{n-1}^M(k(x)) & \to & \cdots \\ & & \varphi_n \downarrow & & \varphi^0 \downarrow & & \varphi^1 \downarrow & & \\ 0 & \to & CH^n(R, n) & \to & CH^n(Frac(R), n) & \to & \oplus_{codim(x)=1} CH^{n-1}(k(x), n-1) & \to & \cdots \end{array}$$

But the maps φ^* are isomorphisms by Nesterenko/Suslin/Totaro, the map φ_n is onto, and the tensor product is a right exact functor. Thus the universal exactness of the bottom row implies the universal exactness of the top row. □

Remark 4.6. Proposition 4.3 gives a new, independant and conceptually different (and perhaps simpler) proof of the exactness of the complex

$$\oplus_{codim(x)=0} \mathrm{K}_n^M(k(x)) \to \oplus_{codim(x)=1} \mathrm{K}_{n-1}^M(k(x)) \to \cdots$$

which was first proved by Rost [38] (see also Kato [25], Sect. 1, p. 242). Furthermore Gabber, in unpublished work[3] [16] has also proved Proposition 4.3 using a different approach (without using higher Chow groups).

Let us define, for R a domain, the following ring

$$\overline{\mathrm{K}}_*^M(R) \stackrel{\text{def}}{=} \mathrm{Im}(\mathrm{K}_*^M(R) \to \mathrm{K}_*^M(Frac(R))).$$

Then, as a consequence of the previous result, the sheaf $\overline{\mathcal{K}}_n^M$ (sheafification of the contravariant functor $\overline{\mathrm{K}}_n^M$) admits Gersten resolutions for all $n \geqslant 0$. Multiplication in $\overline{\mathrm{K}}^M$-theory defines a pairing of sheaves $\overline{\mathcal{K}}_n^M \otimes \overline{\mathcal{K}}_m^M \to \overline{\mathcal{K}}_{n+m}^M$ and hence a pairing in $\overline{\mathcal{K}}^M$-cohomology [18] (p. 242), and if k is an infinite field, we then have the

Corollary 4.7 (Relations with algebraic cycles). *Let X be a smooth quasi-projective variety over k. Then, for all p, n, we have a natural map, compatible with products, from $CH^p(X, n)$ to $H^{p-n}(X, \overline{\mathcal{K}}_p^M)$, which, for all $p \geqslant 0$, is surjective for $n \leqslant 3$ and an isomorphism for $n = 0, 1, 2$.*

[3] The authors were not aware of the proof of Gabber while they were working on this project

Miscellanea 4.8. 1. *The construction of the map* $CH^p(X, n) \to H^{p-n}(X, \overline{\mathcal{K}}_p^M)$ *follows Landsburg* [26] *and uses a norm map. However, it is also induced by the local-to-global spectral sequence for higher Chow groups, see* [34].

2. *For* $n = 3$, *we can give a more precise statement (see* [34]*). The following sequence*

$$H^{p-2}(X, \mathcal{CH}_X^p(p+1)) \longrightarrow CH^p(X, 3) \longrightarrow H^{p-3}(X, \overline{\mathcal{K}}_p^M) \longrightarrow 0,$$

is exact.

3. *The map of Corollary 4.7 factorizes the well known map*

$$CH^p(X, n) \to H^{p-n}(X, \mathcal{K}_p),$$

where $\mathcal{K}$ *denotes the usual K-sheaf. In general, you can expect that* $H^{p-n}(X, \overline{\mathcal{K}}_p^M)$ *will be different from* $H^{p-n}(X, \mathcal{K}_p)$ *for* $n \geqslant 3$. *If* $n = 2$ *and all p, both groups are isomorphic* [38] *(Remark 5.4, p. 357). We can find more details in* [34]. *Notice that the map* $CH^p(X, n) \to H^{p-n}(X, \mathcal{K}_p)$ *is known to be rationally an isomorphism for* $n = 2$ *and all p by* [40] *and* [27].

4. *If* $\mathcal{K}_*^M$ *is the sheaf associated to Milnor K-theory, we have that* $H^n(X, \mathcal{K}_n^M) \cong CH^n(X)$, *up to* $(n-1)!$ *torsion by* [40] *(pp. 526-532), and we also know that this isomorphism holds integrally for* $n = \dim(X)$ *by* [25] *(pp. 251-252).*

5. *("Generalized Kato conjecture"). Using* $CH^2(X, 2) \cong H^0(X, \mathcal{K}_2)$, *and a well known* $\mathcal{K}$*-cohomology exact sequence (see* [42], *Corollary 23.4), we obtain:*

$$0 \to CH^2(X, 2)/\ell^n \to H^2_{ét}(X, \mu_{\ell^n}^{\otimes 2}) \to {}_{\ell^n}CH^2(X, 1) \to 0.$$

Applying this to $X = \mathrm{Spec}(R)$ *for R a smooth semi-local k-algebra of finite type, assuming char(k)* $\neq \ell$, *and using Remark 3.5, we get* $CH^2(X, 2)/\ell^n \cong H^2_{ét}(X, \mu_{\ell^n}^{\otimes 2})$, *in other words a direct proof of the generalized Kato conjecture for* $K_2^M(R)$.

5. The case of a semi-local PID

Let A be a semi-local PID and $\mathfrak{m}_i$ its maximal ideals. Recall that k is our ground field, we will denote by F the quotient field and by k_i the residue fields. We will also denote by π_i a generator of $\mathfrak{m}_i$. Therefore we can identify $\mathrm{Spec}(k_i)$ with a closed subvariety of codimension 1 in $\mathrm{Spec}(A)$. Then, the localization exact sequence for higher Chow groups in this context looks like

$$\cdots \to \oplus_i CH^*(k_i, n) \to CH^{*+1}(A, n) \to CH^{*+1}(F, n) \xrightarrow{\partial^{Chow}} \oplus_i CH^*(k_i, n-1) \to \cdots$$

which, if A is smooth, reduces to short (split) exact sequences

$$0 \to CH^{*+1}(A, n) \to CH^{*+1}(F, n) \xrightarrow{\partial^{Chow}} \oplus_i CH^*(k_i, n-1) \to 0.$$

Observation 5.1. *Notice that as a consequence of the splitting we have $CH^{n+1}(A, n) = 0$. We can also prove this result directly by using the moving lemma of Levine* [29] *(Theorem 4.5.12).*

The compatibility between the maps in Milnor K-theory and higher Chow groups is given by

Lemma 5.2. *For all $n \geqslant 0$, the following diagram is always commutative*

$$\begin{array}{ccc} \mathrm{K}_n^M(F) & \xrightarrow{\partial} & \oplus_i \mathrm{K}_{n-1}^M(k_i) \\ {\scriptstyle \varphi_F}\downarrow & & \downarrow{\scriptstyle \oplus\varphi_{k_i}} \\ CH^n(F, n) & \xrightarrow[\partial^{Chow}]{} & \oplus_i CH^{n-1}(k_i, n-1). \end{array}$$

Proof. see [17] (Lemma 3.2). □

By a generalization of the arguments of [43] (resp. [20]), we have

Proposition 5.3. *Let A be a semi-local PID of geometric type over an infinite field k or such that the map $\mathrm{K}_3(A) \longrightarrow \oplus_i \mathrm{K}_3(A/\mathfrak{m}_i)$ is onto. Then $\mathrm{K}_3^M(A) \cong CH^3(A, 3)$.*

And once again by a generalization of [43], we get the

Proposition 5.4. *If for a given n we have $\ker(\mathrm{K}_n^M(A, \cap_i \mathfrak{m}_i) \longrightarrow \mathrm{K}_n^M(A)) = 0$ and A is a smooth semi-local PID of geometric type over k, then the map $\mathrm{K}_n^M(A) \to CH^n(A, n)$ is an isomorphism.*

Corollary 5.5. *If the maps $A \longrightarrow k_i$ are split, then the map $\mathrm{K}_n^M(A) \to CH^n(A, n)$ is an isomorphism for all $n \in \mathbb{N}$.*

Proof. Indeed, if the maps $A \longrightarrow k_i$ are split then $\ker(\mathrm{K}_n^M(A, \cap_i \mathfrak{m}_i) \longrightarrow \mathrm{K}_n^M(A)) = 0$, and $CH^n(A, n) \to CH^n(k_i, n)$ is a split surjection for all i. □

Remark 5.6. The splitting hypothesis in Corollary 5.5 is not necessary. At least in characteristic 0 we are able to prove that the same result holds in the general (i.e. not necessarily split semi-local PID) case, see [12]. In characteristic p, the techniques of Levine, developed in [30], seem to be applicable.

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