

Averaging Operators and a Generalized Robinson Differential Inequality

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Let $\text{co } E$ denote the convex hull of a set E in \mathbb{C} , let H be the class of analytic functions defined in the unit disc U , and denote the subordination of functions $f, g \in H$ by $f \prec g$. This article deals with the theory of *averaging operators* defined on a set $K \subset H$. These are operators $I: K \rightarrow H$ that satisfy $I[f](0) = f(0)$ and $I[f](U) \subset \text{co } f(U)$ for all $f \in K$. Conditions for and examples of such operators are presented. The proofs of these results are dependent on determining dominants of the second-order differential subordination $Az^2p''(z) + B(z)z p'(z) + C(z)p(z) + D(z) \prec h(z)$. With suitable conditions on A, B, C, D , and h the authors show that $p \prec h$. As an additional application of this differential subordination the authors generalize a differential inequality of R. M. Robinson. © 1993 Academic Press, Inc.

1. INTRODUCTION

Let f and h be continuous on $J = [0, 1]$ with $h(x) \geq 0$ for $x \in J$. If $x \in J$ then by the First Mean-Value Theorem for Riemann Integrals there exists $c \in (0, x)$ such that

$$\int_0^x f(t) h(t) dt = f(c) \int_0^x h(t) dt. \tag{1}$$

If we let $h(x) = g'(x)$ with $g(0) = 0$, then (1) becomes

$$g(x) = \int_0^x f(t) g'(t) dt = f(c) g(x) \in f(J), \tag{2}$$

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for all $x \in J$. In Section 3 of this article we show that this mean-value theorem for integrals of real-valued functions defined on the unit interval J extends very naturally to complex-valued functions defined on the unit disc $U = \{z: |z| < 1\}$. If f and g are analytic on U , then with some simple conditions on f and g we obtain the following analog of (2),

$$g(z) \int_0^z f(w) g'(w) dw \in f(U), \quad (3)$$

for all $z \in U$. If we denote this integral operator by $I[f]$ then (3) can be rewritten as

$$I[f](U) \subset f(U). \quad (3')$$

Similar inclusion relations are proved for several operators besides the mean-value operator given in (3). These results, presented in Section 3, are dependent on the second-order differential subordination theorems of Section 2. Included in this group of differential subordinations is a generalization of the following differential inequality of R. M. Robinson [8]:

$$|T'(z)/S'(z)| < 1 \Rightarrow |T(z)/S(z)| < 1.$$

Since subordination is critical to several of the proofs, we now restate its definition. Let f and F be analytic in U . The function f is *subordinate* to F , written $f < F$ or $f(z) < F(z)$, if F is univalent, $f(0) = F(0)$, and $f(U) \subset F(U)$.

We close this section with a lemma that will be used to prove the main theorems of Section 2.

LEMMA 1. *Let p be analytic in U and let h be convex (univalent) on \bar{U} , with $p(0) = h(0)$. If p is not subordinate to h , then there exist $z_0 \in U$ and $w_0 \in \partial U$, and an $m \geq 1$ for which $p(|z| < |z_0|) \subset h(U)$,*

- (i) $p(z_0) = h(w_0)$,
- (ii) $z_0 p'(z_0) = m w_0 h'(w_0)$, and
- (iii) $\operatorname{Re}[z_0^2 p''(z_0)/w_0 h'(w_0)] \geq -m$.

A more general form of this lemma appears in [7, p. 158].

2. DIFFERENTIAL SUBORDINATIONS

In [7, p. 170] the authors showed that if an analytic function p satisfies the second-order differential subordination

$$Az^2 p''(z) + Bz p'(z) + Cp(z) < z,$$

where $A \geq 0$, $A + B \geq 0$, and $B + C > 0$, then $p(z) \prec z/(B + C)$. Our first theorem extends this result to a more general differential subordination of the form

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z). \tag{4}$$

THEOREM 1. *Let h be convex (univalent) in U with $h(0) = 0$, and let $A \geq 0$. Suppose that $k > 4/|h'(0)|$ and that $B(z)$, $C(z)$, and $D(z)$ are analytic in U and satisfy*

$$\operatorname{Re} B(z) \geq A + |C(z) - 1| - \operatorname{Re}[C(z) - 1] + k|D(z)|, \tag{5}$$

for $z \in U$. If p is analytic in U with $p(0) = 0$ and if p satisfies (4) then $p \prec h$.

Proof. Note that condition (4) implies that $D(0) = 0$. Since $k|h'(0)| > 4$, there is an r_0 , $0 < r_0 < 1$, such that $(1 + r_0)^2/r_0 = k|h'(0)|$ and

$$4 < (1 + r)^2/r < k|h'(0)|, \quad \text{for } r_0 < r < 1. \tag{6}$$

Since h is convex in U , if we set $h_r(z) = h(rz)$ for $r_0 < r < 1$, then h_r is convex on \bar{U} . If we also set $p_r(z) = p(rz)$, then from (4) we obtain

$$\begin{aligned} u_r(z) \equiv Az^2p_r''(z) + B(rz)zp_r'(z) \\ + C(rz)p_r(z) + D(rz) \prec h_r(z), \end{aligned}$$

for $z \in U$ and $r_0 < r < 1$. We will use Lemma 1 to show that $p_r(z) \prec h_r(z)$ for $r_0 < r < 1$.

Assume that p_r is not subordinate to h_r for some r in $(r_0, 1)$. Then by Lemma 1 there exist points $z_0 \in U$ and $w_0 \in U$, and an $m \geq 1$ such that

$$\begin{aligned} p_r(z_0) = h_r(w_0), \quad z_0 p_r'(z_0) = mw_0 h_r'(w_0), \\ \operatorname{Re}[z_0^2 p_r''(z_0)/w_0 h_r'(w_0)] \geq -m. \end{aligned} \tag{8}$$

If we set $V \equiv [u_r(z_0) - h_r(w_0)]/w_0 h_r'(w_0)$ then from (7) we have

$$\begin{aligned} V = \frac{Az_0^2 p_r''(z_0)}{w_0 h_r'(w_0)} + \frac{B(rz_0) z_0 p_r'(z_0)}{w_0 h_r'(w_0)} \\ + \frac{C(rz_0) p_r(z_0) - h_r(w_0)}{w_0 h_r'(w_0)} + \frac{D(rz_0)}{w_0 h_r'(w_0)} \end{aligned} \tag{9}$$

with

$$u_r(z_0) = h_r(w_0) + Vw_0 h_r'(w_0). \tag{10}$$

We first show that $\operatorname{Re} V > 0$, which will then lead to the desired contradiction. Since h_r is convex and $h_r(0) = 0$ we have $\operatorname{Re}[w_0 h'_r(w_0)/h_r(w_0)] \geq 1/2$ for $|w_0| = 1$ [2, p. 176], or equivalently

$$|h_r(w_0)/w_0 h'_r(w_0) - 1| \leq 1. \quad (11)$$

If W and Z are complex numbers and $|Z - 1| \leq 1$ then

$$\operatorname{Re} WZ = \operatorname{Re} W + \operatorname{Re} W(Z - 1) \geq \operatorname{Re} W - |W|.$$

Using this inequality with $W = C(rz_0) - 1$ and $Z = h_r(w_0)/w_0 h'_r(w_0)$, we obtain from (11) that

$$\begin{aligned} & \operatorname{Re}[(C(rz_0) - 1) h_r(w_0)/w_0 h'_r(w_0)] \\ & \geq \operatorname{Re}[C(rz_0) - 1] - |C(rz_0) - 1|. \end{aligned} \quad (12)$$

Since h is convex we have $|h'(z)| \geq |h'(0)|/(1+r)^2$ for $|z| = r < 1$ [2, p. 118]. By setting $z = rw_0$ we obtain

$$|w_0 h'_r(w_0)| \geq r |h'(0)|/(1+r)^2, \quad \text{for } |w_0| = 1. \quad (13)$$

If we use (5), (8), (12), and (13) in (9) we obtain

$$\begin{aligned} \operatorname{Re} V & \geq A(-m) + \operatorname{Re}[B(rz_0)]m \\ & \quad + \operatorname{Re}[(C(rz_0) - 1) h_r(w_0)/w_0 h'_r(w_0)] \\ & \quad - |D(rz_0)/w_0 h'_r(w_0)| \\ & \geq (m-1)[|C(rz_0) - 1| - \operatorname{Re}(C(rz_0) - 1)] \\ & \quad + m[k - (1+r)^2/r |h'(0)|] |D(rz_0)|. \end{aligned}$$

From (6) and the fact that $m \geq 1$ we obtain $\operatorname{Re} V > 0$, or equivalently $|\arg V| < \pi/2$. Using this in (10) together with the fact that $w_0 h'_r(w_0)$ is the outward normal to the boundary of the convex domain $h_r(U)$, we obtain $u_r(z_0) \notin h_r(U)$. Since this contradicts (7) we must have $p_r < h_r$ for all $r_0 < r < 1$. By letting $r \rightarrow 1$ we obtain the desired conclusion $p(z) < h(z)$.

In the special case $C(z) \equiv 1$, the condition $h(0) = p(0) = 0$ can be replaced by $h(0) = p(0)$. In this case the proof of Theorem 1 still holds and we obtain the following result.

THEOREM 2. *Let h be convex (univalent) in U and let $A \geq 0$. Suppose $k > 4/|h'(0)|$ and that B and D are analytic in U , with $D(0) = 0$ and*

$$\operatorname{Re} B(z) \geq A + k |D(z)|, \quad (14)$$

for $z \in U$. If p is analytic in U with $p(0) = h(0)$, and if p satisfies

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) < h(z)$$

then $p(z) < h(z)$.

Let S and T be analytic in U with $S(0) = T(0) = 0$. In addition, suppose that S is starlike (not necessarily univalent) in U , that is, $\operatorname{Re} zS'(z)/S(z) > 0$. In 1947 R. M. Robinson [8, p. 30] proved that

$$|T'(z)/S'(z)| < 1 \text{ for } z \in U \Rightarrow |T(z)/S(z)| < 1 \text{ for } z \in U. \tag{15}$$

In 1965 R. J. Libera [3, p. 755] proved that

$$\operatorname{Re} T'(z)/S'(z) > 0 \text{ for } z \in U \Rightarrow \operatorname{Re} T(z)/S(z) > 0 \text{ for } z \in U. \tag{16}$$

K. Sakaguchi [9, p. 74] also proved this result but with the additional assumption that S is univalent.

In 1973 T. H. MacGregor [4, p. 532] and in 1978 the authors [6, p. 301] extended Libera's result and for real γ proved that

$$\operatorname{Re} T'(z)/S'(z) > \gamma \text{ for } z \in U \Rightarrow \operatorname{Re} T(z)/S(z) > \gamma \text{ for } z \in U. \tag{17}$$

All these results concerning differential inequalities are special cases of the following theorem on differential subordinations.

THEOREM 3. *Let h be convex (univalent) in U , and let R, S , and T be analytic in U with $R(0) = S(0) = T(0) = 0$. If*

$$\operatorname{Re} S(z)/zS'(z) > 4 |R(z)|/|h'(0)|, \tag{18}$$

for $z \in U$, then

$$T'(z)/S'(z) + R(z) < h(z) \Rightarrow T(z)/S(z) < h(z). \tag{19}$$

Proof. From (18) we obtain $\operatorname{Re} zS'(z)/S(z) > 0$ which implies that S maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin. If we let $p = T/S$, then p is analytic in U and satisfies

$$[S(z)/zS'(z)] zp'(z) + p(z) = T'(z)/S'(z).$$

Since $T'/S' + R < h$, if we set $B(z) = S(z)/zS'(z)$ and $D = R$ we obtain

$$B(z) zp'(z) + p(z) + D(z) < h(z),$$

while (18) becomes $\operatorname{Re} B(z) > 4 |D(z)|/|h'(0)|$. The conditions of Theorem 2 with $A = 0$ are satisfied, and from that result we obtain $p = T/S < h$.

In the special case $S(z) = z$, (18) simplifies to $|R(z)| < |h'(0)|/4$ and from (19) we obtain

$$T'(z) + R(z) < h(z) \Rightarrow T(z)/z < h(z).$$

As an example, if $h'(0) = 1$ and $|\alpha| < 1/4$ we have

$$T'(z) + \alpha z < h(z) \Rightarrow T(z)/z < h(z).$$

If we take $R(z) = 0$ in Theorem 3 we obtain:

COROLLARY 3.1. *Let h be convex (univalent) in U , and let S and T be analytic in U with $S(0) = T(0) = 0$. If $\operatorname{Re} zS'(z)/S(z) > 0$, then*

$$T'(z)/S'(z) < h(z) \Rightarrow T(z)/S(z) < h(z). \quad (20)$$

Remarks. 1. If the function $T'(z)/S'(z)$ is itself a convex function, (20) can be replaced by $T/S < T'/S'$.

2. We can eliminate the function h in the corollary and extend the result by using the concept of the convex hull of a set E (denoted by $\operatorname{co} E$) in \mathbf{C} . In this extension, (20) is replaced by

$$\{T(z)/S(z) : z \in U\} \subset \operatorname{co}\{T'(z)/S'(z) : z \in U\}. \quad (20')$$

To check the validity of this we need to consider the cases $\operatorname{co}\{T'/S'\} = \mathbf{C}$ and $\operatorname{co}\{T'/S'\} \neq \mathbf{C}$. In the first case (20') trivially holds, while in the second case there is a convex (univalent) function h mapping U onto $\operatorname{co}\{T'/S'\}$ with $h(0) = T'(0)/S'(0)$, and (20') follows from (20).

3. In the special case when S is also univalent then the corollary coincides with a result of E. Merkes and D. Wright [5, p. 98].

4. We can obtain (15), (16), and (17) by applying (20) with $h(z) = z$, $h(z) = (1+z)/(1-z)$, and $h(z) = [1 + (1-2\gamma)z]/(1-z)$, respectively. As another example, consider the domain D given by $|\arg z| < \alpha\pi/2$, where $0 < \alpha < 1$. The function $h(z) = [(1+z)/(1-z)]^\alpha$ maps U conformally onto D and by the corollary we obtain

$$\frac{T'(z)}{S'(z)} < \left[\frac{1+z}{1-z} \right]^\alpha \Rightarrow \frac{T(z)}{S(z)} < \left[\frac{1+z}{1-z} \right]^\alpha,$$

or equivalently

$$|\arg T'(z)/S'(z)| < \alpha\pi/2 \Rightarrow |\arg T(z)/S(z)| < \alpha\pi/2.$$

3. AVERAGING OPERATORS

In this section we denote by H the set of functions that are analytic in U , we let $H_0 = \{h \in H: h(0) = 0\}$, and when we refer to a convex function we shall assume that it is univalent.

The first result in this section involves an integral analog of Theorem 3.

THEOREM 4. *If h is convex in U , $f \in H$, and $g, w \in H_0$ with $\operatorname{Re} g(z)/rg'(z) > 4 |w(z)|/|h'(0)|$, then*

$$f(z) \prec h(z) \Rightarrow F(z) = g(z)^{-1} \int_0^z [f(t) + w(t)] g'(t) dt \prec h(z). \tag{21}$$

In particular, if f is convex and $\operatorname{Re} g(z)/zg'(z) > 4 |w(z)|/|f'(0)|$ then

$$I[f](z) = F(z) = g(z)^{-1} \int_0^z [f(t) + w(t)] g'(t) dt \prec f(z). \tag{22}$$

Proof. If we set $T(z) = \int_0^z [f(t) + w(t)] g'(t) dt$, $R(z) = -w(z)$, and $S(z) = g(z)$, the $T'/S' + R = f \prec h$ and all the conditions of Theorem 3 are satisfied. Hence we conclude that $F = T/S \prec h$. The subordination (22) follows from (21) by taking $h = f$.

EXAMPLE 1. If we let $h(z) = (1+z)/(1-z)$ then $|h'(0)| = 2$ and we obtain: if $f \in H$ with $f(0) = 1$, and if $w \in H_0$ with $|w(z)| < 1/2$ then

$$\operatorname{Re} f(z) > 0 \Rightarrow \operatorname{Re} \left[z^{-1} \int_0^z [f(t) + w(t)] dt \right] > 0.$$

If $w(z) \equiv 0$ in Theorem 4 then the function h can be eliminated and the result can be extended by considering the convex hull of $f(U)$. The proof of the following result follows from considering the cases $\operatorname{co} f(U) \neq \mathbf{C}$. For the first case there exists a convex function $h: U \rightarrow \operatorname{co} f(U)$ with $h(0) = f(0)$ and the proof follows from Theorem 4, while the second case is trivial.

COROLLARY 4.1. *Let $g \in H_0$ with $\operatorname{Re} zg'(z)/g(z) > 0$. If $f \in H$ and if I is defined by*

$$I[f](z) = g(z)^{-1} \int_0^z f(t) g'(t) dt, \tag{23}$$

then

$$I[f](U) \subset \operatorname{co} f(U). \tag{24}$$

In addition, if f is convex then

$$I[f](U) \subset f(U). \tag{25}$$

The operator I given in (23) is the same as the operator given in (3), which is the analog of the mean-value theorem of real analysis. The inclusion relation given in (25) indicates that I has the mean-value property as required in (3').

EXAMPLE 2. If we let $g(z) = z^\alpha$ in Corollary 4.1 we obtain: if $f \in H$ and $\operatorname{Re} \alpha > 0$ then the operator

$$I_\alpha[f](z) = \alpha z^{-1} \int_0^z f(t) t^{\alpha-1} dt$$

satisfies (24). Operators of this particular type are discussed in [1, p. 85] and are referred to as *averaging operators*. In the special case when $\alpha = 1$ we obtain the classical *average value of f*

$$I_1[f](z) = z^{-1} \int_0^z f(t) dt.$$

In light of the above remarks we make the following definition:

DEFINITION 1. If $K \subset H$ and if an operator $I: K \rightarrow H$ satisfies

$$I[f](0) = f(0) \quad \text{and} \quad I[f](U) \subset \operatorname{co} f(U)$$

for all $f \in K$, then I is said to be an *averaging (or mean-value) operator on K*.

From the last corollary we see that the integral operator I given in (23) is an averaging operator on H . Because of (22) the operator described therein is an averaging operator on the subset of convex functions f for which $|f'(0)| > 4 |w(z)| / \operatorname{Re}[g(z)/zg'(z)]$.

The following lemma provides a simple characterization of averaging operators.

LEMMA 2. Let $K \subset H$ and let an operator $I: K \rightarrow H$ satisfy $I[f](0) = f(0)$ for all $f \in K$. A necessary and sufficient condition for I to be an averaging operator on K is that

$$\{f \in K, h \text{ convex and } f \prec h\} \Rightarrow I[f] \prec h. \quad (26)$$

Proof. If (26) is satisfied then there are two cases to consider. If $\operatorname{co} f(U) = \mathbf{C}$ then $I[f](U) \subset \operatorname{co} f(U)$. If $\operatorname{co} f(U) \neq \mathbf{C}$ then let h be a convex function mapping U onto $\operatorname{co} f(U)$ with $h(0) = f(0)$. Using this h in (26) we obtain $I[f](U) \subset \operatorname{co} f(U)$, which again shows that I is an averaging operator.

If I is an averaging operator then (26) follows since $f \prec h$ implies that $I[f](U) \subset \operatorname{co} f(U) \subset h(U)$.

We can not use this lemma to conclude that the operator given in (21) is an averaging operator on H . This is due to the fact that the implication in (21) does not hold for all convex h ; the operator itself depends on h . We can remove this dependency in special cases. Since $f \prec h$ implies $|f'(0)| \leq |h'(0)|$, if we take $g(z) = z$ and $w(z) = 2\beta f'(0)z$, with $|\beta| < 1/8$, then we can use (21) and Lemma 2 to obtain the following class of averaging operators.

COROLLARY 4.2. *If $f \in H$ and $|\beta| \leq 1/8$ then the integral operator I , defined by*

$$I[f](z) = z^{-1} \int_0^z (f(t) + 2\beta f'(0)t) dt = z^{-1} \int_0^z f(t) dt + \beta f'(0)z$$

is an averaging operator on H .

Our last theorem provides us with a large group of averaging operators on H_0 .

THEOREM 5. *Let $\gamma \in \mathbb{C}$ with $\gamma \neq -1, -2, \dots$, and let $\phi, \Phi \in H$ with $\phi(z) \cdot \Phi(z) \neq 0$ for $z \in U$. If*

$$\operatorname{Re} B(z) \geq |C(z) - 1| - \operatorname{Re} [C(z) - 1], \quad z \in U, \tag{27}$$

where $B(z) = \Phi(z)/\phi(z)$ and $C(z) = (\gamma\Phi(z) + z\Phi'(z))/\phi(z)$, then the integral operator I defined by

$$I[f](z) = z^{-\gamma} \Phi(z)^{-1} \int_0^z f(t) t^{\gamma-1} \phi(t) dt \tag{28}$$

is an averaging operator on H_0 .

Proof. The restrictions on γ, ϕ , and Φ imply that $I[f]$ is analytic on U and $I[f](0) = 0$. Since $f \prec h$, if we let $p = I[f]$ and differentiate (28) we obtain

$$B(z) z p'(z) + C(z) p(z) = f(z) \prec h(z).$$

We now apply Theorem 1 with $A = 0$ and $D(z) = 0$. Since (27) implies (5) we obtain $p = I[f] \prec h$. Hence by Lemma 2 the operator I is an averaging operator on H_0 .

EXAMPLE 3. If we set $\gamma = 0, \phi(z) \equiv 1/2$, and $\Phi(z) \equiv 1$ in Theorem 5 then $B(z) \equiv 2, C(z) \equiv 0$, and (28) is satisfied. Hence the operator

$$I[f](z) = \frac{1}{2} \int_0^z \frac{f(t)}{t} dt$$

is an averaging operator on H_0 , that is $I[f](U) \subset \operatorname{co} f(U)$.

This last operator is very similar to the classical Alexander operator

$$J[f](z) = \int_0^z \frac{f(t)}{t} dt.$$

However, this operator is not an averaging operator on H_0 . If we take $f(z) = z + z^2/4$ and set $F(z) \equiv J[f](z)$ then $F(z) = z + z^2/8$. Since the functions f and F are convex and since $F(-1) < f(-1) < 0 < F(1) < f(1)$ we have $J[f](U) = F(U) \not\subset f(U) = \text{co } f(U)$.

In the special case $\phi(z) \equiv \Phi(z) \equiv 1$ Theorem 5 reduces to the following corollary.

COROLLARY 5.1. *If $\gamma \in \mathbb{C}$ and γ satisfies*

$$|\gamma - 1| \leq \text{Re } \gamma, \quad (29)$$

then $I[f](z) = z^{-\gamma} \int_0^z f(t) t^{\gamma-1} dt$ is an averaging operator on H_0 .

For the remainder of this article it is convenient to introduce the set $Q = \{w = x + iy : |w - 1| \leq \text{Re } w\}$. Note that (29) is equivalent to $\gamma \in Q$ and that Q consists of points inside or on the parabola $2x = y^2 + 1$.

COROLLARY 5.2. *Let $\gamma \in \mathbb{C}$ and let $g \in H_0$ with $g'(0) = 1$. If*

$$|\gamma z g'(z)/g(z) - 1| \leq \text{Re}(\gamma z g'(z)/g(z)) \quad (30)$$

for $z \in U$, then I defined by

$$I[f](z) = [g(z)]^{-\gamma} \int_0^z f(t) t^{-1} [g(t)]^{\gamma} dt$$

is an averaging operator on H_0 .

Proof. Condition (30) is equivalent to $z g'(z)/g(z) \in Q$, which for $z = 0$ also implies $\gamma \in Q$. In addition, (30) implies that $\text{Re } \gamma z g'(z)/g(z) > 0$. Since $g'(0) = 1$, the function g is a spirallike univalent function [2, p. 149]. Since $g(z)/z \neq 0$, if we set $\phi(z) = \Phi(z) = (g(z)/z)^{\gamma}$ then $\phi, \Phi \in H$ and $\phi(z) \cdot \Phi(z) \neq 0$ for $z \in U$. Using these values of ϕ and Φ in Theorem 5 we see that (30) implies (27), and hence the operator I is an averaging operator on H_0 .

We close this article by considering two examples of this last corollary.

EXAMPLE 4. Let γ be real with $\gamma \geq 1/2$ and let $g(z) = ze^{\mu z/\gamma}$. In this case $\gamma z g'(z)/g(z) = \gamma + \mu z$ and condition (30) is equivalent to requiring

$$\gamma + \mu z \in Q, \quad \text{for } z \in U. \quad (31)$$

We need to find the maximum value of $|\mu|$ such that, for a fixed value of γ , condition (31) holds. This problem is equivalent to finding the maximum radius r_γ of a circle with center at γ , which is contained in Q . The solution to this simple calculus problem is given by

$$r = \begin{cases} \gamma - 1/2 & \text{if } 1/2 \leq \gamma \leq 3/2, \\ \sqrt{2(\gamma - 1)} & \text{if } 3/2 < \gamma. \end{cases} \quad (32)$$

Hence, if $|\mu| \leq r_\gamma$, then by Corollary 5.2 the operator I defined by

$$I[f](z) = z^{-\gamma} e^{-\mu z} \int_0^z f(t) t^{\gamma-1} e^{\mu t} dt$$

is an averaging operator on H_0 .

EXAMPLE 5. Let $\gamma = 1$ and let $g(z) = z/(1 + \mu z)$ with $|\mu| < 1$. In this case $zg'(z)/g(z) = 1/(1 + \mu z)$ and in order to satisfy condition (30) we need to find the largest $|\mu| < 1$ such that $w(z) = 1/(1 + \mu z) \in Q$ for all $z \in U$. The function w maps U onto a disc D_μ with center at $c = 1/(1 - |\mu|^2)$ and radius $R = |\mu|/(1 - |\mu|^2)$. Hence we have to find the largest $|\mu|$ so that $D_\mu \subset Q$. We have essentially solved this problem in the last example with formula (32). Using that formula and setting $r_c = R$ we obtain $|\mu| = 1/\sqrt{2}$. Hence if $|\mu| \leq 1/\sqrt{2}$ the operator I defined by

$$I[f](z) = \frac{1 + \mu z}{z} \int_0^z \frac{f(t)}{1 + \mu t} dt$$

is an averaging operator on H_0 .

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