The further discussion of collineations, as well as the group properties of collineations and motions, will be presented in a later paper.

[1] Finsler, P., Dissertation, Göttingen, 1918.

[2] Synge, J. L., *Trans. Amer. Math. Soc.*, **27**, 1925, pp. 61–67.

[3] Taylor, J. H., *Trans. Amer. Math. Soc.*, **27**, 1925, pp. 246–264.

[4] Berwald, L., *Math. Zs.*, **25**, 1926, pp. 40–73.

[5] Eisenhart, L. P., *Riemannian Geometry*, **1926**, p. 234.

[6] Knebelman, M. S., these PROCEEDINGS, **13**, 1927, pp. 396–400. Also Eisenhart, L. P., and Knebelman, M. S., *Ibid.*, **13**, 1927, pp. 38–42.

[7] Veblen, O., and Thomas, T. Y., *Trans. Amer. Math. Soc.*, **25**, 1923, pp. 551–608.

FELIX KLEIN AND THE HISTORY OF MODERN MATHEMATICS

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For more than a quarter of a century Felix Klein was a Foreign Associate of this Academy, and during this period no other foreign mathematician exerted a greater influence on the development of mathematics in our country. This influence extended from investigations relating to various advanced branches of this subject to the teaching and history of the elementary parts thereof. His method of work was in unusually close accord with the motto of a younger large American research organization, viz., "Companions in Zealous Research." He imparted his fruitful notions very freely, especially to his students, and it is now impossible in many cases to determine just what was contributed by Klein himself. Even when his "Collected Works" were prepared for the press he discussed various parts thereof in his University lectures and then added thereto illuminating notes arising from these discussions.

His own work illustrates the fundamental principle of the history of mathematics that the great forward movements are due more and more to collective efforts, and that one of the most effective services which an individual can render is to point out resultants of such collective forces and to work in harmony therewith. This is the work of the scientific statesman, and in the mathematical world Klein was preëminently such a statesman. He took a leading part in an effort to collect and coördinate the extensive body of mathematical advances in the form of large encyclopedias, and during the last few years of his life he gave courses of lectures on the development of mathematics during the nineteenth century. While his work along this line was left in an incomplete form it was in a sufficiently advanced state to exhibit his views on the general methods which should be pursued in such a history. He aimed to popularize mathematics even at the risk of a kind of *pia fraus*, due to making the subject appear easier than it really is. Just as in the case of his treatment of elementary mathematics, he proceeded in his history mainly from the higher point of view instead of toward the higher point of view in the sense of first laying an elaborate basis on which a superstructure leading to this point of view might be erected.

In view of the fact that a treatment of the history of mathematics since the beginning of the nineteenth century presents such great difficulties Klein's method of dealing with this subject will doubtless receive much attention. What is still more important is that he expressed approval of separate efforts along this line by his own example. The mathematical work of the last century has furnished many evidences of the fact that the successful investigator in one field is not necessarily familiar with many of the other fields. Mathematics is not a world empire where one man can exert a dominating influence over work along all the lines, but it is continually splitting up into self-determining republics. There is a growing need for conferences dealing with common affairs of such republics and these common affairs must constitute one of the main elements of a general history of modern mathematics if such a history is to establish closer contact not only between the mathematicians engaged in widely different fields of research but also between the mathematicians and other scientists. The preservation of such contact is largely dependent upon the recognition of serious efforts along this line.

In the introduction to his posthumous work entitled *Vorlesungen über die Entwicklung der Mathematik im* **19**, *Jahrhundent,* 1926, Klein noted that this history did not aim to take the place of a brief mathematical encyclopedia in which the main divisions of the subject are selected and presented in a systematic form. He aimed much more to give selected sketches of the work of eminent individuals and of the purposes and results of definite schools. He explicitly disclaimed completeness of any kind whatsoever, and stated that he did not expect to make minute preparatory studies relating to the subjects treated. He aimed merely to give a tolerably accurate presentation.

These modest claims were doubtless inspired by the fact that a history of the development of mathematics during the nineteenth century from such a standpoint would serve a very useful purpose, since it would enable the reader to see something of the nature and purposes of modern mathematical work without making heavy demands on his attainments. On the other hand, it would naturally inspire others to supply desirable additions and modifications, and thus be conducive to mathematical progress. Such a modification was suggested already by the editors of the work in question in a note to the introduction thereof. It is stated here that one finds frequently in the writings of Euler remarks which aim to awaken on the part of the reader the desire to advance by his own efforts beyond the work in hand, while Klein claimed that such remarks belonged entirely to writings of the nineteenth century, and referred to the writings of Monge, Jacobi and Faraday as illustrative of this point.

A somewhat similar modification applies to the fundamental statement found on page 84 of the same work to the effect that the arithmetization of mathematics was begun by A. L. Cauchy. In an article which appeared in these PROCEEDINGS[1] about two years ago it was noted that this arithmetization dominated various fundamental mathematical developments since the times of the ancient Greeks. In fact, it appears already in the arithmetical books of Euclid's "Elements." Recently a well-known German mathematical historian supported this view.[2] Such principles of the history of mathematics should be formulated as accurately as possible since they enable the student to associate harmoniously many historical facts, and thus to simplify greatly his mastery thereof. Moreover, the various illustrations of the same general principle tend to deepen insight into the nature of the subject, and such deepening is naturally attended by increasing zest. It, therefore, seemed desirable to refer here to the exact status of the principle in question, especially since it has not yet been widely accepted.

Klein's work relating to the history of mathematics is by no means confined to the lectures on this subject to which we referred above. About twenty years before the publication of these lectures one of his students presented a doctor's dissertation on the history of mathematics at Göttingen. Moreover, Klein frequently referred to fundamental historical questions in his writings. For instance, in an article entitled "Grenzfragen der Mathematik und Philosophie," 1906, he said that in common with all the other scientists the mathematicians had begun to doubt everything that was supposed until then to have been fully established, and that everything was then in fermentation. Such doubt obviously relates only to questions of postulates. Otherwise, the mathematician still has the comfortable feeling that many of the contributions which he is able to make will abide, especially if wide contact can be maintained. In view of the common interest in the origin of useful ideas the history of mathematics seems now to present one of the most effective avenues for maintaining such contact, and as Klein was especially interested in this contact he naturally devoted much thought to this subject during the closing years of his influential life. As scientific life becomes more complex the old avenues of intercommunication should not only be improved but new ones should be sought.

[1] G. A. Miller, *Proc. N. A. S.*, **11**, 546 (1925).

[2] H. Wieleitner, *Jahresbericht D. Math. Vr.*, **36**, 74 (1927).