Ukrainian Mathematical Journal, Vol. 54, No. 1, 2002

UPPER AND LOWER BOUNDS OF A SOLUTION OF THE CAUCHY PROBLEM FOR A STOCHASTIC DIFFERENTIAL EQUATION OF PARABOLIC TYPE WITH POWER NONLINEARITIES (WEAK SOURCE)

S. A. Mel'nik

UDC 519.21

We study the time evolution of a solution of the Cauchy problem for a stochastic differential equation of the parabolic type with power nonlinearities. We construct upper and lower bounds for this solution.

1. Introduction

Consider the following Cauchy problem:

$$du(t,x) = \left(a(u^{\sigma+1})_{xx} + bu^{\beta}\right)dt + c\,u\,dw(t),$$

$$t \in [0;T), \quad x \in R^1, \quad u(0;x) = u_0(x). \tag{1}$$

Here, $w(t)$ is a standard Wiener process, σ, a, b, and c are positive constants, $1 < \beta < \sigma + 1$, and $u_0(x)$ is a bounded even function positive on a certain interval $(-l_0; l_0)$ and equal to zero outside this interval.

A solution of problem (1) is defined as a random process $u(t;x)$ defined on a complete probability space (Ω, F, P), subordinated to a flow of σ-algebras $\{F_t\}_{t\ge 0}$ consistent with $w(t)$, and such that

$$u^{\sigma/2+1} \in C\big((0,T); L_2(R^1\times\Omega)\big), \quad u^{\sigma+1} \in L_2\big((0,T)\times\Omega;\, W_2^1(R^1)\big),$$

and, for any $v \in W_2^1(R^1)$ and $t \in [0;T)$, the following equality holds with probability one:

$$\int_{R^1} u(t;x)v(x)dx - \int_{R^1} u_0(x)v(x)dx = \int_0^t\int_{R^1}\left[-a(u^{\sigma+1})_x v_x + bu^{\beta}v\right]dx\,ds + c\int_0^t\int_{R^1} uv\,dx\,dw(s).$$

We consider solutions symmetric with respect to the point $x = 0$ and satisfying the conditions

$$\forall t \in [0;T):\ u_x(t;0) = 0, \quad u\big(t;x_f(t)\big) = 0, \quad (u^{\sigma+1})_x\Big|_{x=x_f(t)} = 0, \tag{2}$$

where $x_f(t) = \inf\limits_{x\ge 0}\{x: u(t;x) = 0\}$ and $x_f(0) = l_0$. The process $x_f(t)$ is called the front of a solution. In view of the reasoning presented above, we consider a solution of problem (1), (2) only on the semiaxis $x \ge 0$.

Donetsk University, Donetsk. Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 54, No. 1, pp. 63–70, January, 2002. Original article submitted April 26, 2000.

 0041–5995/02/5401–0076 $27.00

Remark. In [1], it was shown that if a solution of problem (1) exists and is unique, then, under the initial conditions indicated above, it is nonnegative for all t and x with probability one. Therefore, the power expressions in (1) and (2) are well defined.

The problem of the existence and uniqueness of a solution is considered below.

Equation (1) is nonlinear. In the deterministic case $(c = 0)$, for the investigation of the behavior of a solution, one constructs upper and lower bounds (see [2, Chap. IV]). In the present paper, we construct a pair of random processes that bound a solution of problem (1), (2) from above and from below for any $t \in [0; T)$ and $x \in R^1$ with probability not less than $1 - \alpha$, $0 < \alpha < 1$. As a basis, we use the methods described in [2, Chap. IV]. Using the estimates constructed, we study the behavior of a solution as time increases.

1. Upper Bound of a Solution

Denote $\xi(t) = -c^2 t/2 + cw(t)$ and $g(t) = e^{\xi(t)}$. We introduce a new unknown function according to the formula $u(t; x) = g(t)v(t; x)$. Then the new unknown function $v(t; x)$ is a solution of the problem

$$v_t = p(t)(v^{\sigma+1})_{xx} + q(t)v^{\beta}, \quad t \in [0; T), \quad x \in R^1, \quad v(0; x) = u_0(x). \tag{3}$$

Here, $p(t) = ag^{\sigma}(t)$ and $q(t) = bg^{\beta-1}(t)$. For $v(t; x)$, conditions (2) take the form

$$\forall t \in [0; T): v_x(t; 0) = 0, \quad v(t; x_f(t)) = 0, \quad (v^{\sigma+1})_x\big|_{x=x_f(t)} = 0. \tag{4}$$

The existence and uniqueness of a solution of problem (3), (4) for almost all $\omega \in \Omega$ follows from the results of [2, pp. 37–38]. Since problems (1), (2) and (3), (4) are equivalent, the existence and uniqueness of a solution of problem (1), (2) are thus established.

In problem (3), (4), we perform the change of variable

$$\hat{\tau} = \hat{\tau}(t) = b\int_0^t g^{\beta-1}(s)\,ds.$$

Then the new unknown function $\hat{v}(\hat{\tau}; x) = \hat{v}(\hat{\tau}(t); x) = v(t; x)$ is a solution of the problem

$$\hat{U}(\hat{v}) \equiv \hat{v}_{\hat{\tau}} - \frac{a}{b} g^{\sigma+1-\beta}(t)(\hat{v}^{\sigma+1})_{xx} - \hat{v}^{\beta} = 0,$$

$$\hat{\tau} \in [0; +\infty), \quad x \in R^1, \quad \hat{v}(0; x) = u_0(x); \tag{5}$$

$$\forall \hat{\tau} \in [0; +\infty): \hat{v}_x(\hat{\tau}; 0) = 0, \quad \hat{v}(\hat{\tau}; x_f(\hat{\tau})) = 0, \quad (\hat{v}^{\sigma+1})_x\big|_{x=x_f(\hat{\tau})} = 0. \tag{6}$$

To construct an upper bound for a solution of problem (5), (6), we use the comparison theorem (Theorem 3 in [2, Chap. I]). Let $\alpha \in (0; 1)$. We construct a function $\hat{V}(\hat{\tau}; x)$ such that the following systems of inequalities is satisfied for any $\hat{\tau} \ge 0$ and $x \ge 0$ with probability not less than $1 - \alpha$:

$$\begin{cases} \hat{U}(\hat{V}(\hat{\tau}; x)) \ \ge \ 0, \\ \hat{V}(0; x) \ \ge \ u_0(x). \end{cases} \tag{7}$$

We set $\hat{V}(\hat{\tau}; x) = \hat{L}(\hat{T} - \hat{\tau})_+^{1/(1-\beta)} z^{1/\sigma}$, where

$$z = \left(1 - \frac{x^2}{\hat{l}^2}(\hat{T} - \hat{\tau})_+^{\frac{\sigma+1-\beta}{\beta-1}}\right)_+, \qquad (f)_+ = \max\{f; 0\}.$$

Here, $\hat{L}$, $\hat{l}$, and $\hat{T}$ are certain positive constants chosen so that inequalities (7) are satisfied with probability not less than $1 - \alpha$. Then

$$\hat{U}(\hat{V}) = (\hat{T} - \hat{\tau})_+^{\frac{\beta}{1-\beta}} \frac{\hat{L}}{\sigma} z^{\frac{1-\sigma}{\sigma}} \hat{G}(t; z),$$

where

$$\hat{G}(t; z) = \frac{\sigma + 1 - \beta}{\beta - 1} - \frac{4a(\sigma + 1)}{b\sigma} \frac{\hat{L}^\sigma}{\hat{l}^2} g^{\sigma+1-\beta}(t)$$
$$+ \left(1 + \frac{2a(\sigma+1)(\sigma+2)}{b\sigma} \frac{\hat{L}^\sigma}{\hat{l}^2} g^{\sigma+1-\beta}(t)\right) z - \sigma \hat{L}^{\beta-1} z^{\frac{\sigma+\beta-1}{\sigma}}, \qquad z \in [0; 1].$$

Since the first three factors in the expression for $\hat{U}(\hat{V})$ are nonnegative, the sign of $\hat{U}(\hat{V})$ coincides with that of $\hat{G}(t; z)$. Note that

$$\hat{G}_{zz}(t; z) = -\frac{(\sigma + \beta - 1)(\beta - 1)}{\sigma} \hat{L}^{\beta-1} z^{\frac{\beta-\sigma-1}{\sigma}} \le 0.$$

Therefore, the function $\hat{G}(t; z)$ is convex upward with respect to z and can attain its minimum value only at the endpoints of the segment $z \in [0; 1]$. We have

$$\hat{G}(t; 0) = \frac{\sigma + 1 - \beta}{\beta - 1} - \frac{4a(\sigma + 1)}{b\sigma} \frac{\hat{L}^\sigma}{\hat{l}^2} g^{\sigma+1-\beta}(t),$$

$$\hat{G}(t; 1) = \frac{\sigma + 1 - \beta}{\beta - 1} - \frac{4a(\sigma + 1)}{b\sigma} \frac{\hat{L}^\sigma}{\hat{l}^2} g^{\sigma+1-\beta}(t) + 1 + \frac{2a(\sigma+1)(\sigma+2)}{b\sigma} \frac{\hat{L}^\sigma}{\hat{l}^2} g^{\sigma+1-\beta}(t) - \sigma \hat{L}^{\beta-1}.$$

If $\hat{L}$ is chosen so that $\sigma\hat{L}^{\beta-1} \le 1$, then $\hat{G}(t;0) \le \hat{G}(t;1)$ and $\min_{z\in[0;1]} \hat{G}(t;z) = \hat{G}(t;0)$. We now choose $\hat{l}$ such that the inequality $\hat{G}(t;0) \ge 0$ is satisfied. First, we choose $\hat{l}$ so that $\hat{G}(0;0) \ge 0$, i.e.,

$$\frac{\sigma+1-\beta}{\beta-1} - \frac{4a(\sigma+1)}{b\sigma}\frac{\hat{L}^{\sigma}}{\hat{l}^{2}} > 0. \tag{8}$$

Since $\hat{L}$ has already been chosen, we can obtain the required result by increasing $\hat{l}$. Thus, for chosen $\hat{L}$ and $\hat{l}$, the following inequality holds for any $t \ge 0$ and $z \in [0; 1]$ with probability one:

$$\hat{G}(t;z) \ge \frac{\sigma+1-\beta}{\beta-1} - \frac{4a(\sigma+1)}{b\sigma}\frac{\hat{L}^{\sigma}}{\hat{l}^{2}} g^{\sigma+1-\beta}(t).$$

Since the process $g(t)$ can take any positive values, the right-hand side of the last inequality can take positive values only with certain probability, which can easily be found by using the results of [3, p. 388] as follows:

$$P\left\{\sup_{0\le t<+\infty} g^{\sigma+1-\beta}(t) < \frac{b\sigma(\sigma+1-\beta)}{4a(\sigma+1)(\beta-1)}\frac{\hat{l}^{2}}{\hat{L}^{\sigma}}\right\}$$

$$= P\left\{\sup_{0\le t<+\infty} \xi(t) < \ln\left(\frac{b\sigma(\sigma+1-\beta)}{4a(\sigma+1)(\beta-1)}\frac{\hat{l}^{2}}{\hat{L}^{\sigma}}\right)^{\frac{1}{\sigma+1-\beta}}\right\}$$

$$= 1 - \left(\frac{4a(\sigma+1)(\beta-1)}{b\sigma(\sigma+1-\beta)}\frac{\hat{L}^{\sigma}}{\hat{l}^{2}}\right)^{\frac{1}{\sigma+1-\beta}} = p^{*}. \tag{9}$$

Thus, the first inequality in system (7) is satisfied with probability not less than p^*.

Consider the second inequality. The function $\hat{V}^{\sigma}(\hat{\tau};x)$, regarded as a function of x, is the positive part of the parabola with the branches directed downward, the vertex at the point $x = 0$, and the roots $x_{1,2} = \hat{l}(\hat{T}-\hat{\tau})_{+}^{-(\sigma+1-\beta)/2(\beta-1)}$. The ordinate of the vertex of this parabola $\hat{V}(\hat{\tau};0) = \hat{L}(\hat{T}-\hat{\tau})_{+}^{1/(1-\beta)}$ is a decreasing function of $\hat{T}$. The modulus of the roots of the parabola also decreases as $\hat{T}$ increases. Since $u_0(x)$ is a bounded nonnegative function different from zero for $x \in (-l_0; l_0)$, there exists $\hat{T}$ such that, for any $k \in R^1$, we have $\hat{V}(0;x) \ge u_0(x)$ and the second inequality of system (7) is satisfied.

Theorem 1. *Let* $\alpha \in (0;1)$. *Then there exist constants* $\hat{L}$, $\hat{l}$, *and* $\hat{T}$ *such that*

$$P\left\{\forall t \ge 0,\ \forall x \in R^1 : u(t;x) \le g(t)\hat{V}(\hat{\tau}(t);x)\right\} \ge 1 - \alpha.$$

Proof. It follows from the equality $p^* = 1-\alpha$ that

$$\frac{\hat{L}^{\sigma}}{\hat{l}^{2}} = \frac{b\alpha^{\sigma+1-\beta}\sigma(\sigma+1-\beta)}{4a(\sigma+1)(\beta-1)}.$$

We set $\widehat{L} = \sigma^{1/(1-\beta)}$. Then

$$\widehat{l} = \sqrt{\frac{4a(\sigma+1)(\beta-1)}{b\alpha^{\alpha+1-\beta}\sigma^{\beta(\beta-1)^{-1}}(\sigma+1-\beta)}}.$$

We now choose the maximum $\widehat{T}$ satisfying the condition

$$\forall x \in R^1 : \widehat{T}^{\frac{1}{1-\beta}}\widehat{L}\left(1 - \frac{x^2}{\widehat{l}^2}\widehat{T}^{\frac{\sigma+1-\beta}{\beta-1}}\right)_+^{\frac{1}{\sigma}} \geq u_0(x).$$

For $\widehat{L}$, $\widehat{l}$, and $\widehat{T}$ thus chosen, the system of inequalities (7) is satisfied with probability not less than $1-\alpha$. Then, according to the comparison theorem (see [2, Chap. I]), we get

$$P\left\{\forall t \geq 0,\ \forall x \in R^1 : \widehat{v}(\widehat{\tau}(t); x) \leq \widehat{V}(\widehat{\tau}(t); x)\right\} \geq 1 - \alpha.$$

Since $u(t; x) = g(t)v(t; x) = g(t)\widehat{v}(\widehat{\tau}(t); x)$, we have

$$P\left\{\forall t \geq 0,\ \forall x \in R^1 : u(t; x) \leq g(t)\widehat{V}(\widehat{\tau}(t); x)\right\} \geq 1 - \alpha,$$

and the theorem is proved.

2. Lower Bound of the Solution

In problem (3), (4), we pass to the new variable

$$\breve{\tau}(t) = a\int_0^t g^{\sigma}(s)\,ds.$$

Then the new unknown function $\breve{v}(\breve{\tau}; x) = \breve{v}(\breve{\tau}(t); x) = v(t; x)$ is a solution of the problem

$$\breve{U}(\breve{v}) \equiv \breve{v}_{\breve{\tau}} - (\breve{v}^{\sigma+1})_{xx} - \frac{b}{a}g^{\beta-\sigma-1}(t)\breve{v}^{\beta} = 0, \tag{10}$$

$$\breve{\tau} \in [0; +\infty), \quad x \in R^1, \qquad \breve{v}(0; x) = u_0(x);$$

$$\forall \breve{\tau} \in [0; +\infty) : \breve{v}_x(\breve{\tau}; 0) = 0, \quad \breve{v}(\breve{\tau}; x_f(\breve{\tau})) = 0, \quad (\breve{v}^{\sigma+1})_x\big|_{x=x_f(\breve{\tau})} = 0. \tag{11}$$

Let $\alpha \in (0; 1)$. We construct a function $\breve{V}(\breve{\tau}; x)$ such that the following system of inequalities is satisfied for any $\breve{\tau} \geq 0$ and $x \geq 0$ with probability not less than $1-\alpha$:

$$\begin{cases} \breve{U}\big(\breve{V}(\breve{\tau};x)\big) \le 0, \\ \breve{V}(0;x) \le u_0(x). \end{cases} \tag{12}$$

We set $\breve{V}(\breve{\tau};x) = \breve{L}(\breve{T}-\breve{\tau})_+^{1/(1-\beta)} z^{1/\sigma}$, where

$$z = \left(1 - \frac{x^2}{\breve{l}^2}(\breve{T}-\breve{\tau})_+^{\frac{\sigma+1-\beta}{\beta-1}}\right)_+ .$$

Here, $\breve{L}$, $\breve{l}$, and $\breve{T}$ are certain positive constants chosen so that inequalities (12) are satisfied with probability not less than $1-\alpha$. Then

$$\breve{U}(\breve{V}) = (\breve{T}-\breve{\tau})_+^{\frac{\beta}{1-\beta}} \frac{\breve{L}}{\sigma} z^{\frac{1-\sigma}{\sigma}} \breve{G}(t;z),$$

where

$$\breve{G}(t;z) = \frac{\sigma+1-\beta}{\beta-1} - \frac{4(\sigma+1)}{\sigma}\frac{\breve{L}^\sigma}{\breve{l}^2} + \left(1 + \frac{2(\sigma+1)(\sigma+2)}{\sigma}\frac{\breve{L}^\sigma}{\breve{l}^2}\right)z - \frac{b}{a}\sigma\breve{L}^{\beta-1}g^{\beta-\sigma-1}(t)z^{\frac{\sigma+\beta-1}{\sigma}}, \quad z \in [0;1].$$

Since the first three factors in the expression for $\breve{U}(\breve{V})$ are nonnegative, the sign of $\breve{U}(\breve{V})$ coincides with that of $\breve{G}(t;z)$. Note that the part of the function $\breve{G}(t;z)$ that is nonlinear in z is nonpositive. Let us analyze the linear part. Denote $\breve{k} = \breve{L}^\sigma/\breve{l}^2$. The root of the linear part is the number

$$z_0 = \frac{1+4(\sigma+1)\sigma^{-1}\breve{k} - \sigma(\beta-1)^{-1}}{1+4(\sigma+1)\sigma^{-1}\breve{k} + 2(\sigma+1)\breve{k}}.$$

It is clear that $z_0 < 1$. Thus, the linear part cannot be negative for all $z \in [0;1]$. We choose

$$\breve{k} > \frac{\sigma(\sigma+1-\beta)}{4(\sigma+1)(\beta-1)}. \tag{13}$$

Then $z_0 > 0$ and, for any $t \ge 0$ and $z \in [0;z_0]$, the inequality $\breve{G}(t;z) \le 0$ is satisfied with probability one. Now let $z \in [z_0;1]$. The coefficient of z is positive, and the coefficient of $z^{(\sigma+\beta-1)/\sigma}$ is negative. Therefore, replacing z by 1 in the linear part and z by z_0 in the nonlinear part, we obtain

$$\breve{G}(t;z) \le \sigma\left(\frac{1}{\beta-1} + \frac{2(\sigma+1)}{\sigma}\breve{k} - \frac{b}{a}\breve{L}^{\beta-1}g^{\beta-\sigma-1}(t)z_0^{\frac{\sigma+\beta-1}{\sigma}}\right).$$

For $z \in [z_0;1]$, the event $\{\forall t \ge 0 : \breve{G}(t;z) \le 0\}$ is now a consequence of the event

$$\left\{ \sup_{0 \le t < +\infty} \xi(t) \le \frac{1}{\sigma + 1 - \beta} \ln\left(\frac{b\breve{L}^{\beta-1} z_0^{(\sigma+\beta-1)\sigma^{-1}}}{a\left((\beta-1)^{-1} + 2(\sigma+1)\sigma^{-1}\breve{k}\right)} \right) \right\}.$$

Note that, by virtue of (13), the value of $\breve{k}$ is fixed and, consequently, z_0 is also fixed. In order that the last event have positive probability, we increase $\breve{L}$ (without changing $\breve{k}$) so that the following inequality is satisfied:

$$b\breve{L}^{\beta-1} z_0^{\frac{\sigma+\beta-1}{\sigma}} > a\left(\frac{1}{\beta-1} + \frac{2(\sigma+1)}{\sigma}\breve{k} \right). \tag{14}$$

Then

$$P\{\forall t \ge 0,\ \forall z \in (z_0; 1]: \breve{G}(t; z) \le 0\} \ge P\left\{ \sup_{0 \le t < +\infty} \xi(t) \le \frac{1}{\sigma+1-\beta} \ln\left(\frac{b\breve{L}^{\beta-1} z_0^{(\sigma+\beta-1)\sigma^{-1}}}{a\left((\beta-1)^{-1} + 2(\sigma+1)\sigma^{-1}\breve{k}\right)} \right) \right\}$$

$$= 1 - \left(\frac{a\left((\beta-1)^{-1} + 2(\sigma+1)\sigma^{-1}\breve{k}\right)}{b\breve{L}^{\beta-1} z_0^{\frac{\sigma+\beta-1}{\sigma}}} \right)^{\frac{1}{\sigma+1-\beta}} = p_* . \tag{15}$$

Thus, the first inequality of system (12) holds with probability not less than p_*.

Consider the second inequality. Note that $u_0(x) > 0$ for $x \in (-l_0; l_0)$. We choose $\breve{T}$ so that, for any $x \in R^1$, we have $\breve{V}(0; x) \le u_0(x)$, and the second inequality of system (12) is satisfied.

Theorem 2. *Let* $\alpha \in (0; 1)$. *Then there exist constants* $\breve{L}$, $\breve{l}$, *and* $\breve{T}$ *such that*

$$P\left\{\forall t \ge 0,\ \forall x \in R^1: u(t; x) \ge g(t)\breve{V}(\breve{\tau}(t); x)\right\} \ge 1 - \alpha .$$

Proof. It follows from the equality $p_* = 1 - \alpha$ that

$$\breve{L} = \left(\frac{a}{b} \alpha^{\beta-\sigma-1} z_0^{\frac{1-\sigma-\beta}{\sigma}} \left(\frac{1}{\beta-1} + \frac{2(\sigma+1)}{\sigma}\breve{k} \right) \right)^{\frac{1}{\beta-1}}.$$

Then inequality (14) is satisfied. We choose $\breve{k}$ according to (13). Then $\breve{l}$ is fixed and z_0 is uniquely determined. We now choose the minimum $\breve{T}$ satisfying the condition

$$\forall x \in R^1: \breve{T}^{\frac{1}{1-\beta}} \breve{L} \left(1 - \frac{x^2}{\breve{l}^2} \breve{T}^{\frac{\sigma+1-\beta}{\beta-1}} \right)_+^{\frac{1}{\sigma}} \le u_0(x).$$

For $\breve{L}$, $\breve{l}$, and $\breve{T}$ thus chosen, the system of inequalities (12) is satisfied with probability not less than $1-\alpha$, and, by analogy with Theorem 1, we complete the proof of Theorem 2.

3. Two-Sided Estimate

Theorem 3. *Let* $\alpha \in (0; 1)$. *Then there exist constants* $\hat{L}$, $\hat{l}$, $\hat{T}$, $\breve{L}$, $\breve{l}$, *and* $\breve{T}$ *such that*

$$P\left\{\forall t \geq 0,\ \forall x \in R^1:\ g(t)\breve{V}(\breve{\tau}(t); x) \leq u(t; x) \leq g(t)\hat{V}(\hat{\tau}(t); x)\right\} \geq 1 - 2\alpha.$$

Proof. The validity of the upper bound follows from (9), and the validity of the lower bound follows from (15). Since we have chosen $p^* = p_* = 1-\alpha$, we have

$$\frac{1}{\sigma+1-\beta}\ln\frac{b(\sigma+1-\beta)\sigma}{4a\hat{k}(\sigma+1)(\beta-1)} = \frac{1}{\sigma+1-\beta}\ln\frac{b\breve{L}^{\beta-1}z_0^{\frac{\sigma+\beta-1}{\sigma}}}{a\left((\beta-1)^{-1}+2(\sigma+1)\sigma^{-1}\breve{k}\right)} = L.$$

Consequently, if the event $\left\{\sup\limits_{0\leq t<+\infty}\xi(t) < L\right\}$ occurs, then, by choosing the constants according to Theorems 1 and 2, we establish that the two-sided inequality holds with probability not less than $1-2\alpha$. The theorem is proved.

4. Limit Behavior of the Solution of Problem (1), (2)

The limit behavior of the solution in the deterministic case ($c = 0$) is known (see [2, Chap. IV]), namely, the solution increases unboundedly at every point of the space; in this case, the lifetime of the solution is finite. Consider the stochastic case. The estimates obtained give certain information about the dynamics of the solution of problem (1), (2). According Theorem 1.1.5 in [4], we have

$$P\left\{\omega:\ \lim_{t\to+\infty}\breve{\tau}(t) = \breve{\eta}(\omega)\right\} = 1, \qquad P\left\{\omega:\ \lim_{t\to+\infty}\hat{\tau}(t) = \hat{\eta}(\omega)\right\} = 1,$$

where $\breve{\eta}(\omega)$ and $\hat{\eta}(\omega)$ are random variables with known distributions. Moreover, according to the law of the iterated logarithm, we have $P\left\{\lim\limits_{t\to+\infty}g(t) = 0\right\} = 1$. Thus, as $t \to +\infty$, the upper bound tends to zero with certain positive probability and the lower bound tends to $+\infty$ also with certain positive probability at all points of the space. One can estimate the behavior of the solution itself as follows: Denote

$$\hat{A} = \{\omega: \hat{\tau}(+\infty) < \hat{T}\}, \qquad \hat{B} = \{\omega:\ \forall t \geq 0,\ \forall x \in R^1\ \ u(t; x) \leq \hat{u}(t; x)\},$$

$$C = \left\{\omega:\ \sup_{0\leq t<+\infty}\xi(t) < L\right\}.$$

Since

$$\widehat{A} \cap C \subseteq \widehat{A} \cap \widehat{B} \subseteq \left\{\omega: \forall x \in R^1 \lim_{t\to+\infty} u(t;x)=0\right\},$$

we have

$$P\left\{\omega: \forall x \in R^1 \lim_{t\to+\infty} u(t;x)=0\right\} \geq P\{\widehat{A} \cap C\}.$$

Thus, the required probability is not smaller than the probability that the vector process $(\xi(t), \widehat{\tau}(t))$ does not leave the set $(-\infty; L) \times [0; \widehat{T})$. This probability, in turn, can be found by the known method [5, p. 495].

Considering the events

$$\breve{A} = \left\{\omega: \breve{\tau}(+\infty) \geq \breve{T}\right\}, \qquad \breve{B} = \left\{\omega: \forall t \geq 0,\ \forall x \in R^1\ u(t;x) \geq \breve{u}(t;x)\right\}$$

and reasoning by analogy, we obtain

$$P\left\{\omega: \forall x \in R^1 \lim_{t\to+\infty} u(t;x)=+\infty\right\} \geq P\{\breve{A} \cap C\}.$$

The last probability can be found by the method presented in [5, p. 495] as the probability that the process $(\xi(t), \widehat{\tau}(t))$ leaves the set $(-\infty; L) \times [0; \widehat{T})$ through its upper wall.

5. Conclusions

The results obtained show that the introduction of a linear stochastic term in Eq. (1) produces a substantial effect on the behavior of the process. In the deterministic case $(c = 0)$, only one scenario of the process life is possible, namely, the unbounded expansion of the space support, infinite growth of the solution at every point of the space, and finite lifetime. In the stochastic case $c \neq 0$, such a scenario is also possible, but only with certain positive probability. In addition, another scenario is also possible with positive probability, namely, the process tends to zero at every point of the space, and the support remains bounded. It also follows from the structure of the estimates that there are no other scenarios.

REFERENCES

1. S. A. Mel'nik, *Maximum Principle for Stochastic Equations of Combustion* [in Russian], Dep. in UkrINTÉI, No. 434-Uk 92 (06.04.92).
2. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Modes with Sharpening in Problems for Quasilinear Parabolic Equations* [in Russian], Nauka, Moscow (1987).
3. V. S. Korolyuk (editor), *Handbook of Probability Theory and Mathematical Statistics* [in Russian], Naukova Dumka, Kiev (1978).
4. E. F. Tsar'kov and V. K. Yasinskii, *Quasilinear Stochastic Differential Functional Equations* [in Russian], Orientir, Riga (1992).
5. I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes* [in Russian], Nauka, Moscow (1977).