JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **173**, 459–469 (1993)

Averaging Operators and a Generalized Robinson Differential Inequality

SANFORD S. MILLER*

Department of Mathematics, State University of New York, Brockport, New York 14420

AND

PETRU T. MOCANU

Faculty of Mathematics, Babes-Bolyai University, 3400 Cluj-Napoca, Romania

Submitted by Steven G. Krantz

Received November 26, 1990

Let co E denote the convex hull of a set E in $\mathbb{C}$, let H be the class of analytic functions defined in the unit disc U, and denote the subordination of functions $f, g \in H$ by $f \prec g$. This article deals with the theory of *averaging operators* defined on a set $K \subset H$. These are operators $I: K \to H$ that satisfy $I[f](0) = f(0)$ and $I[f](U) \subset \operatorname{co} f(U)$ for all $f \in K$. Conditions for and examples of such operators are presented. The proofs of these results are dependent on determining dominants of the second-order differential subordination $Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z)$. With suitable conditions on A, B, C, D, and h the authors show that $p \prec h$. As an additional application of this differential subordination the authors generalize a differential inequality of R. M. Robinson. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let f and h be continuous on $J = [0, 1)$ with $h(x) \geq 0$ for $x \in J$. If $x \in J$ then by the First Mean-Value Theorem for Riemann Integrals there exists $c \in (0, x)$ such that

$$\int_0^x f(t)\,h(t)\,dt \;=\; f(c)\int_0^x h(t)\,dt. \tag{1}$$

If we let $h(x) = g'(x)$ with $g(0) = 0$, then (1) becomes

$$g(x)^{-1}\int_0^x f(t)\,g'(t)\,dt \;=\; f(c) \;\in\; f(J), \tag{2}$$

* The first author acknowledges support received from the International Research and Excange Board (IREX), with funds provided by the National Endownment for the Humanities and the United States Information Agency.

0022-247X/93 $5.00

for all $x \in J$. In Section 3 of this article we show that this mean-value theorem for integrals of real-valued functions defined on the unit interval J extends very naturally to complex-valued functions defined on the unit disc $U = \{z : |z| < 1\}$. If f and g are analytic on U, then with some simple conditions on f and g we obtain the following analog of (2),

$$g(z)^{-1} \int_0^z f(w)\, g'(w)\, dw \ \in \ f(U), \tag{3}$$

for all $z \in U$. If we denote this integral operator by $I[f]$ then (3) can be rewritten as

$$I[f](U) \ \subset \ f(U). \tag{3'}$$

Similar inclusion relations are proved for several operators besides the mean-value operator given in (3). These results, presented in Section 3, are dependent on the second-order differential subordination theorems of Section 2. Included in this group of differential subordinations is a generalization of the following differential inequality of R. M. Robinson [8]:

$$|T'(z)/S'(z)| < 1 \ \Rightarrow \ |T(z)/S(z)| < 1.$$

Since subordination is critical to several of the proofs, we now restate its definition. Let f and F be analytic in U. The function f is *subordinate* to F, written $f \prec F$ or $f(z) \prec F(z)$, if F is univalent, $f(0) = F(0)$, and $f(U) \subset F(U)$.

We close this section with a lemma that will be used to prove the main theorems of Section 2.

LEMMA 1. *Let p be analytic in U and let h be convex (univalent) on $\bar{U}$, with $p(0) = h(0)$. If p is not subordinate to h, then there exist $z_0 \in U$ and $w_0 \in \partial U$, and an $m \geqslant 1$ for which $p(|z| < |z_0|) \subset h(U)$,*

(i) $p(z_0) = h(w_0)$,

(ii) $z_0 p'(z_0) = m w_0 h'(w_0)$, *and*

(iii) $\operatorname{Re}[z_0^2 p''(z_0)/w_0 h'(w_0)] \geqslant -m$.

A more general form of this lemma appears in [7, p. 158].

2. DIFFERENTIAL SUBORDINATIONS

In [7, p. 170] the authors showed that if an analytic fuction p satisfies the second-order differential subordination

$$Az^2 p''(z) + Bzp'(z) + Cp(z) \ \prec \ z,$$

where $A \geqslant 0$, $A+B \geqslant 0$, and $B+C>0$, then $p(z) \prec z/(B+C)$. Our first theorem extends this result to a more general differential subordination of the form

$$Az^2p''(z) + B(z)\,zp'(z) + C(z)\,p(z) + D(z) \prec h(z). \tag{4}$$

THEOREM 1. *Let h be convex (univalent) in U with $h(0)=0$, and let $A \geqslant 0$. Suppose that $k > 4/|h'(0)|$ and that $B(z)$, $C(z)$, and $D(z)$ are analytic in U and satisfy*

$$\operatorname{Re} B(z) \geqslant A + |C(z)-1| - \operatorname{Re}[C(z)-1] + k\,|D(z)|, \tag{5}$$

for $z \in U$. If p is analytic in U with $p(0)=0$ and if p satisfies (4) then $p \prec h$.

Proof. Note that condition (4) implies that $D(0)=0$. Since $k\,|h'(0)|>4$, there is an r_0, $0<r_0<1$, such that $(1+r_0)^2/r_0 = k\,|h'(0)|$ and

$$4 < (1+r)^2/r < k\,|h'(0)|, \qquad \text{for} \quad r_0<r<1. \tag{6}$$

Since h is convex in U, if we set $h_r(z)=h(rz)$ for $r_0<r<1$, then h_r is convex on $\bar{U}$. If we also set $p_r(z)=p(rz)$, then from (4) we obtain

$$\begin{aligned} u_r(z) \equiv {} & Az^2p_r''(z) + B(rz)\,zp_r'(z) \\ & + C(rz)\,p_r(z) + D(rz) \prec h_r(z), \end{aligned}$$

for $z \in U$ and $r_0<r<1$. We will use Lemma 1 to show that $p_r(z) \prec h_r(z)$ for $r_0<r<1$.

Assume that p_r is not subordinate to h_r for some r in $(r_0, 1)$. Then by Lemma 1 there exist points $z_0 \in U$ and $w_0 \in U$, and an $m \geqslant 1$ such that

$$\begin{gathered} p_r(z_0) = h_r(w_0), \qquad z_0\,p_r'(z_0) = mw_0h_r'(w_0), \\ \operatorname{Re}[z_0^2\,p_r''(z_0)/w_0h_r'(w_0)] \geqslant -m. \end{gathered} \tag{8}$$

If we set $V \equiv [u_r(z_0) - h_r(w_0)]/w_0h_r'(w_0)$ then from (7) we have

$$\begin{aligned} V = {} & \frac{Az_0^2\,p_r''(z_0)}{w_0h_r'(w_0)} + \frac{B(rz_0)\,z_0\,p_r'(z_0)}{w_0h_r'(w_0)} \\ & + \frac{C(rz_0)\,p_r(z_0) - h_r(w_0)}{w_0h_r'(w_0)} + \frac{D(rz_0)}{w_0h_r'(w_0)} \end{aligned} \tag{9}$$

with

$$u_r(z_0) = h_r(w_0) + Vw_0\,h_r'(w_0). \tag{10}$$

We first show that $\operatorname{Re} V > 0$, which will then lead to the desired contradiction. Since h_r is convex and $h_r(0)=0$ we have $\operatorname{Re}[w_0 h_r'(w_0)/h_r(w_0)] \geqslant 1/2$ for $|w_0|=1$ [2, p. 176], or equivalently

$$|h_r(w_0)/w_0 h_r'(w_0) - 1| \leqslant 1. \tag{11}$$

If W and Z are complex numbers and $|Z-1| \leqslant 1$ then

$$\operatorname{Re} WZ = \operatorname{Re} W + \operatorname{Re} W(Z-1) \geqslant \operatorname{Re} W - |W|.$$

Using this inequality with $W = C(rz_0) - 1$ and $Z = h_r(w_0)/w_0 h_r'(w_0)$, we obtain from (11) that

$$\begin{aligned}\operatorname{Re}[(C(rz_0)-1)\, h_r(w_0)/w_0 h_r'(w_0)] \\ \geqslant \operatorname{Re}[C(rz_0)-1] - |C(rz_0)-1|.\end{aligned} \tag{12}$$

Since h is convex we have $|h'(z)| \geqslant |h'(0)|/(1+r)^2$ for $|z| = r < 1$ [2, p. 118]. By setting $z = rw_0$ we obtain

$$|w_0 h_r'(w_0)| \geqslant r\,|h'(0)|/(1+r)^2, \qquad \text{for} \quad |w_0| = 1. \tag{13}$$

If we use (5), (8), (12), and (13) in (9) we obtain

$$\begin{aligned}\operatorname{Re} V \geqslant{}& A(-m) + \operatorname{Re}[B(rz_0)]m \\ &+ \operatorname{Re}[(C(rz_0)-1)\, h_r(w_0)/w_0 h_r'(w_0)] \\ &- |D(rz_0)/w_0 h_r'(w_0)| \\ \geqslant{}& (m-1)[|C(rz_0)-1| - \operatorname{Re}(C(rz_0)-1)] \\ &+ m[k - (1+r)^2/r\,|h'(0)|]\,|D(rz_0)|.\end{aligned}$$

From (6) and the fact that $m \geqslant 1$ we obtain $\operatorname{Re} V > 0$, or equivalently $|\arg V| < \pi/2$. Using this in (10) together with the fact that $w_0 h_r'(w_0)$ is the outward normal to the boundary of the convex domain $h_r(U)$, we obtain $u_r(z_0) \notin h_r(U)$. Since this contradicts (7) we must have $p_r \prec h_r$ for all $r_0 < r < 1$. By letting $r \to 1$ we obtain the desired conclusion $p(z) \prec h(z)$.

In the special case $C(z) \equiv 1$, the condition $h(0) = p(0) = 0$ can be replaced by $h(0) = p(0)$. In this case the proof of Theorem 1 still holds and we obtain the following result.

THEOREM 2. *Let h be convex (univalent) in U and let $A \geqslant 0$. Suppose $k > 4/|h'(0)|$ and that B and D are analytic in U, with $D(0)=0$ and*

$$\operatorname{Re} B(z) \geqslant A + k\,|D(z)|, \tag{14}$$

for $z \in U$. *If* p *is analytic in* U *with* $p(0) = h(0)$, *and if* p *satisfies*

$$Az^2p''(z) + B(z)\,zp'(z) + p(z) + D(z) \prec h(z)$$

then $p(z) \prec h(z)$.

Let S and T be analytic in U with $S(0) = T(0) = 0$. In addition, suppose that S is starlike (not necessarily univalent) in U, that is, $\operatorname{Re} zS'(z)/S(z) > 0$. In 1947 R. M. Robinson [8, p. 30] proved that

$$|T'(z)/S'(z)| < 1 \text{ for } z \in U \;\Rightarrow\; |T(z)/S(z)| < 1 \text{ for } z \in U. \tag{15}$$

In 1965 R. J. Libera [3, p. 755] proved that

$$\operatorname{Re} T'(z)/S'(z) > 0 \text{ for } z \in U \;\Rightarrow\; \operatorname{Re} T(z)/S(z) > 0 \text{ for } z \in U. \tag{16}$$

K. Sakaguchi [9, p. 74] also proved this result but with the additional assumption that S is univalent.

In 1973 T. H. MacGregor [4, p. 532] and in 1978 the authors [6, p. 301] extended Libera's result and for real γ proved that

$$\operatorname{Re} T'(z)/S'(z) > \gamma \text{ for } z \in U \;\Rightarrow\; \operatorname{Re} T(z)/S(z) > \gamma \text{ for } z \in U. \tag{17}$$

All these results concerning differential inequalities are special cases of the following theorem on differential subordinations.

THEOREM 3. *Let* h *be convex (univalent) in* U, *and let* R, S, *and* T *be analytic in* U *with* $R(0) = S(0) = T(0) = 0$. *If*

$$\operatorname{Re} S(z)/zS'(z) \;>\; 4\,|R(z)|/h'(0)|, \tag{18}$$

for $z \in U$, *then*

$$T'(z)/S'(z) + R(z) \;\prec\; h(z) \;\Rightarrow\; T(z)/S(z) \;\prec\; h(z). \tag{19}$$

Proof. From (18) we obtain $\operatorname{Re} zS'(z)/S(z) > 0$ which implies that S maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin. If we let $p = T/S$, then p is analytic in U and satisfies

$$[S(z)/zS'(z)]\,zp'(z) + p(z) \;=\; T'(z)/S'(z).$$

Since $T'/S' + R \prec h$, if we set $B(z) = S(z)/zS'(z)$ and $D = R$ we obtain

$$B(z)\,zp'(z) + p(z) + D(z) \;\prec\; h(z),$$

while (18) becomes $\operatorname{Re} B(z) > 4\,|D(z)|/|h'(0)|$. The conditions of Theorem 2 with $A = 0$ are satisfied, and from that result we obtain $p = T/S \prec h$.

In the special case $S(z)=z$, (18) simplifies to $|R(z)|<|h'(0)|/4$ and from (19) we obtain

$$T'(z)+R(z) \prec h(z) \Rightarrow T(z)/z \prec h(z).$$

As an example, if $h'(0)=1$ and $|\alpha|<1/4$ we have

$$T'(z)+\alpha z \prec h(z) \Rightarrow T(z)/z \prec h(z).$$

If we take $R(z)=0$ in Theorem 3 we obtain:

COROLLARY 3.1. *Let h be convex (univalent) in U, and let S and T be analytic in U with $S(0)=T(0)=0$. If* $\operatorname{Re} zS'(z)/S(z)>0$, *then*

$$T'(z)/S'(z) \prec h(z) \Rightarrow T(z)/S(z) \prec h(z). \tag{20}$$

Remarks. 1. If the function $T'(z)/S'(z)$ is itself a convex function, (20) can be replaced by $T/S \prec T'/S'$.

2. We can eliminate the function h in the corollary and extend the result by using the concept of the convex hull of a set E (denoted by co E) in $\mathbf{C}$. In this extension, (20) is replaced by

$$\{T(z)/S(z): z\in U\} \subset \operatorname{co}\{T'(z)/S'(z): z\in U\}. \tag{20'}$$

To check the validity of this we need to consider the cases $\operatorname{co}\{T'/S'\}=\mathbf{C}$ and $\operatorname{co}\{T'/S'\}\neq\mathbf{C}$. In the first case (20′) trivially holds, while in the second case there is a convex (univalent) function h mapping U onto $\operatorname{co}\{T'/S'\}$ with $h(0)=T'(0)/S'(0)$, and (20′) follows from (20).

3. In the special case when S is also univalent then the corollary coincides with a result of E. Merkes and D. Wright [5, p. 98].

4. We can obtain (15), (16), and (17) by applying (20) with $h(z)=z$, $h(z)=(1+z)/(1-z)$, and $h(z)=[1+(1-2\gamma)z]/(1-z)$, respectively. As another example, consider the domain D given by $|\arg z|<\alpha\pi/2$, where $0<\alpha<1$. The function $h(z)=[(1+z)/(1-z)]^\alpha$ maps U conformally onto D and by the corollary we obtain

$$\frac{T'(z)}{S'(z)} \prec \left[\frac{1+z}{1-z}\right]^\alpha \Rightarrow \frac{T(z)}{S(z)} \prec \left[\frac{1+z}{1-z}\right]^\alpha,$$

or equivalently

$$|\arg T'(z)/S'(z)| < \alpha\pi/2 \Rightarrow |\arg T(z)/S(z)| < \alpha\pi/2.$$

3. AVERAGING OPERATORS

In this section we denote by H the set of functions that are analytic in U, we let $H_0 = \{h \in H : h(0) = 0\}$, and when we refer to a convex function we shall assume that it is univalent.

The first result in this section involves an integral analog of Theorem 3.

THEOREM 4. *If h is convex in U, $f \in H$, and g, $w \in H_0$ with* $\operatorname{Re} g(z)/rg'(z) > 4\,|w(z)|/|h'(0)|$, *then*

$$f(z) \prec h(z) \;\Rightarrow\; F(z) = g(z)^{-1} \int_0^z [f(t) + w(t)]\, g'(t)\, dt \prec h(z). \tag{21}$$

In particular, if f is convex and $\operatorname{Re} g(z)/zg'(z) > 4\,|w(z)|/|f'(0)|$ *then*

$$I[f](z) = F(z) = g(z)^{-1} \int_0^z [f(t) + w(t)]\, g'(t)\, dt \prec f(z). \tag{22}$$

Proof. If we set $T(z) = \int_0^z [f(t) + w(t)]\, g'(t)\, dt$, $R(z) = -w(z)$, and $S(z) = g(z)$, the $T'/S' + R = f \prec h$ and all the conditions of Theorem 3 are satisfied. Hence we conclude that $F = T/S \prec h$. The subordination (22) follows from (21) by taking $h = f$.

EXAMPLE 1. If we let $h(z) = (1+z)/(1-z)$ then $|h'(0)| = 2$ and we obtain: if $f \in H$ with $f(0) = 1$, and if $w \in H_0$ with $|w(z)| < 1/2$ then

$$\operatorname{Re} f(z) > 0 \;\Rightarrow\; \operatorname{Re}\left[z^{-1} \int_0^z [f(t) + w(t)]\, dt\right] > 0.$$

If $w(z) \equiv 0$ in Theorem 4 then the function h can be eliminated and the result can be extended by considering the convex hull of $f(U)$. The proof of the following result follows from considering the cases $\operatorname{co} f(U) \neq \mathbf{C}$. For the first case there exists a convex function $h : U \to \operatorname{co} f(U)$ with $h(0) = f(0)$ and the proof follows from Theorem 4, while the second case is trivial.

COROLLARY 4.1. *Let $g \in H_0$ with* $\operatorname{Re} zg'(z)/g(z) > 0$. *If $f \in H$ and if I is defined by*

$$I[f](z) = g(z)^{-1} \int_0^z f(t)\, g'(t)\, dt, \tag{23}$$

then

$$I[f](U) \subset \operatorname{co} f(U). \tag{24}$$

In addition, if f is convex then

$$I[f](U) \subset f(U). \tag{25}$$

409 173 2-10

The operator I given in (23) is the same as the operator given in (3), which is the analog of the mean-value theorem of real analysis. The inclusion relation given in (25) indicates that I has the mean-value property as required in (3′).

EXAMPLE 2. If we let $g(z)=z^{\alpha}$ in Corollary 4.1 we obtain: if $f\in H$ and $\operatorname{Re}\alpha>0$ then the operator

$$I_{\alpha}[f](z)=\alpha z^{-\alpha}\int_0^z f(t)\,t^{\alpha-1}\,dt$$

satisfies (24). Operators of this particular type are discussed in [1, p. 85] and are referred to as *averaging operators*. In the special case when $\alpha=1$ we obtain the classical *average value of f*

$$I_1[f](z)=z^{-1}\int_0^z f(t)\,dt.$$

In light of the above remarks we make the following definition:

DEFINITION 1. If $K\subset H$ and if an operator $I\colon K\to H$ satisfies

$$I[f](0)=f(0) \qquad \text{and} \qquad I[f](U)\subset \operatorname{co} f(U)$$

for all $f\in K$, then I is said to be an *averaging (or mean-value) operator on K*.

From the last corollary we see that the integral operator I given in (23) is an averaging operator on H. Because of (22) the operator described therein is an averaging operator on the subset of convex functions f for which $|f'(0)|>4\,|w(z)|/\operatorname{Re}[g(z)/zg'(z)]$.

The following lemma provides a simple characterization of averaging operators.

LEMMA 2. *Let $K\subset H$ and let an operator $I\colon K\to H$ satisfy $I[f](0)=f(0)$ for all $f\in K$. A necessary and sufficient condition for I to be an averaging operator on K is that*

$$\{f\in K,\ h \text{ convex and } f\prec h\} \ \Rightarrow\ I[f]\prec h. \tag{26}$$

Proof. If (26) is satisfied then there are two cases to consider. If $\operatorname{co} f(U)=\mathbf{C}$ then $I[f](U)\subset \operatorname{co} f(U)$. If $\operatorname{co} f(U)\neq\mathbf{C}$ then let h be a convex function mapping U onto $\operatorname{co} f(U)$ with $h(0)=f(0)$. Using this h in (26) we obtain $I[f](U)\subset \operatorname{co} f(U)$, which again shows that I is an averaging operator.

If I is an averaging operator then (26) follows since $f\prec h$ implies that $I[f](U)\subset \operatorname{co} f(U)\subset h(U)$.

We can not use this lemma to conclude that the operator given in (21) is an averaging operator on H. This is due to the fact that the implication in (21) does not hold for all convex h; the operator itself depends on h. We can remove this dependency in special cases. Since $f \prec h$ implies $|f'(0)| \leqslant |h'(0)|$, if we take $g(z)=z$ and $w(z)=2\beta f'(0)z$, with $|\beta|<1/8$, then we can use (21) and Lemma 2 to obtain the following class of averaging operators.

COROLLARY 4.2. *If $f \in H$ and $|\beta| \leqslant 1/8$ then the integral operator I, defined by*

$$I[f](z) = z^{-1}\int_0^z (f(t)+2\beta f'(0)t)\,dt = z^{-1}\int_0^z f(t)\,dt + \beta f'(0)z$$

is an averaging operator on H.

Our last theorem provides us with a large group of averaging operators on H_0.

THEOREM 5. *Let $\gamma \in \mathbf{C}$ with $\gamma \neq -1, -2, \ldots,$ and let $\phi, \Phi \in H$ with $\phi(z)\cdot\Phi(z) \neq 0$ for $z \in U$. If*

$$\operatorname{Re} B(z) \;\geqslant\; |C(z)-1| - \operatorname{Re}\,[C(z)-1], \qquad z \in U, \tag{27}$$

where $B(z)=\Phi(z)/\phi(z)$ and $C(z)=(\gamma\Phi(z)+z\Phi'(z))/\phi(z)$, then the integral operator I defined by

$$I[f](z) = z^{-\gamma}\Phi(z)^{-1}\int_0^z f(t)\,t^{\gamma-1}\phi(t)\,dt \tag{28}$$

is an averaging operator on H_0.

Proof. The restrictions on γ, ϕ, and Φ imply that $I[f]$ is analytic on U and $I[f](0)=0$. Since $f \prec h$, if we let $p=I[f]$ and differentiate (28) we obtain

$$B(z)\,zp'(z)+C(z)\,p(z) \;=\; f(z) \;\prec\; h(z).$$

We now apply Theorem 1 with $A=0$ and $D(z)=0$. Since (27) implies (5) we obtain $p=I[f]\prec h$. Hence by Lemma 2 the operator I is an averaging operator on H_0.

EXAMPLE 3. If we set $\gamma=0$, $\phi(z)\equiv 1/2$, and $\Phi(z)\equiv 1$ in Theorem 5 then $B(z)\equiv 2$, $C(z)\equiv 0$, and (28) is satisfied. Hence the operator

$$I[f](z) = \frac{1}{2}\int_0^z \frac{f(t)}{t}\,dt$$

is an averaging operator on H_0, that is $I[f](U) \subset \operatorname{co} f(U)$.

This last operator is very similar to the classical Alexander operator

$$J[f](z)=\int_0^z \frac{f(t)}{t}\,dt.$$

However, this operator is not an averaging operator on H_0. If we take $f(z)=z+z^2/4$ and set $F(z)\equiv J[f](z)$ then $F(z)=z+z^2/8$. Since the functions f and F are convex and since $F(-1)<f(-1)<0<F(1)<f(1)$ we have $J[f](U)=F(U)\not\subset f(U)=\operatorname{co} f(U)$.

In the special case $\phi(z)\equiv\Phi(z)\equiv 1$ Theorem 5 reduces to the following corollary.

COROLLARY 5.1. *If $\gamma\in\mathbf{C}$ and γ satisfies*

$$|\gamma-1| \leqslant \operatorname{Re}\gamma, \tag{29}$$

then $I[f](z)=z^{-\gamma}\int_0^z f(t)\,t^{\gamma-1}\,dt$ is an averaging operator on H_0.

For the remainder of this article it is convenient to introduce the set $Q=\{w=x+iy\colon |w-1|\leqslant \operatorname{Re} w\}$. Note that (29) is equivalent to $\gamma\in Q$ and that Q consists of points inside or on the parabola $2x=y^2+1$.

COROLLARY 5.2. *Let $\gamma\in\mathbf{C}$ and let $g\in H_0$ with $g'(0)=1$. If*

$$|\gamma z g'(z)/g(z)-1| \leqslant \operatorname{Re}(\gamma z g'(z)/g(z)) \tag{30}$$

for $z\in U$, then I defined by

$$I[f](z)=[g(z)]^{-\gamma}\int_0^z f(t)\,t^{-1}[g(t)]^{\gamma}\,dt$$

is an averaging operator on H_0.

Proof. Condition (30) is equivalent to $zg'(z)/g(z)\in Q$, which for $z=0$ also implies $\gamma\in Q$. In addition, (30) implies that $\operatorname{Re}\gamma z g'(z)/g(z)>0$. Since $g'(0)=1$, the function g is a spirallike univalent function [2, p. 149]. Since $g(z)/z\neq 0$, if we set $\phi(z)=\Phi(z)=(g(z)/z)^{\gamma}$ then ϕ, $\Phi\in H$ and $\phi(z)\cdot\Phi(z)\neq 0$ for $z\in U$. Using these values of ϕ and Φ in Theorem 5 we see that (30) implies (27), and hence the operator I is an averaging operator on H_0.

We close this article by considering two examples of this last corollary.

EXAMPLE 4. Let γ be real with $\gamma\geqslant 1/2$ and let $g(z)=ze^{\mu z/\gamma}$. In this case $\gamma z g'(z)/g(z)=\gamma+\mu z$ and condition (30) is equivalent to requiring

$$\gamma+\mu z\in Q, \qquad \text{for} \quad z\in U. \tag{31}$$

We need to find the maximum value of $|\mu|$ such that, for a fixed value of γ, condition (31) holds. This problem is equivalent to finding the maximum radius r_γ of a circle with center at γ, which is contained in Q. The solution to this simple calculus problem is given by

$$r_\gamma = \begin{cases} \gamma - 1/2 & \text{if } 1/2 \leqslant \gamma \leqslant 3/2, \\ \sqrt{2(\gamma - 1)} & \text{if } 3/2 < \gamma. \end{cases} \tag{32}$$

Hence, if $|\mu| \leqslant r_\gamma$ then by Corollary 5.2 the operator I defined by

$$I[f](z) = z^{-\gamma} e^{-\mu z} \int_0^z f(t)\, t^{\gamma - 1} e^{\mu t}\, dt$$

is an averaging operator on H_0.

EXAMPLE 5. Let $\gamma = 1$ and let $g(z) = z/(1 + \mu z)$ with $|\mu| < 1$. In this case $zg'(z)/g(z) = 1/(1 + \mu z)$ and in order to satisfy condition (30) we need to find the largest $|\mu| < 1$ such that $w(z) = 1/(1 + \mu z) \in Q$ for all $z \in U$. The function w maps U onto a disc D_μ with center at $c = 1/(1 - |\mu|^2)$ and radius $R = |\mu|/(1 - |\mu|^2)$. Hence we have to find the largest $|\mu|$ so that $D_\mu \subset Q$. We have essentially solved this problem in the last example with formula (32). Using that formula and setting $r_c = R$ we obtain $|\mu| = 1/\sqrt{2}$. Hence if $|\mu| \leqslant 1/\sqrt{2}$ the operator I defined by

$$I[f](z) = \frac{1 + \mu z}{z} \int_0^z \frac{f(t)}{1 + \mu t}\, dt$$

is an averaging operator on H_0.

REFERENCES

1. A. V. BALAKRISHNAN, "Applied Functional Analysis," Springer-Verlag, New York, 1976.
2. A. W. GOODMAN, "Univalent Functions," Vol. 1, Mariner Publishing, Tampa, Florida, 1983.
3. R. J. LIBERA, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* **16** (1965), 755–758.
4. T. H. MACGREGOR, A subordination for convex functions of order α, *J. London Math. Soc.* **9** (1975), 530–536.
5. E. P. MERKES AND D. J. WRIGHT, On the univalence of a certain integral, *Proc. Amer. Math. Soc.* **27** (1971), 97–100.
6. S. S. MILLER AND P. T. MOCANU, Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.* **65** (1978), 289–305.
7. S. S. MILLER AND P. T. MOCANU, Differential subordinations and univalent functions, *Michigan Math. J.* **28** (1981), 157–171.
8. R. M. ROBINSON, Univalent majorants, *Trans. Amer. Math. Soc.* **61** (1947), 1–35.
9. K. SAKAGUCHI, On a certain univalent mapping, *J. Math. Soc. Japan* **2** (1959), 72–75.