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Some integral inequalities for harmonic *h*-convex functions involving hypergeometric functions

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ABSTRACT

The aim of this paper is to establish some new Hermite–Hadamard type inequalities for harmonic *h*-convex functions involving hypergeometric functions. We also discuss some new and known special cases, which can be deduced from our results. The ideas and techniques of this paper may inspire further research in this field.

1. Introduction

In recent years, much attention have been given to theory of convexity because of its great utility in various fields of pure and applied sciences. Many researchers have extended and generalized the classical concepts of convex sets and convex functions in various directions using novel and innovative techniques. For more information, see [1–4,6,9,14–17,19]. To unify the classes of classical convex functions, *s*-Breckner convex functions [1], Godunova–Levin functions [6] and *P*-functions [4], Varošanec [19] introduced the concept of *h*-convex functions. Işcan [9] introduced another new class of convex functions which is called harmonically convex functions. For some recent investigations on harmonically convex functions, see [5,18]. Noor et al. [16] introduced the concept of harmonically *h*-convex functions, which generalizes several new and known class of harmonically convex functions.

A very interested inequality associated with convex functions is called the Hermite–Hadamard type inequality. This inequality provides a necessary and sufficient condition for a function to be convex.

Let $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ a convex function, where $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if and only if f is convex.

The inequality (1.1) has been extended and generalized for various classes of convex functions via different approaches, see [3–5,7,9–13,15–18]. We derive some new Hermite–Hadamard type inequalities for harmonically *h*-convex functions. Results proved continue to hold for various known and new classes of convex functions. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

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2. Preliminaries

In this section, we recall some known concepts.

An important generalization of convex functions was considered by Varošanec in [19] which is called the h-convex functions.

Definition 2.1. Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a h-convex function ($f \in SX(h, I)$), if f is nonnegative and

$$f(tx + (1-t)y) \leqslant h(t)f(x) + h(1-t)f(y), \quad \forall x, \quad y \in I \quad \text{and} \quad t \in [0,1]. \tag{2.1}$$

If (2.1 holds in the reversed sense, then f is h-concave, ($f \in SV(h, I)$).

For $h(t) = t, h(t) = t^s, h(t) = \frac{1}{t}, h(t) = 1$ and $h(t) = \frac{1}{t^s}$, the class of h-convex functions reduces to the class of convex functions, s-Breckner convex functions [1], Godunova–Levin functions [6], P-functions [4] and s-Godunova–Levin functions [3] respectively. This shows that the class of h-convex functions is quite general and unifying one.

Işcan [9] obtained several inequalities of Hermite–Hadamard type for harmonic convex functions.

Definition 2.2 [9]. Let $f : I = [a,b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where I an real interval. The function f is said to be harmonic convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant tf(y) + (1-t)f(x), \quad \forall x, \quad y \in I \quad \text{and} \quad t \in [0,1]. \tag{2.2}$$

For this class of functions, Işcan [9] obtained the following Hermite–Hadamard type inequality.

Theorem 2.3. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic convex and $a, b \in I, a < b$. If $f \in L(a,b)$, then*

$$f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx \leqslant \frac{f(a)+f(b)}{2}. \tag{2.3}$$

Motivated and inspired by the research going on this dynamic field, Noor et al. [16] introduced and considered a new class of harmonically convex functions, which is called the harmonic h-convex function. For the recent results and details, see [5,9,15] and the references therein.

Definition 2.4 [16]. Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}, I$ an real interval. We say that f be a harmonic h-convex function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leqslant h(t)f(y) + h(1-t)f(x), \quad \forall x, \quad y \in I \quad \text{and} \quad t \in [0,1]. \tag{2.4}$$

Theorem 2.5 [16]. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a harmonic h-convex function, where $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then, for $h(\frac{1}{2}) \neq 0$*

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{2ab}{a+b}\right) \leqslant \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leqslant [f(a)+f(b)]\int_0^1 h(t)dt.$$

The following results play an important role in obtaining some new Hermite–Hadamard type inequalities for harmonic h-convex functions.

Lemma 2.6 ([9], Theorem 4). *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^0 of an interval $I, a, b \in I, a < b$ and $f' \in L[a,b]$. Then*

$$\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt, \tag{2.5}$$

where $A_t = tb + (1-t)a$.

Lemma 2.7 ([18], Lemma 1). *Let $f : I \subseteq (0, +\infty) \to \mathbb{R}$ be a differentiable function on the interior I^0 of an interval $I, a, b \in I, a < b$ and $f' \in L[a,b]$. Then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx - f\left(\frac{2ab}{a+b}\right) = ab(b-a)\left[\int_0^{1/2} \frac{t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt + \int_{1/2}^1 \frac{t-1}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\right], \tag{2.6}$$

where $A_t = tb + (1-t)a$.

For the reader's convenience, we recall the definitions of the Gamma function $\Gamma(.)$ and Beta function $B(.,.)$ respectively, which are as:

$$\Gamma(x) = \int_0^\infty e^{-x} t^{x-1} \mathrm{d}t,$$

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ \mathrm{d}t.$$

It is known [8] that

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x,y;c;z) = \frac{1}{\mathrm{B}(y,c-y)} \int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}\mathrm{d}t$$

for $|z| < 1, c > y > 0$.

3. Main results

In this section, we derive our main results.

Theorem 3.1. *Let $f : I \subseteq (0,+\infty) \to \mathbb{R}$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L[a,b]$, where $a,b \in I^0, a < b$. If $|f'|^q$, is a harmonic h-convex function for $q > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| \leqslant \frac{ab(b-a)}{2}(c(a,b))^{1-1/q}\left(|f'(a)|^q \int_0^1 |1-2t|h(t)A_t^{-2}\mathrm{d}t + |f'(b)|^q \int_0^1 |1-2t|h(1-t)A_t^{-2}\mathrm{d}t\right)^{1/q}, \tag{3.1}$$

where

$$c(a,b) = a^{-2}\left[{}_2F_1\left(2,2;3;1-\frac{b}{a}\right) - {}_2F_1\left(2,1;2;1-\frac{b}{a}\right) + \frac{1}{2}\cdot {}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)\right],$$

and $A_t = tb + (1-t)a$.

Proof. Using Lemma 2.6, the power mean inequality and the harmonic h-convexity of $|f'|^q$, we have

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| &\leqslant \frac{ab(b-a)}{2}\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t \\
&\leqslant \frac{ab(b-a)}{2}\left(\int_0^1 \frac{|1-2t|}{A_t^2}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q \mathrm{d}t\right)^{1/q} \\
&\leqslant \frac{ab(b-a)}{2}(c(a,b))^{1-1/q}\left(\int_0^1 \frac{|1-2t|}{A_t^2}(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q)\mathrm{d}t\right)^{1/q} \\
&= \frac{ab(b-a)}{2}(c(a,b))^{1-1/q}\left(|f'(a)|^q\int_0^1 |1-2t|h(t)A_t^{-2}\mathrm{d}t + |f'(b)|^q \int_0^1 |1-2t|h(1-t)A_t^{-2}\mathrm{d}t\right)^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
c(a,b) &= \int_0^1 \frac{|1-2t|}{A_t^2}\mathrm{d}t = \int_0^{1/2} \frac{1-2t}{A_t^2}\mathrm{d}t + \int_{1/2}^1 \frac{2t-1}{A_t^2}\mathrm{d}t \\
&= a^{-2}\left({}_2F_1\left(2,2;3;1-\frac{b}{a}\right) - {}_2F_1\left(2,1;2;1-\frac{b}{a}\right) + \frac{1}{2}\cdot {}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)\right).
\end{aligned}$$

This completes the proof. □

We now discuss some special cases of Theorem 3.1.

I. If $h(t) = t^s$ and the function f is harmonic s-convex, then the inequality (3.1) becomes

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\ \mathrm{d}x\right| \leqslant \frac{ab(b-a)}{2}(c(a,b))^{1-1/q}\left[c_1(s;a,b)|f'(a)|^q + c_2(s;a,b)|f'(b)|^q\right]^{1/q},$$

where

$$c_1(s;a,b)=a^{-2}\left[\frac{2}{s+2}\cdot{}_2F_1\left(2,s+2;s+3;1-\frac{b}{a}\right)-\frac{1}{s+1}\cdot{}_2F_1\left(2,s+1;s+2;1-\frac{b}{a}\right)+\frac{1}{2^s(s+1)(s+2)}\cdot{}_2F_1\left(2,s+1;s+3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)\right]$$

and

$$c_2(s;a,b)=a^{-2}\left[\frac{2}{(s+1)(s+2)}\cdot{}_2F_1\left(2,2;s+3;1-\frac{b}{a}\right)-\frac{1}{s+1}\cdot{}_2F_1\left(2,1;s+2;1-\frac{b}{a}\right)+\frac{1}{2}\cdot{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)\right].$$

This result is due to Chen and Wu [5].

II. If $h(t)=t^{-s}$ and the function f is harmonic s-Godunova–Levin function, then inequality (3.1) becomes

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx\right|\leqslant\frac{ab(b-a)}{2}(c(a,b))^{1-1/q}\left[\mu_1(s;a,b)|f'(a)|^q+\mu_2(s;a,b)|f'(b)|^q\right]^{1/q},$$

where

$$\mu_1(s;a,b)=a^{-2}\left[\frac{2}{2-s}\cdot{}_2F_1\left(2,2-s;3-s;1-\frac{b}{a}\right)-\frac{1}{1-s}\cdot{}_2F_1\left(2,1-s;2-s;1-\frac{b}{a}\right)+\frac{2^s}{(1-s)(2-s)}\cdot{}_2F_1\left(2,1-s;3-s;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)\right],$$

and

$$\mu_2(s;a,b)=a^{-2}\left[\frac{1}{(1-s)(2-s)}\cdot{}_2F_1\left(2,2;3-s;1-\frac{b}{a}\right)-\frac{1}{1-s}\cdot{}_2F_1\left(2,1;2-s;1-\frac{b}{a}\right)+\frac{1}{2}\cdot{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)\right].$$

To the best of our knowledge, this result is a new one.

Theorem 3.2. *Let $f:I\subseteq(0,+\infty)\to\mathbb{R}$ be a differentiable function on the interior I^0 of an interval I such that $f'\in L[a,b]$, where $a,b\in I^0$ with $a<b$. If the function $|f'|^q$, is a harmonic h-convex function for $q>1$, then the following inequality holds:*

$$\left|\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx-f\left(\frac{2ab}{a+b}\right)\right|\leqslant ab(b-a)\left[c_3(a,b)^{1-1/q}\left(\int_0^{1/2}\frac{t}{A_t^2}(h(t)|f'(a)|^q+h(1-t)|f'(b)|^q)dt\right)^{1/q}\right.$$
$$\left.+c_4(a,b)^{1-1/q}\left(\int_{1/2}^1\frac{1-t}{A_t^2}(h(t)|f'(a)|^q+h(1-t)|f'(b)|^q)dt\right)^{1/q}\right],\qquad(3.2)$$

where

$$c_3(a,b)=\frac{1}{8a^2}\cdot{}_2F_1\left[2,2;3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right],$$

$$c_4(a,b)=\frac{1}{2(a+b)^2}\cdot{}_2F_1\left[2,1;3;\frac{a-b}{a+b}\right],$$

and $A_t=tb+(1-t)a$.

Proof. From Lemma 2.7, the power mean inequality and the harmonic h-convexity of $|f'|^q$ where $q>1$, we have

$$\left|\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx-f\left(\frac{2ab}{a+b}\right)\right|\leqslant ab(b-a)\left[\int_0^{1/2}\frac{t}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt+\int_{1/2}^1\frac{|t-1|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right]$$
$$\leqslant ab(b-a)\left[\left(\int_0^{1/2}\frac{t}{A_t^2}dt\right)^{1-1/q}\left(\int_0^{1/2}\frac{t}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}+\left(\int_{1/2}^1\frac{1-t}{A_t^2}dt\right)^{1-1/q}\left(\int_{1/2}^1\frac{1-t}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right]$$
$$\leqslant ab(b-a)\left[c_3(a,b)^{1-1/q}\left(\int_0^{1/2}\frac{t}{A_t^2}(h(t)|f'(a)|^q+h(1-t)|f'(b)|^q)dt\right)^{1/q}\right.$$
$$\left.+c_4(a,b)^{1-1/q}\left(\int_{1/2}^1\frac{1-t}{A_t^2}(h(t)|f'(a)|^q+h(1-t)|f'(b)|^q)dt\right)^{1/q}\right],$$

where

$$c_3(a,b)=\int_0^{1/2}\frac{t}{A_t^2}dt=\frac{1}{8a^2}\cdot{}_2F_1\left(2,2;3;\frac{1}{2}\left(1-\frac{b}{a}\right)\right)$$

and

$$c_4(a,b) = \int_{1/2}^{1} \frac{1-t}{A_t^2}\mathrm{d}t = \frac{1}{2(a+b)^2}\cdot {}_2F_1\left(2,1;3;\frac{a-b}{a+b}\right).$$

This completes the proof. □

Theorem 3.3. *Let $f : I \subseteq (0,+\infty) \to \mathbb{R}$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L[a,b]$, where $a,b \in I^0$ with $a < b$. If the function $|f'|^q$ is a harmonic h-convex function for $q > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| \leqslant \frac{ab(b-a)}{2(p+1)^{1/p}}\left(|f'(a)|^q\int_0^1 h(t)A_t^{-2q}\mathrm{d}t + |f'(b)|^q\int_0^1 h(1-t)A_t^{-2q}\mathrm{d}t\right)^{1/q}, \tag{3.3}$$

where $A_t = tb + (1-t)a, \frac{1}{p}+\frac{1}{q} = 1$.

Proof. Using Lemma 2.6, Hölder's inequality and the harmonic h-convexity of $|f'|^q$, we have

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| &\leqslant \frac{ab(b-a)}{2}\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t \\ &\leqslant \frac{ab(b-a)}{2}K_1^{1/p}\left(\int_0^1 \frac{1}{A_t^{2q}}(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q)\mathrm{d}t\right)^{1/q} \\ &\leqslant \frac{ab(b-a)}{2(p+1)^{1/p}}\left(|f'(a)|^q\int_0^1 h(t)A_t^{-2q}\mathrm{d}t + |f'(b)|^q\int_0^1 h(1-t)A_t^{-2q}\mathrm{d}t\right)^{1/q},\end{aligned}$$

where

$$K_1 = \int_0^1 |1-2t|\mathrm{d}t = \frac{1}{p+1}.$$

This completes the proof. □

We discuss some special cases of Theorem 3.3.

I. If $h(t) = t^s$ and the function f is harmonic s-convex, then the inequality (3.3) reduces to

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| &\leqslant \frac{b(b-a)}{2a}\cdot\left(\frac{1}{p+1}\right)^{1/p}\cdot\left(\frac{1}{s+1}\right)^{1/q} \\ &\times\left({}_2F_1\left(2q,s+1;s+2;1-\frac{b}{a}\right)|f'(a)|^q + {}_2F_1\left(2q,1;s+2;1-\frac{b}{a}\right)|f'(b)|^q\right)^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$. This result was obtained by Chen and Wu [5].

II. If $h(t) = t^{-s}$ and the function f is harmonic s-Godunova–Levin function, then the inequality (3.3) reduces to the following new result.

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x\right| &\leqslant \frac{b(b-a)}{2a}\cdot\left(\frac{1}{p+1}\right)^{1/p}\cdot\left(\frac{1}{1-s}\right)^{1/q} \\ &\times\left({}_2F_1\left(2q,1-s;2-s;1-\frac{b}{a}\right)|f'(a)|^q + {}_2F_1\left(2q,1;2-s;1-\frac{b}{a}\right)|f'(b)|^q\right)^{1/q},\end{aligned}$$

where $1/p + 1/q = 1$.

Theorem 3.4. *Let $f : I \subseteq (0,+\infty) \to \mathbb{R}$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L[a,b]$, where $a,b \in I^0$ with $a < b$. If $|f'|^q$ is a harmonic h-convex function for $q > 1$, then the following inequality hold:*

$$\begin{aligned}\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x - f\left(\frac{2ab}{a+b}\right)\right| &\leqslant \frac{ab(b-a)}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\left[\left(\int_0^{1/2}\frac{1}{A_t^{2q}}(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q)\mathrm{d}t\right)^{1/q}\right. \\ &\left.+\left(\int_{1/2}^1 \frac{1}{A_t^{2q}}(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q)\mathrm{d}t\right)^{1/q}\right],\end{aligned}$$

where $A_t = tb + (1-t)a$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. From Lemma 2.7, Hölder's inequality and the harmonic h-convexity of $|f'|^q$, we get

$$\begin{aligned}
\left|\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\,dx - f\left(\frac{2ab}{a+b}\right)\right| &\leqslant ab(b-a)\left[\int_0^{1/2}\frac{t}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt + \int_{1/2}^1 \frac{t-1}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right]\\
&\leqslant ab(b-a)\left[\left(\int_0^{1/2} t^p dt\right)^{1/p}\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} + \left(\int_{1/2}^1 |t-1|^p dt\right)^{1/p}\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right]\\
&\leqslant ab(b-a)\left[\left(\frac{1}{2^{p+1}(p+1)}\right)^{1/p}\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q\right)dt\right)^{1/q}\right.\\
&\quad\left. + \left(\frac{1}{2^{p+1}(p+1)}\right)^{1/p}\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q\right)dt\right)^{1/q}\right]\\
&= \frac{ab(b-a)}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\left[\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q\right)dt\right)^{1/q}\right.\\
&\quad\left. + \left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left(h(t)|f'(a)|^q + h(1-t)|f'(b)|^q\right)dt\right)^{1/q}\right].
\end{aligned}$$

This completes the proof. □

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