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Letter to the Editor

On a Kummer-type transformation for the generalized hypergeometric function ${}_2F_2$

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Abstract

A Kummer-type transformation formula for the generalized hypergeometric function ${}_2F_2$ deduced by Exton is rederived in two simple and transparent ways.

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Several years ago Exton [1] deduced four new reduction formulas for the Kampé de Fériet function all of which are unfortunately garbled, but retrievable. One of these results is (cf. [1, Eq. (8)])

$$F^{p:2;0}_{q:2;0}\left[\begin{matrix}(a_p): & a, 1+\frac{1}{2}a & ; & — & ; \\ (b_q): & b, \frac{1}{2}a & ; & — & ;\end{matrix}\, y, -y\right]$$
$$= {}_{p+2}F_{q+2}\left[\begin{matrix}(a_p), b-a-1,\ 2+a-b & ; \\ (b_q), b,\ 1+a-b & ;\end{matrix}\, -y\right], \tag{1}$$

where (a_p) represents the sequence of parameters $a_1, a_2, \ldots, a_p$ and each member of Eq. (1) either converges or terminates. (See, for example, [6] for an introduction to generalized hypergeometric and Kampé de Fériet functions and [5, pp. 28–32] for a listing of many other reducible cases of the latter.) Furthermore, Exton noted that when $p = q = 0$ (or equivalently when $(a_p) = (b_q)$), then

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Eq. (1) provides the transformation formula (cf. [1, Eq. (12)])

$$ {}_2F_2\left[\begin{matrix} a,\ 1+\frac{1}{2}a & ; \\ b,\ \frac{1}{2}a & ; \end{matrix}\ y\right] = \mathrm{e}^{y}{}_2F_2\left[\begin{matrix} b-a-1,\ 2+a-b & ; \\ b,\ 1+a-b & ; \end{matrix}\ -y\right]. \tag{2} $$

The latter result is an analog and as we shall see actually a corollary of the so-called Kummer's first transformation formula (1836–1837) for the confluent hypergeometric function:

$$ {}_1F_1\left[\begin{matrix} a; \\ b; \end{matrix}\ y\right] = \mathrm{e}^{y}{}_1F_1\left[\begin{matrix} b-a & ; \\ b & ; \end{matrix}\ -y\right]. \tag{3} $$

With respect to its analog, Eq. (3), which has been known for over a century and a half, it is noteworthy that the result given by Eq. (2) has apparently not appeared in the literature prior to 1997 and certainly deserves further attention. (See [2, Section 7.12, p. 585] for an extensive list of identities for the generalized hypergeometric function ${}_2F_2$ known up to 1990.) Thus, we shall give below two simple and transparent derivations of Eq. (2) that do not rely on Eq. (1) which contains a Kampé de Fériet function whose notation can be daunting to the uninitiated.

Following Rainville [3, p. 124] we have

$$ \mathrm{e}^{-y}{}_2F_2\left[\begin{matrix} a,c; \\ b,d; \end{matrix}\ y\right] = \sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix} -n,a,c & ; \\ b,d & ; \end{matrix}\ 1\right] \frac{(-y)^n}{n!}, \tag{4} $$

which is easily obtained by using series rearrangement. (See also [6, Eq. (19), p. 141]) for a generalization of Eq. (4).) Now set $c = 1+\frac{1}{2}a$ and $d = \frac{1}{2}a$ so that

$$ {}_3F_2\left[\begin{matrix} -n,a,1+\frac{1}{2}a & ; \\ b,\frac{1}{2}a & ; \end{matrix}\ 1\right] = \frac{(b-a-1)_n(2+a-b)_n}{(b)_n(1+a-b)_n}, \tag{5} $$

(cf. [2, Eq. (106), p. 540]) which is also used in [1] to obtain Eq. (1). Thus it is evident that Eqs. (4) and (5) yield Eq. (2).

Finally, we show in an elementary way that Eq. (2) is essentially a consequence of Kummer's first transformation formula Eq. (3). Thus, since $(1+\alpha)_n/(\alpha)_n = 1+n/\alpha$ letting respectively $\alpha = \frac{1}{2}a$, $\alpha = 1+a-b$ an easy computation reveals that

$$ {}_2F_2\left[\begin{matrix} a,1+\frac{1}{2}a & ; \\ b,\frac{1}{2}a & ; \end{matrix}\ y\right] = {}_1F_1\left[\begin{matrix} a; \\ b; \end{matrix}\ y\right] + \frac{2y}{b}{}_1F_1\left[\begin{matrix} a+1; \\ b+1; \end{matrix}\ y\right] \tag{6} $$

and

$$ {}_2F_2\left[\begin{matrix} b-a-1,2+a-b & ; \\ b,1+a-b & ; \end{matrix}\ -y\right] $$

$$ = {}_1F_1\left[\begin{matrix} b-a-1 & ; \\ b & ; \end{matrix}\ -y\right] + \frac{y}{b}{}_1F_1\left[\begin{matrix} b-a; \\ b+1; \end{matrix}\ -y\right]. \tag{7} $$

Now applying Kummer's theorem Eq. (3) to the confluent functions in Eq. (7) gives

$$e^{y}{}_2F_2\left[\begin{matrix} b-a-1,\ 2+a-b & ; \\ b,\ 1+a-b & ; \end{matrix} -y\right] = {}_1F_1\left[\begin{matrix} a+1 & ; \\ b & ; \end{matrix} y\right] + \frac{y}{b}\,{}_1F_1\left[\begin{matrix} a+1; \\ b+1; \end{matrix} y\right]. \tag{8}$$

However, by using the recurrence relation (cf. [4, Eq. (2.2.4), p. 19])

$${}_1F_1\left[\begin{matrix} a; \\ b; \end{matrix} y\right] + \frac{y}{b}\,{}_1F_1\left[\begin{matrix} a+1; \\ b+1; \end{matrix} y\right] = {}_1F_1\left[\begin{matrix} a+1 & ; \\ b & ; \end{matrix} y\right],$$

(which can easily be verified by expanding each confluent function in ascending powers of y) we see that the left members of Eqs. (6) and (8) are equal thus giving Eq. (2).

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