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Domination of ordered weighted averaging operators over t-norms

R. Mesiar, S. Saminger

Abstract The fusion of transitive fuzzy relations preserving the transitivity is linked to the domination of the involved aggregation operator. The aim of this contribution is to investigate the domination of OWA operators over t-norms whereas the main emphasis is on the domination over the Łukasiewicz t-norm. The domination of OWA operators and related operators over continuous Archimedean t-norms will also be discussed.

Keywords Domination, OWA operators

1 Motivation

In several applications of fuzzy logics and fuzzy systems the processing of data based on the strongest t-norm, i.e., the minimum T_M suffers from the increase of uncertainty. Recall only the addition of fuzzy numbers where the final sum spreads equal to the sum of all incoming spreads. In order to avoid this undesirable effect, alternative approaches have to be taken into account. One of the most promising t-norms for reducing the enormous growth of uncertainty is the Łukasiewicz t-norm T_L. The addition of triangular (trapezoidal) fuzzy numbers based on T_L leads to output spreads equal to the maximal (left and right) incoming spreads – a property which is often required in models dealing with uncertainty. Moreover, the Łukasiewicz t-norm T_L is often applied when fuzzy rule based systems are designed with the purpose to reduce redundancy, and especially in clustering algorithms (likeness relations of Bezdek and Harris [2]). Recall also the approximate solutions of fuzzy relational equations [6] or several relations possessing T_L-transitivity as a genuine counterpart of the classical triangle inequality of the Euclidean metric on $\mathbb{R}$. Furthermore, when solving complex problems we sometimes need either to refine several fuzzy relations or to introduce their Cartesian product. However, we expect that the new fuzzy relation will be again T_L-transitive if the original fuzzy relations have also been T_L-transitive.

As it has been shown in [11], the preservation of T-transitivity of fuzzy relations during an aggregation process is guaranteed if the involved aggregation operator dominates the corresponding t-norm T. Several special operators dominating the Łukasiewicz t-norm are already known, e.g. T_L itself, the minimum and the arithmetic mean, see [11]. The later two are special cases of so called OWA operators, one of the most important family of aggregation operators applied in many domains (see [14, 16]), which are used for summarizing singular data into a single output where inputs are ordered with respect to their value and are assigned certain weights before being aggregated.

If we want to preserve the T_L-transitivity of fuzzy relations the domination of an OWA operator $\mathbf{O}$ over T_L should be checked before fusing the T_L-transitive fuzzy relations by means of this OWA operator $\mathbf{O}$. Therefore the characterization of OWA operators dominating the Łukasiewicz t-norm T_L seemed to be important. The extension of the obtained results for some other t-norms and other dominating aggregation operators will be the final task of this contribution.

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R. Mesiar (✉)
Department of Mathematics and Descriptive Geometry,
Faculty of Civil Engineering,
Slovak University of Technology, SK-81 368 Bratislava, Slovakia
Institute of Information Theory and Automation,
Czech Academy of Sciences,
CZ-182 08 Prague 8, Czech Republic
e-mail: mesiar@math.sk

S. Saminger
Fuzzy Logic Laboratorium Linz-Hagenberg,
Department of Algebra,
Stochastics and Knowledge-Based Mathematical Systems,
Johannes Kepler University, A-4040 Linz, Austria

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2 Preliminaries

2.1 Aggregation operators, t-norms, t-conorms

Definition 1 *A function* $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ *is called an* aggregation operator *if it fulfills the following properties* ([3, 8]):

(AO1) $\mathbf{A}(x_1,\ldots,x_n) \le \mathbf{A}(y_1,\ldots,y_n)$ *whenever* $x_i \le y_i$ *for all* $i \in \{1,\ldots,n\}$,
(AO2) $\mathbf{A}(x) = x$ *for all* $x \in [0,1]$,
(AO3) $\mathbf{A}(0,\ldots,0) = 0$ *and* $\mathbf{A}(1,\ldots,1) = 1$.

Each aggregation operator $\mathbf{A}$ can be represented by a family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, i.e., $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ given by

$$\mathbf{A}_{(n)}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n) \ .$$

In that case, $\mathbf{A}_{(1)} = \mathrm{id}_{[0,1]}$ and, for $n \geq 2$, each $\mathbf{A}_{(n)}$ is non-decreasing satisfying $\mathbf{A}_{(n)}(0, \ldots, 0) = 0$ and $\mathbf{A}_{(n)}(1, \ldots, 1) = 1$. Usually, the aggregation operator $\mathbf{A}$ and the corresponding family $(\mathbf{A}_{(n)})_{n \in \mathbb{N}}$ of n-ary operations are identified with each other.

Note that, for $n \geq 2$, n-ary operations $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ which fulfill properties **(AO1)** and **(AO3)** are referred to as *n-ary aggregation operators*.

In general, for $n \neq m$ the operators $\mathbf{A}_{(n)}$ and $\mathbf{A}_{(m)}$ need not be related. This observation does not hold in several special cases, e.g. by quasi-arithmetic means [3] or in the case of associative aggregation operators, where the binary operator $\mathbf{A}_{(2)}$ already contains all the information about $\mathbf{A}$.

With only simple and obvious modifications, aggregation operators acting on any closed interval $I = [a, b] \subseteq [-\infty, \infty]$ can be defined. While **(AO1)** and **(AO2)** basically remain the same, only **(AO3)** has to be modified accordingly

$$\textbf{(AO3')}\,\mathbf{A}(a, \ldots, a) = a \quad \text{and} \quad \mathbf{A}(b, \ldots, b) = b \ .$$

Consequently, we will speak of an *aggregation operator acting on I*. Unless explicitly mentioned otherwise, we will restrict our further considerations to aggregation operators acting on the unit interval (according to Definition 1).

Consider an aggregation operator $\mathbf{A} : \bigcup_{n \in \mathbb{N}} [a, b]^n \to [a, b]$ on $[a, b]$ and a monotone bijection $\varphi : [c, d] \to [a, b]$. The operator $\mathbf{A}_\varphi : \bigcup_{n \in \mathbb{N}} [c, d]^n \to [c, d]$ defined by

$$\mathbf{A}_\varphi(x_1, \ldots, x_n) = \varphi^{-1}(\mathbf{A}(\varphi(x_1), \ldots, \varphi(x_n)))$$

is an aggregation operator on $[c, d]$, which is *isomorphic* to $\mathbf{A}$.

The isomorphic transformation of an aggregation operator $\mathbf{A} : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$ with respect to the specific decreasing bijection $\varphi : [0, 1] \to [0, 1]$, $\varphi(x) = 1 - x$ is called the *dual* of the aggregation operator $\mathbf{A}$, $\mathbf{A}_\varphi = \mathbf{A}^d$, i.e.,

$$\mathbf{A}^d(x_1, \ldots, x_n) = 1 - \mathbf{A}(1 - x_1, \ldots, 1 - x_n) \ .$$

Triangular norms were originally introduced in the context of probabilistic metric spaces ([9, 12, 13]), but they are in fact nothing else than associative, and symmetric aggregation operators with 1 as neutral element. Due to their associativity it is enough to discuss only their binary form.

Definition 2 *A* triangular norm (*t-norm for short*) *is a binary operation T on the unit interval which is commutative, associative, non-decreasing in each component, and has 1 as a neutral element.*

Example 1 *The following are the four basic t-norms*:

$$\textit{Minimum t-norm}: \quad T_{\mathbf{M}}(x, y) = \min(x, y),$$

$$\textit{Product t-norm}: \quad T_{\mathbf{P}}(x, y) = x \cdot y,$$

$$\textit{Łukasiewicz t-norm}: \quad T_{\mathbf{L}}(x, y) = \max(x + y - 1, 0),$$

Drastic product:

$$T_{\mathbf{D}}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Remark 1 *A huge class of t-norms is closely linked to the product t-norm $T_{\mathbf{P}}$ by so called* multiplicative generators. *Because of the isomorphism of the semigroups $([0, 1], T_{\mathbf{P}})$ and $([0, \infty], +)$, also additive generators can be introduced.*

Recall that any continuous strictly decreasing mapping $t : [0, 1] \to [0, \infty]$, $t(1) = 0$ is called an additive generator *and the mapping $T : [0, 1]^2 \to [0, 1]$ given by*

$$T(x, y) = t^{-1}\big(\min(t(0), t(x) + t(y))\big)$$

is a continuous Archimedean t-norm [8]. *Any continuous Archimedean t-norm is either isomorphic to $T_{\mathbf{P}}$* (strict *t-norms*) *and then its additive generator is unbounded, whereas bounded additive generators are characteristic for t-norms isomorphic to $T_{\mathbf{L}}$* (nilpotent *t-norms*). *Additive generators assigned to a continuous Archimedean t-norm T are unique up to a positive multiplicative constant* [8].

Closely related to t-norms are t-conorms, defined by the following properties.

Definition 3 *A* t-conorm *is a binary operation S on the unit interval which is commutative, associative, non-decreasing in each component, and has 0 as a neutral element.*

A t-norm T and a t-conorm S are said to be *dual* (according to the duality of aggregation operators in general), if the following relationship holds

$$T(x, y) = 1 - S(1 - x, 1 - y) \ .$$

Example 2 *Corresponding to the four basic t-norms the four basic t-conorms are*

$$\textit{Maximum}: \quad S_{\mathbf{M}}(x, y) = \max(x, y),$$

$$\textit{Probabilistic sum}: \quad S_{\mathbf{P}}(x, y) = x + y - x \cdot y,$$

$$\textit{Bounded sum}: \quad S_{\mathbf{L}}(x, y) = \min(x + y, 1),$$

Drastic sum:

$$S_{\mathbf{D}}(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \,]0, 1]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Observe that if a (continuous Archimedean) t-norm T is generated by an additive generator t, then $s : [0, 1] \to [0, \infty]$ given by $s(x) = t(1 - x)$ is an additive generator of the corresponding dual t-conorm S

$$S(x, y) = s^{-1}\big(\min(s(0), s(x) + s(y))\big) \ .$$

2.2 OWA operators

A special class of aggregation operators are the so called OWA operators (ordered weighted averaging operators) introduced in [14] and related to the Choquet integral [7].

Definition 4 *The operator $\mathbf{A} : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$ given by*

$$\mathbf{A}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_{in} \cdot x'_i$$

where $(x'_1, \ldots, x'_n)$ is a non-decreasing permutation of the input n-tuple $(x_1, \ldots, x_n)$ is called an OWA operator *associated with the weighting triangle $\triangle = (w_{in})$.*

A weighting triangle $\triangle = (w_{in} \mid n \in \mathbb{N}, i \in \{1, \ldots, n\})$ such that all $w_{in} \in [0,1]$ and $\sum_{i=1}^{n} w_{in} = 1$ for all $n \in \mathbb{N}$, collects all necessary weights for an OWA operator. For n-ary operators, the weights $w_1, \ldots, w_n$ form an n-dimensional weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0,1]^n$ with $\sum_{i=1}^{n} w_i = 1$. For more details on weighting triangles see also [4, 5, 15].

If $\triangle$ is a weighting triangle and $\triangle^r$ is the corresponding reversed weighting triangle, then the corresponding OWA operators form a couple of dual aggregation operators. Therefore an OWA operator is self-dual if the corresponding weighting triangle is symmetric. Starting from a binary OWA operator, we cannot construct an n-ary OWA operator, in general. Note also that the composition of OWA operators need not be an OWA operator in general.

2.3 Domination

Definition 5 *Consider an n-ary aggregation operator* $\mathbf{A}_{(n)}$ *and an m-ary aggregation operator* $\mathbf{B}_{(m)}$. *We say that* $\mathbf{A}_{(n)}$ dominates $\mathbf{B}_{(m)}$, $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$, *if for all* $x_{i,j} \in [0,1]$ *with* $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ *the following property holds*:

$$\mathbf{B}_{(m)}\left(\mathbf{A}_{(n)}(x_{1,1}, \ldots, x_{1,n}), \ldots, \mathbf{A}_{(n)}(x_{m,1}, \ldots, x_{m,n})\right) \\ \leq \mathbf{A}_{(n)}\left(\mathbf{B}_{(m)}(x_{1,1}, \ldots, x_{m,1}), \ldots, \mathbf{B}_{(m)}(x_{1,n}, \ldots, x_{m,n})\right) . \tag{1}$$

Note that if either n or m or both are equal to 1, because of the boundary condition **(AO2)**, $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$ is trivially fulfilled for any two aggregation operators $\mathbf{A}, \mathbf{B}$.

Definition 6 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be aggregation operators. We say that* $\mathbf{A}$ dominates $\mathbf{B}$, $\mathbf{A} \gg \mathbf{B}$, *if* $\mathbf{A}_{(n)}$ *dominates* $\mathbf{B}_{(m)}$ for all $n, m \in \mathbb{N}$.

It has been shown in [11] that because of the associativity of a t-norm T it is sufficient to show that an aggregation operator $\mathbf{A}_{(n)}$ dominates $T_{(2)}$ for arbitrary $n \in \mathbb{N}$ in order to prove that it dominates T for all $n \in \mathbb{N}$, i.e., that the following inequality holds for arbitrary $x_1, \ldots, x_n \in [0,1]$, $y_1, \ldots, y_n \in [0,1]$ and $n \in \mathbb{N}$

$$T(\mathbf{A}(x_1, \ldots, x_n), \mathbf{A}(y_1, \ldots, y_n)) \\ \leq \mathbf{A}(T(x_1, y_1), \ldots, T(x_n, y_n)) . \tag{2}$$

Another result for dominating aggregation operators which is necessary for our proofs and which is proven in [11] is summarized in the following lemma.

Lemma 1 *Let* $[a,b]$ *and* $[c,d]$ *be non-trivial subintervals of* $[-\infty, \infty]$ *and consider two aggregation operators* $\mathbf{A}$ *and* $\mathbf{B}$ *both acting on* $[a,b]$.

(i) $\mathbf{A} \gg \mathbf{B}$ *if and only if* $\mathbf{A}_\varphi \gg \mathbf{B}_\varphi$ *for all non-decreasing bijections* $\varphi : [c,d] \to [a,b]$ *if and only if* $\mathbf{A}_\varphi \gg \mathbf{B}_\varphi$ *for some non-decreasing bijection* $\varphi : [c,d] \to [a,b]$.
(ii) $\mathbf{A} \gg \mathbf{B}$ *if and only if* $\mathbf{B}_\varphi \gg \mathbf{A}_\varphi$ *for all non-increasing bijections* $\varphi : [c,d] \to [a,b]$ *if and only if* $\mathbf{B}_\varphi \gg \mathbf{A}_\varphi$ *for some non-increasing bijections* $\varphi : [c,d] \to [a,b]$.

3 OWA operators and Łukasiewicz t-norm

3.1 General observations

It is well known that the minimum dominates any t-norm (see [8]), i.e., also the Łukasiewicz t-norm ($T_\mathbf{M} \gg T_\mathrm{L}$). On the other hand, the minimum can be interpreted as an OWA operator with weights $w_{1n} = 1$ and $w_{in} = 0$ for all $i \neq 1$ and $n \in \mathbb{N}$. So the minimum is one of the OWA operators dominating Łukasiewicz t-norm.

Secondly, the Łukasiewicz t-norm is dominated by the arithmetic mean (for a proof see e.g. [11]), which is an OWA operator with weights $w_{in} = \frac{1}{n}$.

These considerations about OWA operators and domination lead to the assumption that an OWA operator will dominate Łukasiewicz t-norm if (and only if) the weights for arbitrary n form a non-increasing sequence, i.e., $w_{1n} \geq w_{2n} \geq \cdots \geq w_{nn}$. Taking into account that $\sum_{i=1}^{n} w_{in} = 1$, we see that the minimum and the arithmetic mean would be the two extremal cases of OWA operators dominating Łukasiewicz t-norm.

3.2 Transformation of the problem

In the sequel, we will restrict our considerations to n-ary aggregation operators with arbitrary $n \in \mathbb{N}$. Subadditive functions will play an important role in the following considerations.

Definition 7 *A function* $F : [0,c]^n \to [0,c]$ *is* subadditive *on* $[0,c]$, *if the following inequality holds for all* $x_i, y_i \in [0,c]$ *with* $x_i + y_i \in [0,c]$:

$$F(x_1+y_1, \ldots, x_n+y_n) \leq F(x_1, \ldots, x_n) + F(y_1, \ldots, y_n) .$$

Let $\mathbf{A}$ be an n-ary OWA operator with weights $w_1, \ldots, w_n$. If we want to show that $\mathbf{A} \gg T_\mathrm{L}$ it is equivalent to prove that $S_\mathrm{L} \gg \mathbf{A}^d$ because of the isomorphism property (see Lemma 1), i.e., for arbitrary $x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$, the following inequality must hold

$$S_\mathrm{L}(\mathbf{A}^d(x_1, \ldots, x_n),\ \mathbf{A}^d(y_1, \ldots, y_n)) \\ \geq \mathbf{A}^d(S_\mathrm{L}(x_1, y_1), \ldots, S_\mathrm{L}(x_n, y_n))$$

which is furthermore equivalent to

$$\min(\mathbf{A}^d(x_1, \ldots, x_n) + \mathbf{A}^d(y_1, \ldots, y_n), 1) \\ \geq \mathbf{A}^d(\min(x_1 + y_1, 1), \ldots, \min(x_n + y_n, 1)) .$$

Since the OWA operator $\mathbf{A}$ and its dual $\mathbf{A}^d$ are acting on $[0,1]$ and have therefore always values smaller or equal to 1, the last inequality can be rewritten in the following form

$$\mathbf{A}^d(x_1, \ldots, x_n) + \mathbf{A}^d(y_1, \ldots, y_n) \\ \geq \mathbf{A}^d(\min(x_1 + y_1, 1), \ldots, \min(x_n + y_n, 1)) .$$

If $x_i + y_i \leq 1$ for all $i \in \{1, \ldots, n\}$, then we can derive the following formula

$$\mathbf{A}^d(x_1, \ldots, x_n) + \mathbf{A}^d(y_1, \ldots, y_n) \geq \mathbf{A}^d(x_1+y_1, \ldots, x_n+y_n)$$

expressing that $\mathbf{A}^d$ is a subadditive function on $[0,1]$ (compare also [10]). The sufficiency of the subadditivity of $\mathbf{A}^d$ to ensure $S_L \gg \mathbf{A}^d$ follows easily from the monotonicity of $\mathbf{A}^d$.

Observe that the subadditivity of an n-ary aggregation operator $\mathbf{A} : [0,\infty]^n \to [0,\infty]$ is equivalent to its concavity, however, on the domain of the unit interval the subadditivity of $\mathbf{A}$ is a more general property then the concavity.

3.3 Considering weights

Until this point we have said nothing about the properties of the weights of the OWA operator. Corresponding to above we want to prove the following assumption.

Proposition 1 *Let $\mathbf{O}_{(n)}$ be an n-ary OWA operator with weights $w_1,\ldots,w_n$. Then S_L dominates $\mathbf{O}_{(n)}$ if and only if $w_1 \le w_2 \le \cdots \le w_n$.*

Proof. First, we want to show, that if $S_L \gg \mathbf{O}_{(n)}$, then $w_1 \le w_2 \le \cdots \le w_n$. Therefore suppose that there exists some $i \in \{1,\ldots,n\}$ with $w_i > w_{i+1}$. Then we have that

$$\mathbf{O}_{(n)}\left(0,\ldots,0,0,\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right) = \frac{1}{2}\sum_{j=i+1}^{n} w_j,$$

$$\mathbf{O}_{(n)}\left(0,\ldots,0,\frac{1}{2},0,\frac{1}{2},\ldots,\frac{1}{2}\right) = \frac{1}{2}\sum_{j=i+1}^{n} w_j,$$

$$\mathbf{O}_{(n)}\left(0,\ldots,0,\frac{1}{2},\frac{1}{2},1,\ldots,1\right) = \sum_{j=i+2}^{n} w_j + \frac{1}{2}w_{i+1} + \frac{1}{2}w_i \ ,$$

and therefore

$$S_L\left(\left(\mathbf{O}_{(n)}(0,\ldots,0,0,\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right),\mathbf{O}_{(n)}\left(0,\ldots,0,\frac{1}{2},0,\frac{1}{2},\ldots,\frac{1}{2}\right)\right)$$
$$= \sum_{j=i+1}^{n} w_j < \sum_{j=i+2}^{n} w_j + \frac{1}{2}w_{i+1} + \frac{1}{2}w_i$$
$$= \mathbf{O}_{(n)}\left(S_L(0,0),\ldots,S_L(0,0),S_L\left(0,\frac{1}{2}\right),S_L\left(\frac{1}{2},0\right), S_L\left(\frac{1}{2},\frac{1}{2}\right),\ldots,S_L\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

contradictory to $S_L \gg \mathbf{O}_{(n)}$.

Secondly, we will show that for any n-ary OWA operator $\mathbf{O}_{(n)}$ with $w_1 \le w_2 \le \ldots \le w_n$, it holds that it is dominated by S_L. Let us consider a very simple class of n-ary OWA operators $\mathbf{O}_k$ characterized by its weighting vector

$$\mathbf{w} = \Big(\underbrace{0,\ldots,0}_{n-k},\underbrace{\frac{1}{k},\ldots,\frac{1}{k}}_{k}\Big) \ .$$

For arbitrary $x_i, y_i \in [0,1]$ with $i \in \{1,\ldots,n\}$ and $x_i + y_i \le 1$ for all i we have

$$\mathbf{O}_k(S_L(x_1,y_1),\ldots,S_L(x_n,y_n)) = \mathbf{O}_k(x_1+y_1,\ldots,x_n+y_n) = \frac{1}{k}\sum_{j=k}^{n}(x_i+y_i)'_j \ ,$$

whereas $\sum_{j=k}^{n}(x_i+y_i)'_j$ is an element of the set

$$\mathscr{A} = \left\{\sum_{i\in I} x_i + \sum_{j\in J} y_j \mid |I| = |J| = k\right\}$$

and is therefore smaller as the maximum of the set, i.e.

$$\sum_{j=k}^{n}(x_i+y_i)'_j \le \max(\mathscr{A}) = \sum_{j=k}^{n} x'_j + \sum_{j=k}^{n} y'_j = k\cdot(\mathbf{O}_k(x_1,\ldots,x_n) + \mathbf{O}_k(y_1,\ldots,y_n)) \ .$$

Since $\mathbf{O}_k(S_L(x_1,y_1),\ldots,S_L(x_n,y_n)) \le 1$ is always fulfilled due to the boundary conditions of an aggregation operator, we can easily see that the domination relation for S_L and $\mathbf{O}_k$ is fulfilled.

What happens, if there exists some $l \in \{1,\ldots,n\}$ with $x_l + y_l > 1$? Then we have that

$$\mathbf{O}_k(S_L(x_1,y_1),\ldots,S_L(x_n,y_n)) = \mathbf{O}_k(x_1+y_1,\ldots,1,\ldots,x_n,y_n)$$
$$= \frac{1}{k} + \frac{1}{k}\sum_{j=k}^{n-1}(x_i+y_i)'_j$$
$$\le \frac{1}{k}(x_l+y_l) + \frac{1}{k}\sum_{j=k}^{n-1}(x_i+y_i)'_j = \frac{1}{k}\sum_{j=k}^{n}(x_i+y_i)'_j$$

and the same arguments hold as in the previous case.

What is still missing is the fact whether any n-ary OWA operator with non-decreasing weights can be constructed from the set of operators $\mathbf{O}_k$ and whether this construction process does not change the domination property, i.e., yields again a dominated n-ary OWA operator.

Therefore, we deal with the question if there exist $c_k \in [0,1]$ for an arbitrary OWA operator $\mathbf{O}$ with $w_1 \le w_2 \le \ldots \le w_n$ such that $\mathbf{O} = \sum_{k=1}^{n} c_k \mathbf{O}_k$. Since all involved operators are OWA operators of the same arity we do not have to care about the ordering algorithm but have to investigate the weights themselves. An operator constructed as a convex sum of $\mathbf{O}_k$ has the following weighting vector.

$$\mathbf{w} = \left(\frac{c_n}{n},\frac{c_n}{n}+\frac{c_{n-1}}{n-1},\ldots,\sum_{k=1}^{n}\frac{c_k}{k}\right) \ ,$$

Therefore we see that each c_i can be computed from the original weights w_i by

$$c_n = n\cdot w_1 \text{and } c_j = j\cdot(w_{n-j+1} - w_{n-j}) \ .$$

Since $w_{n-j+1} \ge w_{n-j}$ all $c_i \ge 0$. Furthermore,

$$\sum_{j=1}^{n} c_j = \sum_{j=1}^{n} j \cdot (w_{n-j+1} - w_{n-j})$$
$$= w_1 + w_2 + \ldots + w_n = 1$$

yields that all $c_i \leq 1$.

Suppose $\mathbf{O}$ to be an n-ary aggregation operator with non-decreasing weights and $\sum_{k=1}^{n} c_k \mathbf{O}_k$ its representation with respect to the operators $\mathbf{O}_k$ with c_k derived from the weights as described above. We want to show that $\mathbf{O}$ is dominated by S_{L}, therefore consider arbitrary $x_i, y_i \in [0,1]$, $i \in \{1, \ldots, n\}$

$$\mathbf{O}(S_{\mathrm{L}}(x_1,y_1),\ldots,S_{\mathrm{L}}(x_n,y_n))$$
$$= \mathbf{O}(\min(x_1+y_1,1),\ldots,\min(x_n+y_n,1))$$
$$= \sum_{k=1}^{n} c_k \mathbf{O}_k(\min(x_1+y_1,1),\ldots,\min(x_n+y_n,1))$$
$$\leq \sum_{k=1}^{n} c_k \min \mathbf{O}_k(x_1,\ldots,x_n) + \mathbf{O}_k(y_1,\ldots,y_n),1)$$
$$= \min\left(\sum_{k=1}^{n} c_k \cdot (\mathbf{O}_k(x_1,\ldots,x_n) + \mathbf{O}_k(y_1,\ldots,y_n)), \sum_{k=1}^{n} c_k \cdot 1\right)$$
$$= \min\left(\sum_{k=1}^{n} c_k \cdot \mathbf{O}_k(x_1,\ldots,x_n) + \sum_{k=1}^{n} c_k \cdot \mathbf{O}_k(y_1,\ldots,y_n)), 1\right)$$
$$= S_{\mathrm{L}}(\mathbf{O}(x_1,\ldots,x_n), \mathbf{O}(y_1,\ldots,y_n))$$

which completes the proof.

Corollary 1 *Consider an n-ary OWA operator* $\mathbf{A}_{(n)}$ *with weights* $w_1, \ldots, w_n$. *Then* $\mathbf{A}_{(n)}$ *dominates* T_{L} *if and only if* $w_1 \geq w_2 \geq \ldots \geq w_n$.

Proof. It is equivalent for $\mathbf{A}_{(n)} \gg T_{\mathrm{L}}$ to prove that $S_{\mathrm{L}} \gg \mathbf{A}^d_{(n)}$. But the weights of the dual operator $\mathbf{A}^d_{(n)}$ are nothing else then the original weights in reversed order, i.e.,

$$w_1^d = w_n, \quad w_2^d = w_{n-1}, \ldots, w_n^d = w_1 \ ,$$

and are therefore fulfilling $w_1^d \leq w_2^d \leq \ldots \leq w_n^d$.

3.4 Extension to the general case

If we consider an OWA operator

$$\mathbf{A} : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1] \ ,$$

it is clear that $\mathbf{A} \gg T_{\mathrm{L}}$ if and only if $\mathbf{A}_{(n)} \gg T_{\mathrm{L}}$ for all $n \in \mathbb{N}$.

It has been proposed in [15] to derive the weights for an OWA operator from some quantifier function $q : [0,1] \to [0,1]$, which is a monotone real function such that $\{0,1\} \subseteq \mathrm{Ran}\, q$. As a consequence, q can either be non-decreasing with $q(0) = 0$ and $q(1) = 1$ or can be non-increasing with $q(0) = 1$ and $q(1) = 0$.

Since we are looking for aggregation operators dominating T_{L}, the corresponding weights for each n-ary operator must be non-increasing. Therefore we are looking for additional properties for the quantifier function, such that the non-increasingness of the weights is guaranteed. It will turn out, that non-increasingness of the weights is closely related to the concavity, resp. the convexity of the involved quantifier.

Definition 8 *A function f on some convex domain A is* convex, *if the following inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $\lambda \in [0,1]$ *and* $x, y \in A$. *The function is said to be* concave, *if the inequality*

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y)$$

holds for all $\lambda \in [0,1]$ *and* $x, y \in A$.

3.4.1 Non-decreasing quantifiers

First, we will restrict our considerations to non-decreasing quantifiers. Some examples for such functions are shown in Fig. 1. The weights derived from such a quantifier can be computed by

$$w_{in} = q\left(\frac{i}{n}\right) - q\left(\frac{i-1}{n}\right) \ .$$

Lemma 2 *If* $q : [0,1] \to [0,1]$ *is a non-decreasing quantifier for some OWA operator and the generated weights fulfill* $w_{1,n} \geq \cdots \geq w_{n,n}$ for all $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, n\}$, *then q is continuous on* $]0,1]$.

Proof. Suppose that q is not continuous on $]0,1]$, then there exists some $u \in]0,1]$ such that either $\lim_{x \to u^-} q(x) < q(u)$ or $\lim_{x \to u^+} q(x) > q(u)$ (see also Fig. 1).

Supposing that $\lim_{x \to u^-} q(x) < q(u)$, we can define

$$\varepsilon := q(u) - \lim_{x \to u^-} q(x) > 0.$$

For any $n \in \mathbb{N}$ there is $i_n \in \{1, \ldots, n\}$ such that

$$\frac{i_n - 1}{n} < u \leq \frac{i_n}{n} \ .$$

Then we know that

$$q\left(\frac{i_n - 1}{n}\right) \leq \lim_{x \to u^-} q(x) = q(u) - \varepsilon \leq q\left(\frac{i_n}{n}\right) - \varepsilon \ ,$$

concluding that

$$w_{i_n,n} = q\left(\frac{i_n}{n}\right) - q\left(\frac{i_n - 1}{n}\right) \geq \varepsilon \ .$$

Analogously, if $\lim_{x \to u^+} q(x) > q(u)$, we define

$$\varepsilon := \lim_{x \to u^+} q(x) - q(u) > 0 \ ,$$

and for $i_n \in \{1, \ldots, n\}$ such that

$$\frac{i_n - 1}{n} \leq u < \frac{i_n}{n}$$

we can conclude

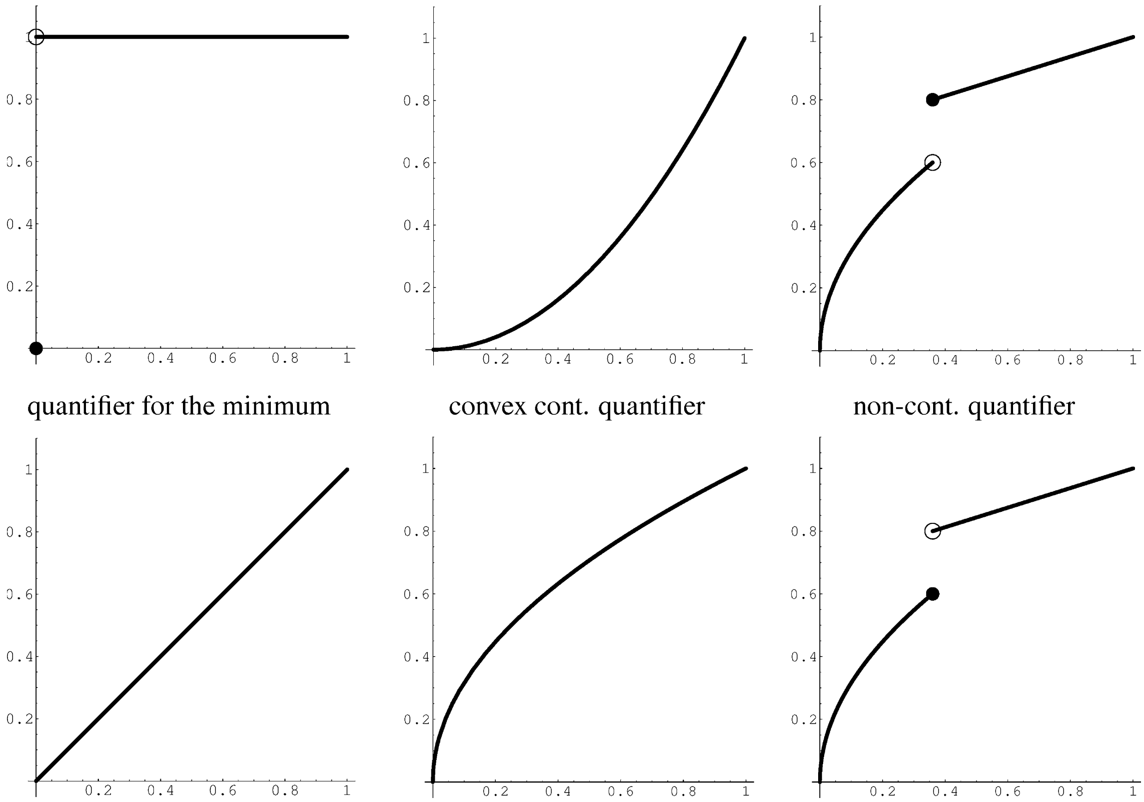


Fig. 1. Some examples of non-decreasing quantifier functions

$$q\left(\frac{i_n}{n}\right) \geq \lim_{x \to u^+} = \varepsilon + q(u) \geq \varepsilon + q\left(\frac{i_n - 1}{n}\right)$$

and

$$w_{i_n,n} = q\left(\frac{i_n}{n}\right) - q\left(\frac{i_n - 1}{n}\right) \geq \varepsilon \ .$$

Therefore we have shown that in both cases $w_{i_n,n} \geq \varepsilon$. Since the weights must be non-increasing we can conclude that $w_{j,n} \geq \varepsilon$ for all $j \in \{1, \ldots, i_n\}$.

Further in both cases we have that $i_n \geq n \cdot u$, such that we can choose $n \in \mathbb{N}$ with $n \cdot u \cdot \varepsilon > 1$. Then the following holds

$$q\left(\frac{i_n}{n}\right) = \sum_{j=1}^{i_n} w_{j,n} \geq i_n \cdot \varepsilon \geq n \cdot u \cdot \varepsilon > 1 \ ,$$

contradictory to the non-decreasingness of q and $q(1) = 1$. Therefore q has to be continuous on $]0, 1]$. □

Proposition 2 *Consider some OWA operator with non-decreasing quantifier* $q : [0, 1] \to [0, 1]$ *and generated weights* $w_{1,n}, \ldots, w_{n,n}$ *for all* $n \in \mathbb{N}$. *Then these weights fulfill* $w_{1,n} \geq \ldots \geq w_{n,n}$ *for all* $n \in \mathbb{N}$ *if and only if* q *is concave on* $]0, 1]$, i.e., $\forall x, y \in [0, 1], \forall \lambda \in [0, 1]$

$$q(\lambda x + (1 - \lambda)y) \geq \lambda q(x) + (1 - \lambda)q(y) \ .$$

Proof. First we will show that concavity of q implies the non-increasingness of the weights, therefore choose some $n \in \mathbb{N}$ and arbitrary $i \in \{2, \ldots, n - 1\}$. Next we define $x = \frac{i-1}{n}$ and $y = \frac{i+1}{n}$, then it holds that

$$q\left(\frac{x + y}{2}\right) = q\left(\frac{i}{n}\right) \geq \frac{1}{2}\left(q\left(\frac{i - 1}{n}\right) + q\left(\frac{i + 1}{n}\right)\right)$$

$$2q\left(\frac{i}{n}\right) \geq q\left(\frac{i - 1}{n}\right) + q\left(\frac{i + 1}{n}\right)$$

$$q\left(\frac{i}{n}\right) - q\left(\frac{i - 1}{n}\right) \geq q\left(\frac{i + 1}{n}\right) - q\left(\frac{i}{n}\right)$$

$$w_{i,n} \geq w_{i+1,n}$$

showing the non-increasingness of the weights for arbitrary $n \in \mathbb{N}$.

For $i = 1$, denote by $u = q(0^+) = \lim_{x \to 0^+} q(x)$. Applying previous ideas and the concavity of q on $]0, 1]$, we see that $q(\frac{1}{n}) - u \geq q(\frac{2}{n}) - q(\frac{1}{n}) = w_{2,n}$. However, then $w_{1,n} \geq q(\frac{1}{n}) - u \geq w_{2,n}$.

Secondly, we will prove that the Jensen inequality holds for all $x, y \in \mathbb{Q} \cap]0, 1]$, i.e.,

$$q\left(\frac{x + y}{2}\right) \geq \frac{q(x) + q(y)}{2} \ ,$$

if we assume that the weights are non-increasing.

If $x, y \in \mathbb{Q} \cap]0, 1]$, then we can find some $n \in \mathbb{N}$ and some $i, j, \in \{1, \ldots, n\}$ such that $x = \frac{i}{n}$ and $y = \frac{j}{n}$. Without loss of generality we suppose that $x \leq y$, i.e., $i \leq j$ and $2i \leq i + j \leq 2j$.

We know that $w_{r,2n} \leq w_{s,2n}$ whenever $r \geq s$, but also

$$\forall t \in \{0, \ldots, 2n-r\} : w_{r+t,2n} \leq w_{s+t,2n} \ ,$$

moreover the inequality holds also for sums with the same amount of summands all of them fulfilling the ordering property mentioned before, i.e., for all $t \in \{0, \ldots, 2n-r\}$

$$\sum_{v=0}^{t} w_{r+v,2n} \leq \sum_{v=0}^{t} w_{s+v,2n} \ ,$$

whenever $r \geq s$.

Now choose $r = i+j+1, s = 2i+1$ and $t = j-i-1$, then we have $r \geq s$ and therefore $w_{r,2n} \leq w_{s,2n}$, but also

$$\sum_{v=0}^{j-i-1} w_{i+j+1+v,2n} = \sum_{v=i+j+1}^{2j} w_{v,2n} \leq \sum_{v=2i+1}^{i+j} w_{v,2n} = \sum_{v=0}^{j-i-1} w_{2i+1+v,2n} \ .$$

Note that

$$\sum_{v=i+j+1}^{2j} w_{v,2n} = q\left(\frac{2j}{2n}\right) - q\left(\frac{2j-1}{2n}\right) + q\left(\frac{2j-1}{2n}\right) - + \cdots - q\left(\frac{i+j}{2n}\right) = q\left(\frac{j}{n}\right) - q\left(\frac{i+j}{2n}\right)$$

and

$$\sum_{v=2i+1}^{i+j} w_{v,2n} = q\left(\frac{i+j}{2n}\right) - q\left(\frac{i+j-1}{2n}\right) + q\left(\frac{i+j-1}{2n}\right) - + \cdots - q\left(\frac{2i}{2n}\right) = q\left(\frac{i+j}{2n}\right) - q\left(\frac{i}{n}\right) \ ,$$

therefore

$$q\left(\frac{j}{n}\right) - q\left(\frac{i+j}{2n}\right) \leq q\left(\frac{i+j}{2n}\right) - q\left(\frac{i}{n}\right) q\left(\frac{i}{n}\right) + q\left(\frac{j}{n}\right)$$
$$\leq 2q\left(\frac{i+j}{2n}\right) \frac{1}{2}\left(q\left(\frac{i}{n}\right) + q\left(\frac{j}{n}\right)\right)$$
$$\leq q\left(\frac{i+j}{2n}\right) \frac{1}{2}(q(x) + q(y))$$
$$\leq q\left(\frac{x+y}{2}\right) \ ,$$

proving the Jensen inequality for all $x, y \in \mathbb{Q} \cap]0,1]$.

We will now extend our results for arbitrary $x, y \in]0,1]$. If $x, y \in]0,1] \setminus \mathbb{Q}$ we can find sequences (x_n) and (y_n) with $x_n, y_n \in]0,1] \cap \mathbb{Q}$ for all $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y \ .$$

We know because of the non-increasingness of the weights that q is continous on $]0,1]$, following that $\lim_{n\to\infty} q(x_n) = q(x)$ and $\lim_{n\to\infty} q(y_n) = q(y)$, but also

$$\lim_{n\to\infty} q\left(\frac{x_n + y_n}{2}\right) = q\left(\frac{x+y}{2}\right) \ .$$

Since for all $n \in N$ the inequality

$$\frac{1}{2}(q(x_n) + q(y_n)) \leq q\left(\frac{x_n + y_n}{2}\right)$$

holds, the following property is fulfilled for arbitrary $x, y \in]0,1]$ due to the continuity of q on $]0,1]$

$$\frac{1}{2}(q(x) + q(y)) \leq q\left(\frac{x+y}{2}\right) \ .$$

Following Aczél [1], the above inequality is equivalent to the concavity of q on $]0,1]$.

Example 3 *A typical example of an OWA operator* **O** *dominating* T_L *is generated by the quantifier function* $q(x) = 2x - x^2$. *Observe that for any* $n \in \mathbb{N}$ *the corresponding weights are given by*

$$w_{in} = \frac{2(n-i)+1}{n^2}, \quad i \in \{1, \ldots, n\} \ .$$

3.4.2
Non-increasing quantifiers

If a quantifier function is non-increasing then the weights can be computed by

$$w_{in} = q\left(\frac{i-1}{n}\right) - q\left(\frac{i}{n}\right) \ .$$

For a few examples of non-increasing quantifiers see Fig. 2.

The following properties can be shown analogously to the case of non-decreasing quantifiers.

Corollary 2 *If* $q : [0,1] \to [0,1]$ *is a non-increasing quantifier for some OWA operator and the generated weights fulfill* $w_{1n} \geq \cdots \geq w_{nn}$ *for all* $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, n\}$, *then* q *is continuous on* $]0,1]$.

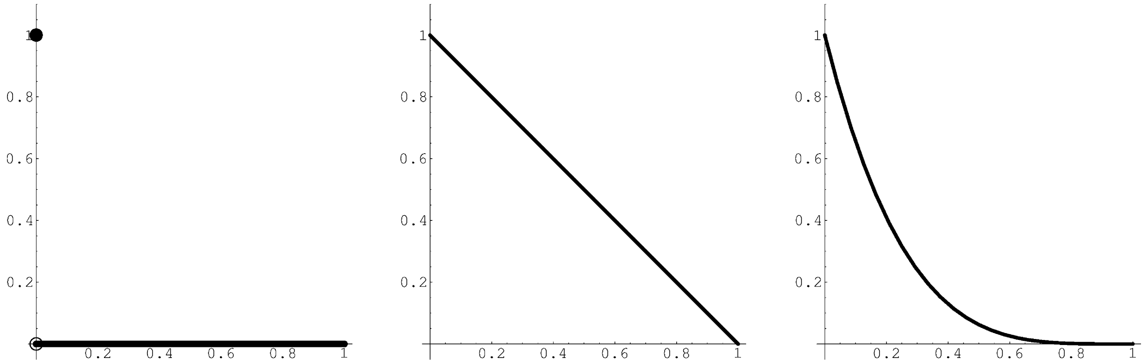
Corollary 3 *Consider some OWA operator with non-increasing quantifier* $q : [0,1] \to [0,1]$. *Then the generated weights fulfill* $w_{1n} \geq \cdots \geq w_{nn}$ *for all* $n \in \mathbb{N}$ *if and only if* q *is convex on* $]0,1]$, *i.e.,* $\forall x, y \in [0,1], \forall \lambda \in [0,1]$

$$q(\lambda x + (1-\lambda)y) \leq \lambda q(x) + (1-\lambda)q(y) \ .$$

4
Transformation to continuous Archimedean t-norms

4.1
Transformation to nilpotent continuous Archimedean t-norms

Any nilpotent t-norm T is isomorphic to the Łukasiewicz t-norm T_L (see [8]), i.e., $T = (T_L)_\varphi$ with $\varphi : [0,1] \to [0,1]$ a strictly increasing bijection. According to Lemma 1, we know that if T_L is dominated by an OWA operator **O** then an isomorphic t-norm $T = (T_L)_\varphi$ is dominated by the aggregation operator $\mathbf{O}_\varphi$. In fact $\mathbf{O}_\varphi$ is nothing else than an ordered weighted quasi-arithmetic mean (OWQA) with



quantifier for the minimum — quantifier for the arithmetic mean — convex continuous quantifier

Fig. 2. Some examples of non-increasing quantifier functions

respect to the strictly increasing bijection $\varphi : [0,1] \to [0,1]$ with corresponding weights $w_{1n} \geq w_{2n} \geq \cdots \geq w_{nn}$ for all $n \in \mathbb{N}$, i.e.,

$$\mathbf{O}_\varphi(x_1,\ldots,x_n) = \varphi^{-1}(\mathbf{O}(\varphi(x_1),\ldots,\varphi(x_n)))$$
$$= \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} w_{in}\varphi(x_i)'\right) = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} w_{in}\varphi(x_i')\right).$$

4.2 Transformation to strict continuous Archimedean t-norms

If we are looking for some aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ which dominates the product t-norm $T_\mathbf{P}$ we can apply once again Lemma 1, i.e., $\mathbf{A} \gg T_\mathbf{P}$ and therefore $\mathbf{A}_\varphi \ll (T_\mathbf{P})_\varphi$ for some strictly decreasing bijection $\varphi : [0,\infty] \to [0,1]$. If we choose the bijection φ by

$$\varphi : [0,\infty] \to [0,1], \varphi(x) = \exp(-x) \ ,$$

we get that

$$T_{\mathbf{P}\varphi}(x,y) = \varphi^{-1}(\varphi(x)\cdot\varphi(y))$$
$$= -\log(\exp(-x)\cdot\exp(-y)) = x + y$$

ensuring that an aggregation operator $\mathbf{A}$ dominates $T_\mathbf{P}$ if and only if its isomorphic transformation $\mathbf{A}_\varphi$ is dominated by the sum, which means in fact that the aggregation operator $\mathbf{A}_\varphi$ is subadditive on $[0,\infty]$.

Applying the proof as in Proposition 1 we see that an OWA operator $\mathbf{B}$ is subadditive if its weights are non-decreasing, i.e., $w^B_{1n} \leq w^B_{2n} \leq \cdots \leq w^B_{nn}$ for all $n \in \mathbb{N}$. As a consequence $\mathbf{A} = \mathbf{B}_{\varphi^{-1}}$ will be an ordered weighted geometric mean with non-increasing weights, i.e., $w^A_{1n} = w^B_{nn} \geq w^A_{2n} \geq \cdots \geq w^A_{nn} = w^B_{1n}$, and it will dominate $T_\mathbf{P}$. Since all strict t-norms are isomorphic to the product t-norm $T_\mathbf{P}$, OWQA operators dominating a given strict t-norm can once again be constructed by applying Lemma 1.

4.3 Considerations about additive generators

As already mentioned in Remark 1 all continuous Archimedean t-norms can be constructed by means of some additive generator. In this part of our contribution we investigate whether we can derive any information about the weights of a dominating OWA operator depending on the involved additive generator.

Therefore let us consider a continuous Archimedean t-norm T with additive generator t and an OWA operator $\mathbf{O} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ which is supposed to dominate T, i.e., for all $n \in \mathbb{N}$ and for all $x_i, y_i \in [0,1], i \in \{1,\ldots,n\}$

$$\mathbf{O}(T(x_1,y_1),\ldots,T(x_n,y_n))$$
$$\geq T(\mathbf{O}(x_1,\ldots,x_n),\mathbf{O}(y_1,\ldots,y_n))$$

If we concentrate on the binary case and choose $x_1 = 0$, $y_1 = 1$, $x_2 = 1$, $y_2 > 0$ then we see that necessarily

$$\mathbf{O}(0,y_2) = w_2 y_2 \geq T(w_2, w_1 y_2 + w_2)$$
$$= t^{-1}(\min(t(0), t(w_2) + t(w_1 y_2 + w_2))) \ ,$$

i.e., for all $y_2 \in [0,1[$

$$t(w_2 y_2) \leq t(w_2) + t(w_1 y_2 + w_2) \ .$$

Evidently if $t(0) = +\infty$ then we get that $w_2 = 0$ because of the continuity of t. Similarly we can show in the general case with $n \in \mathbb{N}$ that $w_i = 0$ for $i > 1$. It follows that for any strict t-norm T only one OWA dominates T, namely the minimum.

In the case of nilpotent t-norms, equation (5) gives a necessary condition for $\mathbf{O} \gg T$.

For $y_2 \to 0^+$ we get that for normed additive generators $1 \leq 2t(w_2)$, i.e., $w_2 \leq t^{-1}(\frac{1}{2})$ holds. This fact can be exploited in determination of OWA operators dominating a specific t-norm. For example, using similar methods as in Section 2, it can be conjectured that an OWA operator with weights $(w_1,\ldots,w_n)$ dominates

- Yager's t-norm T^Y_p [8] with parameter $p \in]0,\infty[$ and normed additive generator $t_p(x) = (1-x)^p$ if and only if

$$w_i \geq \frac{1}{2^{1/p}-1} w_{i+1}, i = 1,\ldots,n-1 \ ,$$

- Schweizer-Sklar's t-norm T^{SS}_λ [8] with parameter $\lambda \in]0,\infty[$ and normed additive generator $t_\lambda(x) = 1 - x^\lambda$ if and only if

$$w_i \geq (2^{1/\lambda} - 1) w_{i+1} \ .$$

Observe that the arithmetic mean $\mathbf{M} \gg T_p^Y$ if and only if $p \leq 1$ and $\mathbf{M} \gg T_\lambda^{SS}$ if and only if $\lambda \geq 1$. Recall that $T_{\mathrm{L}} = T_1^Y = T_1^{SS}$.

5 Summary

Aggregation operators appropriate for fusion of T_{L}-based fuzzy equivalences relations, fuzzy preorders and similar structures based on T_{L}-transitivity have been discussed. We have shown that in the class of OWA operators, those operators with non-increasing weighting vectors dominate T_{L} and thus are appropriate for the above mentioned fusion. Some other distinguished cases of continuous Archimedean t-norms have also been discussed.

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