ARITHMETIZATION IN THE HISTORY OF MATHEMATICS

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Arithmetization is doubtlessly the most dominating principle in the history of mathematics and hence this history can be studied most effectively by keeping this principle in the foreground. In particular, the ancient Greeks developed a geometrical algebra which the later Greeks themselves began to arithmetize as may be seen in the *Arithmetica* of Diophantus. The Indian mathematicians extended this arithmetization, especially as regards the use of negative numbers, while the Arabs brought the algebraic equation into greater prominence; in particular, by reducing the Archimedean problem to the solution of a cubic equation. On the other hand, they extended the geometric algebra of the Greeks in their successful efforts to find one root of this equation geometrically in the important case when it has a positive root. The extension of the Greek geometrical algebra thus established was later arithmetized by Italian mathematicians during the sixteenth century by the discovery of a general formula which they could use for the algebraic determination of one root of the cubic equation when it has one positive and two imaginary roots.

The Italian mathematicians began about the same time a much more fruitful arithmetization by the occasional use of complex numbers. This use was suggested, in particular, by the fact that when a cubic equation has three positive roots their formula, due to Ferro (1465–1526), but commonly known as *Cardan's formula*, presents these roots in the form of complex numbers. In fact, this is always the case when the three roots are real numbers, but negative roots were usually excluded by the mathematicians of the sixteenth century, and hence the fact that they also appeared in the form of complex numbers did not arouse much interest at that time. Even the noted English mathematician T. Harriot (1560–1621) thought he could prove that an algebraic equation could have only positive roots. Cf. *Artis analyticae praxis*, page 89. This is the more noteworthy because Harriot was not only a renowned mathematician but he was also prominently connected with the early settlement of our country, having surveyed and mapped the region now known as North Carolina.

Even the theory of the ordinary complex numbers, which were used tentatively from about the middle of the sixteenth century until the close of the eighteenth century when they were placed on a firm basis by Caspar Wessel and others, was arithmetized about thirty years later by means of a theory of number couples due to the noted Irish mathematician W. R. Hamilton (1805–1865). This arithmetic theory was greatly extended later by means of the modular systems due to L. Kronecker, who emphasized the arithmetic foundation of the entire field of mathematics by the following words: "God made the integers, all else is the work of man." Cf. *Juhresbericht der deutschen Mathematiker-Vereinigung*, volume 2, 1893, page 19. The extent to which algebra had been arithmetized by the close of the nineteenth century is clearly exhibited in the standard work entitled *Lehrbuch der Algebra* by H. Weber.

The arithmetization of trigonometry is most clearly seen in the change as regards the definitions of the elementary trigonometric functions. With possibly a few isolated exceptions these functions were regarded as line segments up to the time of L. Euler (1707–1783). What is more important is the fact that they were not usually regarded up to this time as functions of a single variable, viz., the magnitude of the angle, but as functions of two variables, composed of this magnitude and the length of the radius of the circle or the hypotenuse of the triangle with respect to which the angle was considered. In fact, the main advantage which the arithmetization of the definitions of the elementary trigonometric functions entailed was due to the fact that these functions thereby became functions of a single variable. It may be added that even Euler did not define the elementary trigonometric functions as abstract numbers but he obviously used them with this meaning and this usage was soon followed by our modern definitions. Such definitions began to appear before the death of Euler but they were not firmly established until after the beginning of the nineteenth century.

It is interesting to note that the definitions of the elementary trigonometric functions as abstract numbers entailed the use of negative numbers when angles larger than 90° were considered. The imperfect knowledge as regards negative numbers even in the eighteenth century doubtless retarded the arithmetization of the definitions of the elementary trigonometric functions, and the usefulness of these numbers for this arithmetization helped to secure for them a permanent and prominent position among the elements of our subject. Even before the use of negative numbers was firmly established in elementary trigonometry the complex numbers began to enter the more advanced parts thereof. In particular, in 1740, Euler communicated the elegant formula $2 \cos x = e^{ix} + e^{-ix}$ to John Bernoulli. There was no generally accepted theory of the elementary operations with negative numbers until the more general subject of operating with the ordinary complex numbers had been placed on a firm basis.

The history of the arithmetization of analytic geometry is, as might have been expected, much more extensive than that of the arithmetization of trigonometry. It is frequently said that systems of coördinates are ancient, but these ancient systems had to be arithmetized. What is now generally understood by a system of coördinates, i.e., a system in which pairs of real numbers with their proper signs represent points in the plane, and triplets of such numbers represent points in space, is comparatively modern, having been slowly introduced after the time of R. Descartes (1596–1650). "Descartes, Fermat and their successors regarded like the ancients their coördinates as line segments, which had a geometric connection with the separate points of the curves under consideration. Only since Euler, Lagrange and Monge does the purely analytic character of the chosen coördinates present itself more clearly." J. Tropfke, *Geschichte der Elementar-Mathematik*, volume 6, 1924, page 122.

It should be noted that while the arithmetization of systems of coördinates entailed the equivalence of the coördinates as well as the use of negative coördinates, this equivalence and this use were not dependent upon the arithmetical character of the systems of coördinates. In fact, negative coördinates were employed correctly as early as 1656 by J. Wallis, and the equivalence of the four quadrants was implied in I. Newton's *Enumeratio linearum tertii ordinis*, 1704, but many later writers did not adopt these advanced views. It is a matter of fundamental importance in the history of mathematics to determine when certain modern views became so well known that practically all of the best writers on the subject adopted them. Earlier isolated cases of such views are also of great interest, and instances where later writers deviated from these views are often instructive, but such isolated cases are usually of secondary importance. Barring such cases it may be said that the arithmetization of coördinate systems was effected about the same time as the arithmetization of the definitions of the elementary trigonometric functions; i.e., about the close of the eighteenth century.

The arithmetizations to which we have referred relate to fundamental subjects and hence they imply arithmetizations in the more advanced subjects based thereon. The main object of the present note is to direct attention to a unifying principle in the history of mathematics, and to cite a sufficient number of instances where it may be used to advantage to exhibit its value even in the study of the history of elementary mathematics. Incidentally the importance of accurate definitions of some of the terms used in historical discussions is also illustrated, for it is practically futile to state when systems of coördinates were first used without noting at the same time what is to be understood by such a system. Similar remarks apply to many other historical questions relating to the development of mathematics. Without such explanations the technical terms should be used only with their generally accepted modern meanings.