A PROBLEM FOR THE COEFFICIENTS OF p-FOLD SYMMETRIC UNIVALENT FUNCTIONS

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1. Introduction. We consider the class S of the regular and univalent functions $f(z) = z + c_2 z^2 + \ldots$ in the unit disk. Each function $f(z) \in S$ generates the sequence of functions

$$f_p(z) = \sum_{k=0}^{\infty} c_{pk+1}^{(p)} z^{pk+1} \tag{1}$$

with p-fold symmetry of rotation with the help of the transformation

$$f_p(z) = \sqrt[p]{f(z^p)} \quad (p = 1, 2, \ldots), \quad f(z) \in S. \tag{2}$$

It has already been proved up to the beginning of the thirties that for the coefficients of p-fold symmetric univalent functions we have $|c_n| < en$ if $p = 1$ (Littlewood [1]) and $|c_n| = O(1)$ if $p = 2$ (Littlewood and Paley [2]), where, for brevity, n denotes an arbitrary exponent of a power of z in the expansion (1). These results and the behavior of the coefficients of the function

$$\frac{z}{(1 - z^p)^{2/p}} = \sum_{k=0}^{\infty} c_{pk+1}^{(p)*} z^{pk+1},$$

obtained from the Koebe function $k(z) = z(1 - z)^{-2}$ by (2), have led to the following hypothesis: The greatest order of growth of the coefficients of an arbitrary function $f_p(z)$, defined in (2), is given by the equality

$$|c_n| = O(n^{2/p-1}) \qquad (p = 1, 2, \ldots). \tag{3}$$

Let us observe that the coefficients $c_{pk+1}^{(p)*}$ of expansion of the function $\sqrt[p]{k(z^p)}$ are binomial, for which we use the following notation for arbitrary $\lambda > 0$:

$$\frac{1}{(1 - z)^{\lambda}} = \sum_{n=0}^{\infty} d_n(\lambda) z^n, \quad d_n(\lambda) \sim \frac{n^{\lambda-1}}{\Gamma(\lambda)}.$$

In the literature, the hypothesis (3) is sometimes called the Szegö hypothesis (see [3, 4]). Lewin [3] has proved the hypothesis (3) for p = 3 and has obtained the similar result $|c_n| = O(n^{-1/2} \ln n)$ for p = 4.

Littlewood [5] has given a simple example of a p-fold symmetric bounded univalent function that corroborates the hypothesis (3) for sufficiently large p. In Littlewood's example, the derivative of the desired function $F_p(z)$ is constructed for integral $p > 1$ by the formula

$$F'_p(z) = \prod_{m=1}^{\infty} \Phi(z^{p^m}), \quad \Phi(z) = \frac{1 + \frac{1}{3} z}{\left(1 - \frac{1}{3} z\right)^3},$$

and it is easily seen that the coefficients of the function $F_p(z) = \int_0^z F'_p(z)\, dz$ satisfy the following inequality for the infinite sequence of the numbers $n_\nu = p + p^2 + \ldots + p^\nu$ ($\nu = 1, 2, \ldots$):

$$|c_n| > A(p) n^{-1+a/\ln p}, \tag{4}$$

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where a is an absolute positive constant and A(p) is a positive constant that depends only on p.

Pommerenke [6] has constructed an example of a p-fold symmetric bounded univalent function for whose coefficients of expansion we have

$$|c_n| > n^{-1+0.17} \tag{5}$$

for infinitely many numbers n for each $p \geq 1$. It is interesting to observe that in the cited examples of Littlewood and Pommerenke the constructed functions have all the coefficients of expansion c_n nonnegative.

Since $0.17 > 2/p$ for $p \geq 12$, it follows from Pommerenke's result that for each $p \geq 12$ there exists a p-fold symmetric univalent function whose coefficients do not satisfy the hypothesis (3). The problem remains open for $4 \leq p \leq 11$. The best-known estimate for the order of growth of the coefficients for $p \geq 5$ has been obtained by Pommerenke [7]:

$$|c_n| = O(n^{-1/2-\varepsilon}), \quad \varepsilon = 1/1600.$$

Thus, the class S contains functions f(z) that generate the sequence $f_p(z)$ by (2) and for which the hypothesis (3) is fulfilled for each p = 1, 2, ... (e.g., the Koebe function, etc.), as well as functions that do not have this property.

The aim of the present note is to elucidate as to in what cases the hypothesis (3) is fulfilled for the whole sequence $f_p(z)$ (p = 1, 2, ...).

2. Connection of the Hypothesis with the Logarithmic Coefficients. Let us recall that the coefficients of the expansion

$$\ln \frac{f(z)}{z} = \sum_{k=1}^{\infty} 2\gamma_k z^k, \quad |z| < 1,$$

are called the logarithmic coefficients of the function $f(z) \in S$.

THEOREM 1. Let a function $f(z) \in S$ generate the sequence of univalent functions

$$f_p(z) = \sum_{n=0}^{\infty} c_n^{(p)} z^{pn+1} \quad (p = 1, 2, \ldots) \tag{6}$$

by Eq. (2). If the logarithmic coefficients of f(z) satisfy the condition

$$|\gamma_n| \leqslant a/n \quad (n = 1, 2, \ldots) \tag{7}$$

with a certain positive constant a, then the hypothesis (3) is valid for all the functions $f_p(z)$ (p = 1, 2, ...), and the following inequality is fulfilled for arbitrary p, n = 1, 2, ...:

$$|c_n^{(p)}| \leqslant \begin{cases} d_n\left(\frac{2}{p}a\right), & a \leqslant 1, \\ e^{\delta/p} \cdot a \cdot d_n\left(\frac{2}{p}\right), & a > 1, \end{cases} \tag{8}$$

where δ is an absolute constant less than 0.312. The sign of equality holds in (8) for $a \leq 1$ and given $p \geq 1$ and $n \geq 1$ for the function

$$f(z) = z\left[\frac{k(\eta z)}{\eta z}\right]^a, \quad |\eta| = 1; \tag{9}$$

for $a > 1$ and each p = 1, 2, ... the order of growth of the coefficients $c^{(p)}_n$ for $n \to \infty$, indicated in (8), is sharp.

Proof. Since the functions $f_p(z)$ (p = 1, 2, ...) are defined by Eq. (2), it follows that $f^p_p(z^{1/p}) = f(z)$. Therefore, the identity

$$\ln \frac{f_p(z^{1/p})}{z^{1/p}} = \frac{1}{p} \ln \frac{f(z)}{z} = \sum_{k=1}^{\infty} \frac{2}{p} \gamma_k z^k$$

of, after potentiation,

$$\sum_{k=0}^{\infty} c_k^{(p)} z^k = \exp\left[\sum_{k=1}^{\infty} \frac{2}{p} \gamma_k z^k\right] \tag{10}$$

holds. The following recurrence relations for the coefficients $c^{(p)}_k$ from (10) are well known:

$$n c_n^{(p)} = \sum_{k=1}^{n} \frac{2k\gamma_k}{p} c_{n-k}^{(p)}, \tag{11}$$

which, together with condition (7), give

$$n\,|c_n^{(p)}| \leqslant \frac{2a}{p} \sum_{k=0}^{n-1} |c_k^{(p)}| \quad (p, n = 1, 2, \ldots). \tag{12}$$

Let us estimate the quantity $\sum_{k=0}^{n-1} |c_k^{(p)}|$. At first, let $a \leq 1$. With regard for (12), we can write

$$\sum_{k=0}^{n} |c_k^{(p)}| = \sum_{k=0}^{n-1} |c_k^{(p)}| + |c_n^{(p)}| \leqslant \left(1 + \frac{2a}{p\cdot n}\right) \sum_{k=0}^{n-1} |c_k^{(p)}|.$$

Repeating this procedure several times, we get

$$\sum_{k=0}^{n} |c_k^{(p)}| \leqslant \prod_{k=1}^{n} \left(1 + \frac{2a}{pk}\right) = d_n\left(1 + \frac{2}{p} a\right).$$

Replacing n by n − 1 and using the identity

$$d_{n-1}(1+\lambda) = \frac{n}{\lambda} d_n(\lambda)$$

for the binomial coefficients for arbitrary $\lambda > 0$, we get

$$\sum_{k=0}^{n-1} |c_k^{(p)}| \leqslant \frac{p}{2a} n d_n\left(\frac{2a}{p}\right), \tag{13}$$

and this and (12) lead to (8). The sign of equality holds in (8) for $a \leq 1$ and given $p \geq 1$ and $n \geq 1$ only if equality holds in (12) and (13) for given $p \geq 1$ and $n = 1, 2, \ldots, n$, and this is possible only for a function whose logarithmic coefficients have the form

$$\gamma_k = \frac{a}{k} \eta^k \quad (k = 1, 2, \ldots, n), \quad |\eta| = 1,$$

which is fulfilled for function (9). It is easily verified that function (9) is actually univalent for $a \leq 1$. But if $a > 1$, then function (9) is univalent in $|z| < 1$ and, therefore, although estimates (13) and $|c_n^{(p)}| \leqslant d_n\left(\frac{2}{p} a\right)$ remain valid, they become nonsharp. We proceed to refine them for large values of n as follows: It follows from (10) that

$$\sum_{k=0}^{\infty} |c_k^{(p)}| z^k \ll \exp\left[\sum_{k=1}^{\infty} \frac{2}{p} |\gamma_k| z^k\right],$$

where the expression $\sum_{n=0}^{\infty} x_n z^n \ll \sum_{n=0}^{\infty} y_n z^n$ means that $x_n \leq y_n$ for all $n = 0, 1, \ldots$. Therefore,

$$\frac{1}{1-z} \sum_{k=0}^{\infty} |c_k^{(p)}| z^k \ll \exp\left[\sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{2}{p} |\gamma_k|\right) z^k\right],$$

or

$$\sum_{k=0}^{\infty}\Big(\sum_{\nu=0}^{k}|c_{\nu}^{(p)}|\Big)z^k \leqslant \exp\Big[\sum_{k=1}^{\infty}\Big(\frac{1}{k}+\frac{2}{p}|\gamma_k|\Big)z^k\Big]. \tag{14}$$

Setting

$$\begin{gathered}\frac{1}{k}+\frac{2}{p}|\gamma_k| = A_k \quad (k=1,2,\ldots),\\ \exp\big[\sum_{k=1}^{\infty}A_k z^k\big] = \sum_{k=0}^{\infty}D_k z^k,\end{gathered} \tag{15}$$

from (14) we get

$$\sum_{k=0}^{n}|c_k^{(p)}| \leqslant D_n \quad (n=0,1,\ldots). \tag{16}$$

But the following exponentiation inequality has been proved for $\lambda \geq 1$ in [8, p. 52] for the coefficients A_k and D_k, connected by the relation (15):

$$|D_n| \leqslant d_n(\lambda)\exp\Big[\frac{\lambda}{2}\delta_n^+(\lambda)\Big], \quad n=1,2,\ldots, \tag{17}$$

where

$$\begin{gathered}\delta_n^+(\lambda) = \delta_n^+(\lambda; A_1, A_2, \ldots, A_n) = \max\{0, \delta_n(\lambda)\},\\ \delta_n(\lambda) = \delta_n(\lambda; A_1, A_2, \ldots, A_n) = \max_{1\leqslant\nu\leqslant n}\{\Delta_\nu(\lambda)\},\\ \Delta_n(\lambda) = \Delta_n(\lambda; A_1, A_2, \ldots, A_n) = \frac{1}{\lambda^2}\sum_{k=1}^{n}k|A_k|^2 - \sum_{k=1}^{n}\frac{1}{k}.\end{gathered} \tag{18}$$

By virtue of (15) and the known inequality [9]

$$\sum_{k=1}^{n}k|\gamma_k|^2 \leqslant \sum_{k=1}^{n}\frac{1}{k}+\delta, \quad n=1,2,\ldots,$$

for the logarithmic coefficients of the functions $f(z)\in S$ we get

$$\sum_{k=1}^{n}k|A_k|^2 \leqslant \Big(1+\frac{2}{p}\Big)\sum_{k=1}^{n}\frac{1}{k}+\frac{2}{p}\Big(1+\frac{2}{p}\Big)\delta. \tag{19}$$

Setting $\lambda = 1 + 2/p$, by (18) we get

$$\delta_n^+(\lambda) = \frac{2}{2+p}\delta. \tag{20}$$

It follows from (16), (17), and (20) that for $n \geq 1$ we have

$$\sum_{l=0}^{n-1}|c_k^{(p)}| \leqslant d_{n-1}\Big(1+\frac{2}{p}\Big)\exp\Big[\frac{\delta}{p}\Big] = \frac{p}{2}\exp\Big[\frac{\delta}{p}\Big]\; nd_n\Big(\frac{2}{p}\Big),$$

and we prove (8) for $\alpha > 1$ with the help of (12). The example of the function $\sqrt[p]{k(z^p)}$ shows that the order of growth of the coefficients for $n \to \infty$, given in (8), is sharp. But the asymptotic equation

$$d_n(\lambda) \sim \frac{n^{\lambda-1}}{\Gamma(\lambda)}$$

for the binomial coefficients is well known for arbitrary $\lambda > 0$; it, together with (8), gives Eq. (3). The theorem is proved.

COROLLARY. If a function $f(z)\in S$ and $\arg f(z)$ or $\ln|f(z)|$ on $|z| = 1$ are functions of bounded variation on the interval $[0, 2\pi]$, then the coefficients of the function $\sqrt[p]{f(z^p)}$ satisfy the hypothesis (3) for arbitrary $p = 1, 2, \ldots$.

Proof. Duren and Leung [10] have proved that the logarithmic coefficients of the functions $f(z) \in S$ for which arg f(z) or ln $|f(x)|$ on the unit circle (i.e., the boundary values) are functions of bounded variation on $[0, 2\pi]$, satisfy the condition (7). Hence (3) follows by the conclusion of Theorem 1.

Let us observe that the class of the functions $f(z) \in S$, indicated in the Corollary, contains, among others, all the functions that map $|z| < 1$ onto domains with rectifiable boundary.

THEOREM 2. Let a function $f(z) \in S$ generate the sequence of functions $f_p(z)$ by Eq. (2). If the hypothesis (3), i.e., the inequality

$$|c_n^{(p)}| \leqslant A(p)\, n^{2/p-1} \quad (n = 1, 2, \ldots) \tag{21}$$

[A(p) is a positive constant that depends only on p], is fulfilled for infinitely many values of p and, moreover,

$$\overline{\lim_{p\to\infty}}\, pA(p) = 2a < \infty, \tag{22}$$

then the logarithmic coefficients of f(x) satisfy relation (7) and, consequently, hypothesis (3) is valid for all functions $f_p(z)$ $(p = 1, 2, \ldots)$.

Proof. We rewrite (21) in the form

$$p\,|c_n^{(p)}| \leqslant pA(p)\, n^{2/p-1}$$

and take limit as $p \to \infty$. As a result, we get

$$\overline{\lim_{p\to\infty}}\, p\,|c_n^{(p)}| \leqslant \overline{\lim_{p\to\infty}}\, p\cdot A(p)\cdot \frac{1}{n} = \frac{2a}{n} \quad (n = 1, 2, \ldots). \tag{23}$$

On the other hand, we easily discern from Eq. (10) that for each $n \geq 1$

$$c_n^{(p)} = \frac{2}{p}\gamma_n + O(1/p^2) \quad \text{or} \quad pc_n^{(p)} = 2\gamma_n + O(1/p),$$

whence the following limit equation follows:

$$\lim_{p\to\infty} [pc_n^{(p)}] = 2\gamma_n. \tag{24}$$

Combining (23) and (24), for each $n \geq 1$ we get

$$|\gamma_n| \leqslant a/n,$$

which coincides with (7). Hence, by Theorem 1, the desired conclusion follows. The theorem is proved.

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