Natural Internal Forcing Schemata extending ZFC. Truth in the Universe?

Garvin Melles*
Abraham Fraenkel Center for Mathematical Logic
Bar Ilan University
Institute of Mathematics
Hebrew University of Jerusalem

April 25, 2011

INTRODUCTION

Mathematicians are one over on the physicists in that they already have a unified theory of mathematics, namely set theory. Unfortunately the plethora of independence results since the invention of forcing has taken away some of the luster of set theory in the eyes of many mathematicians. Will man's knowledge of mathematical truth be forever limited to those theorems derivable from the standard axioms of set theory, ZFC? This author does not think so, he feels that set theorists intuition about the universe is stronger than ZFC. Here in this paper, using part of this intuition, he introduces some axiom schemata which he feels are very natural candidates for being considered as part of the axioms of set theory. These schemata assert the existence of many generics over simple inner models. The main purpose of this article is to present arguments for why the assertion of the existence of such generics belongs to the axioms of set theory.

Our central guiding principle in justifying the axioms is what Maddy called the rule of thumb maximize in her survey article on the axioms of set theory, [BAI] and [BAII]. More specifically, our intuition conforms with that expressed by Mathias in his article "What is Maclane Missing?" challenging Mac Lane's view of set theory.

*Would like to thank Ehud Hrushovski for supporting him with funds from NSF Grant DMS 8959511

> This might be a good moment to challenge one of Mac Lane's opinions, which I believe to rest on a misconception. On page 359 of his book he writes, after reflecting on the plethora of independence results, that "for these reasons 'set' turns out to have many meanings, so that the purported foundations of all Mathematics upon set theory totters." Elsewhere, on page 385, he remarks that "the Platonic notion that there is somewhere the ideal realm of sets, not yet fully described, is a glorious illusion."
>
> I would suggest a contrary view: independence results within set theory are generally achieved either by examining an inner model of the universe (an inner model being a transitive class containing all ordinals) or by utilizing forcing to obtain a larger universe of which the original one is an inner model. The conception that begins to seem more and more reasonable with the advance of the inner model program on the one hand and a deeper understanding of iterated forcing on the other is that within one enormous universe there are many inner models, and the various "independence arguments" may be reworked to give positive information about the way various inner models relate to one another. Far from undermining the set theoretic point of view, the various techniques available for building models actually promote that unity.

One of this author's reasons for having an intuition about sets similar to that expressed by Mathias is given by a very short look back at the history of the development of mathematics. Mathematics began with the study of mathematical objects very physical and concrete in nature and has progressed to the study of things completely imaginary and abstract. Most mathematicians now accept these objects as mathematically legitimate as any of their more concrete counterparts. It is enough that these objects are consistently imaginable, i.e., exist in the world of set theory. Applying the same intuition to set theory itself, we get some motivation for why we should accept as sets generic objects over inner models.

Using the rule of thumb maximize, the principle of reflection and esthetics as a basis we will now procede to give further more concrete arguments which say that theories T extending ZFC implying the existence of many generics over inner models reflect truth about the universe of sets V. While the arguments are not technically sophisticated, the author hopes this will not detract from the axioms intuitive appeal.

SET THEORETIC PRELIMINARIES

In order to make this article accessible to the general reader, we review in this section some of the set theoretic basics needed to understand the arguments in favor of the schemata. As this review is sparse, any gaps in the readers understanding can be filled in with reading the appropriate sections of [Jech2] or [Kunen]. Set theorists can safely skip this section except for the last three definitions, which are not standard.

A formula in the language of set theory is a formula in the first order predicate claculus built from atomic formulas of the form $x = y$ and $x \in y$. We denote the universe of sets by V and the satisfaction relation by $\models$. A class is a collection of sets satisfying some formula of set theory. (So all sets are classes.) We denote the class of ordinals by Ord. A set $Y \subseteq X$ is a definable subset of X if for some formula of set theory $\varphi(x)$,

$$\forall y (y \in Y \;\leftrightarrow\; X \models \varphi(y))$$

If X is a set, then $Def(X)$ is the set of all definable subsets of X and $P(X)$ the set of all subsets of X. If we assume the axiom of foundation then the universe of sets V can be written as

$$V = \bigcup_{\alpha \in Ord} V_\alpha$$

where $V_{\alpha+1} = V_\alpha \cup P(V_\alpha)$ and if α is a limit ordinal then

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta$$

By a real we will usually mean a subset of ω, but within set theory a real can also mean an element of $V_{\omega+1}$, ω^ω or $\mathbb{R}$. The constructible universe L (the class of constructible sets) can be written as

$$L = \bigcup_{\alpha \in Ord} L_\alpha$$

where $L_\emptyset = \emptyset$, $L_{\alpha+1} = L_\alpha \cup Def(L_\alpha)$ and if α is a limit ordinal then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$. L is a model of $ZFC + GCH$. L is absolute in the sense that if M and N are transitive models of ZF with the same ordinals, then $L^M = L^N$, i.e., the class of things M thinks is constructible is the class of things which N thinks is constructible. Similarly, if X is a transitive set, $L(X)$ the class of sets constructible from X, can be written as

$$L(X) = \bigcup_{\alpha \in Ord} L(X)_\alpha$$

where $L(X)_\emptyset = X$, $L(X)_{\alpha+1} = L(X)_\alpha \cup Def(L(X)_\alpha)$ and if α is a limit ordinal, then $L(X)_\alpha = \bigcup_{\beta<\alpha} L(X)_\beta$. In general $L(X)$ is a model of ZF but not of AC. If M is a transitive model and $x \in M$, we say x is definable in M if for some formula $\varphi(v)$ of set theory, $M \models \exists! v\varphi(v) \wedge \varphi(x)$. Now we turn to a short review of forcing. If P is a partial order, a subset D of P is said to be dense if for every $p \in P$, there is a $d \in D$ such that $d \leq p$. If $p, q \in P$ such that there is no $r \in P$ such that $r \leq p$ and $r \leq q$, then we say p and q are incompatible, written, $p \perp q$. If P is a partial order in a transitive model M of ZFC, then a subset G of M is said to be M generic if G intersects every dense subset of P in M. If M is a transitive model of ZFC and P is a partial order in M such that for every $p \in P$ there is r and q in P such that $r \leq p$, $q \leq p$, $r \perp q$, and if G is an M generic subset of P, then there is a transitive model $M[G]$ of ZFC such that

1. $G \in M[G]$ and $G \notin M$
2. $Ord^M = Ord^{M[G]}$
3. If N is a transitive model such that $M \subseteq N$, $Ord^M = Ord^N$, and $G \in N$, then $M[G] \subseteq N$.

$M[G]$ is called a forcing extension of M. A partial order P is separative if for every $p, q \in P$, $p \not\leq q \rightarrow \exists r(r \leq p \wedge r \perp q)$. Without loss of generality we can assume all the partial orders we use are separative. Associated with every separative partial order is the set of regular open subsets of P, denoted $r.o.(P)$.

$$r.o.(P) = \Big\{ S \subset P \mid \forall p, q \in P\ p \in S \wedge q \leq p \rightarrow q \in S \wedge$$
$$\forall p \in P(\forall r \in P(r \leq p \rightarrow \exists q \in S(q \leq r)) \rightarrow p \in S) \Big\}$$

With the appropriate interpretation of $+$ and $\bullet$, $r.o.(P)$ is a complete Boolean algebra. If κ and λ are cardinals, a Boolean algebra B is (κ, λ)

distributive iff every collection of κ partitions of B of size at most λ has a common refinement.

Theorem 1. If M is a transitive model of ZFC, κ is a cardinal in M and $M[G]$ is a forcing extension of M via the partial order P, then $M[G]$ has no functions $f : \kappa \to \kappa$ not in the ground model if and only if $r.o.(P)$ is (κ, κ) distributive.
PROOF See [Jech2].

Definition 1. A subset r of ω is said to be absolutely definable if for some Π_1 formula $\theta(x)$,

1. $V \models \theta(r)$
2. $V \vdash \exists! \, x\theta(x)$

A canonical example of an absolutely definable real is $0^{\#}$.

Definition 2. $x \in V$ is said to be weakly absolutely definable if for some Σ_1 formula $\psi(x)$

$$V \models \forall y(y \in x \ \leftrightarrow \ \psi(y)\,)$$

Definition 3. $x \in V$ is said to be weakly absolutely definable of the form V_α if for some ordinal α definable in L,

$$V \models \forall y(y \in x \ \leftrightarrow \ \rho(y) \leq \alpha)$$

THE SCHEMATA

In this section we list the Schemata. Schemata we know how to prove the consistency of assuming the existence of a countable transitive model of ZFC we label with a (*). The consistency of the other schemata are just conjectured, and we write (Conj) besides those. Schemata which follow from large cardinal assumptions we write (FLC) besides. Note that the definitions are informal as the formal versions are unwieldy.

Definition 4. (*) (FLC) $IFS(L)$ is the axiom schema which says for every formula $\phi(x)$, if $L \models$ there is a unique partial order P such that $\phi(P)$, then there is a L generic subset of P in the universe V.

Definition 5. (*) (FLC) $IFS(L[r])$ is the axiom schema of set theory which says if r is an absolutely definable real then every partial order P definable in $L[r]$ has an $L[r]$ generic subset.

Definition 6. (*) $IFS(L(\mathbb{R})$ is the axiom schema of set theory which says if P is a partial order definable in $L(\mathbb{R})$ such that

$$\Vdash \mathbb{R} = \mathbb{R}^{V^P}$$

(By $\Vdash$ we mean $\Vdash$ in V) then there exists a $L(\mathbb{R})$ generic subset G of P.

Definition 7. (*) $IFS(L(V_\alpha))$ is the axiom schema which says if V_α is weakly absolutely definable of the form V_α and P is a partially ordered set definable in $L(V_\alpha)$ such that

$$\Vdash V_\alpha = V_\alpha^{V^P}$$

then there is a $L(V_\alpha)$ generic subset G of P.

Definition 8. (Conj) $IFS(\forall L(V_\alpha))$ is the axiom schema which says for all $\alpha \in Ord$, if P is a partially orderd set definable in $L(V_\alpha)$ such that

$$\Vdash V_\alpha = V_\alpha^{V^P}$$

then there is a $L(V_\alpha)$ generic subset G of P.

Definition 9. (*) $IFS^{-}(\forall L(V_\alpha))$ is the axiom schema which says for each ordinal α if P is a partial order definable in $L(V_{\omega+\alpha})$ such that P is $\aleph_\beta$ closed for each $\beta < \alpha$, then there is a $L(V_{\omega+\alpha})$ generic subset G of P.

Definition 10. (Conj) IFS is the axiom schema of set theory which says for every weakly absolutely definable set X, for every partial order P definable in $L(X)$, if

$$\Vdash \ X^{V^P} = X$$

then there exists an $L(X)$ generic subset G of P.

Definition 11. (FLC + CH) $IFS \restriction X \subseteq \mathbb{R}$ makes the same claim as IFS but only for $X \subseteq \mathbb{R}$.

Definition 12. (*) $CIFS$ is the axiom schema of set theory which says that for each regular cardinal $\aleph_\alpha$ all definable subsets of $L(V_{\omega+\alpha})$ are of size at most $\aleph_\alpha$.

We conjecture that $IFS(\forall L(V_\alpha))$ is consistent, since it is a natural generalization of $IFS(L(V_\alpha))$. The intuition behind the conjecture that IFS is consistent is somewhat more nebulous. One can look upon $IFS(L(V_\alpha))$

as saying the universe has a sort of minimal largeness with respect to the $L(V_\alpha)$ because it implies there are many $L(V_\alpha)$ generics in the universe. The $L(V_\alpha)$ provide a reference frame from which to measure the size of the universe, since $L(V_\alpha)$ is absolute for any class model of ZFC containing V_α and $L(V_\alpha)$ generic subsets for partial orders definable in $L(V_\alpha)$ maintain their genericity under extensions as long as V_α (and the class of ordinals) is not changed. If X is weakly absolutely definable, $L(X)$ is also absolute for any class model of ZFC containing X. So IFS is a natural generalization of $IFS(L(V_\alpha))$ implying the universe has a minimal largeness with respect to each of the $L(X)$. For a given weakly absolutely definable set X the consistency of $IFS \restriction X$ is easy to show.

WHY SHOULD THE SCHEMATA SHOULD HOLD IN V ?

Our version of the 'rule of thumb maximize' will take the form of the following three principles:

1. $V \models ZFC$
2. V is large with respect to Ord
3. V is large with respect to each of the V_α

In order to get a better handle on what principles two and three mean, we shall use countable transitive sets as models for transitive classes scaled down to a countable size. We will take a look at countable transitive models of ZFC satisfying principles two and three and look for common and esthetically pleasing properties among them, i.e., properties that we think V itself should satisfy. In order to see what principle two gives us, we fix the height of the models under consideration i.e., we assume all our models have the same set of ordinals α. (And we also assume of course that countable transitive models of ZFC with ordinals α exist.) So we are using α as a model for Ord. Now there is a unique countable transitive model of $ZFC + V = L$ with height α, namely the set L_α. $L_\alpha \subseteq M$ for every M which is a countable transitive model of ZFC of height α and $L^M = L_\alpha$. So the statement $V = L$ expresses a kind of minimal property, the opposite of what we are looking for. On the other hand, the statement $IFS(L)$ is a kind of minimal maximality condition among the countable transitive models of ZFC with given height. Why? Suppose M, N are countable transitive models of ZFC such that $M \models IFS(L)$ with $M \subseteq N$ and

$Ord^N = Ord^M$. Then $N \models IFS(L)$ since the interpretation of L and of the L generics for the various partial orders definable in L are absolute. Furthermore, larger models tend toward $IFS(L)$, i.e., given any countable transitive model M of ZFC and any finite list $P_1, \ldots, P_n$ definable in L^M, if we let N be the forcing extension of M by the partial order $P_1 \times \ldots \times P_n$ then N has the same height as M and satisfies $IFS(L \restriction \{P_1, \ldots, P_n\})$. So $IFS(L)$ is a natural closure condition on the countable transitive models of ZFC of given height. The arguments for the axiom schema $IFS(L[r])$ have similar justifications. As we consider that the relationships among countable transitive models of ZFC are reflections of the relationships among transitive class models of ZFC, we argue that $IFS(L)$ and $IFS(L[r])$ should hold in V. If $V \not\models IFS(L)$, it would be as if the universe had an artificial boundary. It seems it would be an artificial constraint on V if for some P a partial order definable in L there is no L generic. Note that under $ZFC + IFS(L[r])$, all the generics asserted to exist by the axioms of $IFS(L[r])$ are in $L(\mathbb{R}) = L(V_{\omega+1})$, so $IFS(L[r])$ is really a schema about the structure of $L(\mathbb{R})$. Note also that to be more formal and to work strictly within ZFC we could have made our arguments using countable transitive models of arbitrarily large finite parts of ZFC and the schemata.

Why do we work with partial orders P definable in L and not all $P \in L$? In the first place axioms asserting the existence of generics for all $P \in L$ are inconsistent with ZFC, but the main point is that we are interested not in countable transitive models but in proper class models of ZFC and forcing only gives the relative consistency of extensions of L of the form $L[G]$ only for those $G \subseteq P$ where P is a partial order definable (without parameters) in L. In keeping with our principal of maximality we reinterpret this to mean that such generic extensions of L actually exist.

To investigate the consequences of principle three, we fix both the height and the width at stage $\omega + 1$ (i.e., $V_{\omega+1}$) among the models (which we can assume satisfy $ZFC + IFS(L[r])$) under consideration. (All transitive models of ZFC have the same first ω stages in the cumulative heirarchy.) So we are using some countable ordinal as a model for Ord and some countable set of reals as a model for the reals. Arguing as before we see that $IFS(L(\mathbb{R}))$ is a natural closure condition on this class of models, implying a minimal kind of maximality. Similarly, we argue that among the countable transitive models of given height and set of reals satisfying $ZFC+IFS(L[r])$ a natural closure property is that all sets definable in $L(\mathbb{R})$ are of size at most $\aleph_1$ since the canonical forcing which collapse definable elements of

$L(\mathbb{R})$ to size $\aleph_1$ are ω closed and therefore do not add reals. Continuing to make use of our third principle, similar reasoning works for all the definable stages V_α so we are lead to $IFS(L(V_\alpha))$, $IFS^-(\forall L(V_\alpha))$, $IFS(\forall L(V_\alpha))$, and $CIFS$.

Another justification for the schemata (see [BA I] page 492-493) is that they are a way of making the power set thick. More precisely, insteads of making $P(V_\alpha) - V_\alpha$ large, they make $L(V_{\alpha+1}) - L(V_\alpha)$ large, a slight variant of the notion that the power set operation should be large. This is one of the appeals behind $IFS(\forall L(V_\alpha))$, $IFS^-(\forall L(V_\alpha))$ and $CIFS$. Even under $IFS^-(\forall L(V_\alpha))$, for all regular cardinals $\aleph_\alpha$, $L(V_{\omega+\alpha+1}) - L(V_{\omega+\alpha}) \neq \emptyset$.

CONNECTIONS WITH LARGE CARDINALS

It is not hard to see that $0^\#$ exists implies $IFS(L)$ since as we show later that $IFS(L)$ is equivalent to the assumption that every set definable in L is countable. So $IFS(L)$ is a kind of intrinsic support for the large cardinal axiom $0^\#$ exists. Similarily the picture of the universe given by $IFS(L[r])$ is related to that under the assumption of a measurable cardinal. If a measurable cardinal exists than $r^\#$ exists for every $r \subseteq \omega$, so that means for every $r \subseteq \omega$, every set definable in $L[r]$ is countable. As we shall soon prove, $IFS(L[r])$ holds if and only if for every r which is an absolutely definable real, every set definable in $L[r]$ is countable. So again $IFS(L[r])$ provides a kind of intrinsic support for large cardinal axioms, in that they give at some level similar pictures of the universe, even though the consistency strength of the large cardinal axioms are much greater than that of $IFS(L)$ or $IFS(L[r])$. The most important connection between large cardinals and the schemata known to the author is the fact which was pointed out to him by Woodin that under large cardinal hypotheses, $IFS(\restriction X \subseteq \mathbb{R})$ is equivalent to CH.

CONSISTENCY FROM A COUNTABLE TRANSITIVE MODEL OF ZFC

Theorem 2. Let $\langle \theta_i \mid i < n \rangle$ and $\{\varphi_{ij}(x) \mid i < n, j < m, \}$ be finite sets of formulas with the θ_i being Π_1. Let M be a countable transitive model of ZFC. Then there exists a countable transitive model M' of M with the same ordinals as M such that for each $i < n$,

$$M' \models \exists! r \theta_i(r) \ \wedge \ \theta_i(r_i) \ \rightarrow$$

$$\bigwedge_{j<m} \Big((\, L[r_i] \models \exists! P(\varphi_{ij}(P)) \ \wedge \ \varphi_{ij}(P_i)\,) \ \rightarrow \ \exists G \subseteq P_{ij}(G \text{ is } L[r_i] \text{ generic})\Big)$$

PROOF Let $\alpha^* \in M$ such that

$$M \models \alpha^* > sup\Big\{ |\mathcal{D}_{ij}| \mid \exists! x \theta_i(x) \ \wedge \ \theta(r_i) \ \wedge \ r_i \subseteq \omega \ \wedge$$

$$L[r_i] \models \exists! P(\, \varphi_{ij}(P) \ \wedge \ \varphi_{ij}(P_{ij}) \ \wedge \ \mathcal{D}_{ij} \text{ is the set of dense subsets of } P_{ij})\Big\}$$

Let P be the set of finite partial one to one functions from α^* to ω. Let $M' = M[G]$ where G is a M generic subset of P. Note that by the Levy-Shoenfield absoluteness lemma, if $M \models \theta_i(r_i)$ then also $M' \models \theta_i(r_i)$. Since all the $\mathcal{D}_{ij}$ are countable in M' the P_{ij} have $L[r_i]$ generic subsets in M'. To finish the proof it is enough to prove the following claim.

Claim: If a formula $\psi(x)$ defines a real in $M[G]$ then it is in M.
PROOF Suppose r is the unique real satisfying $\psi(x)$ in $M[G]$. Since P is separative, if $p \in P$ and π is an automorphism of P, then by [Jech 2] lemma 19.10, for every formula $\varphi(v_1, \ldots, v_n)$ and names $x_1, \ldots, x_n$

$$* \quad p \Vdash \varphi(x_1, \ldots, x_n) \ \text{ iff } \ \pi p \Vdash \varphi(\pi x_1, \ldots, \pi x_n)$$

Let $\varphi(x) = \exists\, Y(\psi(Y) \wedge x \in Y)$. Let $n \in \omega$. We will show that $||\varphi(\check{n})|| = 0$ or $||\varphi(\check{n})|| = 1$. If for no $p \in P$ does $p \Vdash ||\varphi(\check{n})||$ then $||\varphi(\check{n})|| = 0$. So let $p \in P$ such that $p \Vdash ||\varphi(\check{n})||$. By $*$ if π is an automorphism of P then $\pi p \Vdash ||\varphi(\check{n})||$. Let π be a permutation of ω. π induces an automorphism of P by letting for $p \in P$, $dom\, \pi p = dom\, p$ and letting $\pi p(\alpha) = \pi(p(\alpha))$. By letting π vary over the permutations of ω it follows that $||\varphi(\check{n})|| = 1$. Let $\dot{r}$ be the name with domain $\{\check{n} \mid n < \omega\}$ and such that

$$\dot{r}(\check{n}) = ||\varphi(\check{n})||$$

$i_G(\dot{r}) = r$, but then $r = \{n \mid ||\varphi(\check{n})|| = 1\}$ which means it is in M.

Corollary 3. If there is a countable transitive model of ZFC then

$$Con(ZFC \ + \ IFS(L[r]))$$

Corollary 4. $ZFC + IFS(L[r])$ + 'there are no absolutely definable non-constructible reals' is consistent. (Relative to the assumption of a countable transitive model of ZFC)

Theorem 5. If there is a countable transitive model of ZFC then

$$Con(ZFC + IFS(L(V_\alpha)))$$

PROOF Let $\langle \theta_i \mid i < n \rangle$ and $\{\varphi_{ij}(x) \mid i < n, j < m,\}$ be finite sets of formulas. Let M be a countable transitive model of ZFC. Without loss of generality we can assume there exists ordinals $\alpha_0, \dots \alpha_{n-1}$ such that

$$L^M \models \exists! \alpha \theta_i(\alpha) \ \wedge \ \theta_i(\alpha_i)$$

and $\alpha_j < \alpha_k$ for $j < k$. It is enough to find a forcing extension N of M such that for each $i < n$ and $j < m$ for some partial order $P_{ij} \in N \quad N \models$

$$L(V_{\alpha_i}) \models \exists! x \psi_{ij}(x) \ \wedge \ L(V_{\alpha_i}) \models \psi_{ij}(P_{ij})$$

$$\wedge \Vdash V_{\alpha_i} = V_{\alpha_i}^{P_{ij}}$$

$$\rightarrow \ \exists G(G \text{ is a } L(V_{\alpha_i}) \text{ generic subset of } P_{ij})$$

We define by induction on the lexicographical order of $n \times m$ sets G_{ij}. Suppose P_{ij} is a partial order definable in $L(V_{\alpha_i})^{M[\{G_{h,l} | h \leq i, l < j\}]}$ by $\varphi_{ij}(x)$ and there exists a $M[\{G_{h,l} | h \leq i, l < j\}]$ generic subset of P_{ij} not increasing

$$V_{\alpha_i}^{M[\{G_{h,l} | h \leq i, l < j\}]}$$

Then let G_{ij} be such a $M[\{G_{h,l} | h \leq i, l < j\}]$ generic subset of P_{ij}. (If not, let $G_{ij} = \emptyset$.) Let

$$N = M[\{G_{ij} | i < n, j < m\}]$$

Theorem 6. If there is a countable transitive model of ZFC, then

$$Con(ZFC + IFS(L(V_\alpha)) + IFS(L[r]))$$

PROOF Similar, just start with a model of enough of $IFS(L[r])$.

Theorem 7. If there is a countable transitive model of ZFC then

$$Con(ZFC + IFS^-(\forall L(V_\alpha))) \ \wedge \ Con(ZFC + CIFS)$$

PROOF See the companion paper.

SOME CONSEQUENCES AND SOME NICER FORMS

Below we give some consequences and equivalents assuming ZFC holds in V.

Theorem 8. $IFS(L(\mathbb{R})) \vdash CH$

PROOF Every bijection between a countable ordinal and a subset of $\mathbb{R}$ is an element of $L(\mathbb{R})$ and $\omega_1 = \omega_1^{L(\mathbb{R})}$. So if $P =$ the set of bijections from countable ordinals into $\mathbb{R}$ then P is a definable element of $L(\mathbb{R})$. Since P is σ closed, a P generic over V will not add any reals, so by $IFS(L(\mathbb{R}))$ there is a $G \subseteq P$ which is $L(\mathbb{R})$ generic. If α is an ordinal less than ω_1 and r is a real, let $D_\alpha = \{p \in P \mid \alpha \in dom\, p\}$ and $D_r = \{p \in P \mid r \in ran\, p\}$. For each $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$ and for each $r \in \mathbb{R}$, $G \cap D_r \neq \emptyset$, so $\bigcup G$ is a bijection from ω_1 to $\mathbb{R}$.

Theorem 9. $IFS(L(\mathbb{R}))$ iff every P definable in $L(\mathbb{R})$ such that $r.o.(P)$ is (ω, ω) distributive has an $L(\mathbb{R})$ generic subset.

PROOF By Theorem 1.

Theorem 10. $IFS(\forall L(V_\alpha))$ iff every $\alpha \in Ord$ and P definable in $L(V_{\omega+\alpha})$ such that $r.o.(P)$ is $(\aleph_\beta, \aleph_\beta)$ distributive for each $\beta < \alpha$, has an $L(V_\alpha)$ generic subset.

PROOF By Theorem 1.

Theorem 11. $IFS(L) \leftrightarrow$ every set definable in L is countable.

PROOF Certainly if every set definable in L is countable, then if P is a partial order definable in L then so is $\mathcal{D}$ the set of dense subsets of P in L, so $\mathcal{D}$ is countable and therefore P has a generic subset over L in the universe. In the other direction, if s is a set definable in L, then so is the partially ordered set consisting of maps from distinct finite subsets of s to distinct finite subsets of ω, so a L generic subset over the partial ordering is a witness to $|s| = \omega$.

So $CIFS$ is a natural generalization of $IFS(L)$.

Theorem 12. $IFS(L[r]) \leftrightarrow$ for every absolutely definable real r, every set definable in $L[r]$ is countable.

PROOF Similar to the previous proof.

Theorem 13. $IFS^-(\forall L(V_\alpha)) \rightarrow \aleph_\alpha = \aleph_\alpha^{L(V_{\omega+\alpha})} \wedge L(V_{\omega+\alpha+1}) \models |V_{\omega+\alpha}| = \aleph_\alpha$

PROOF By induction on α. If α is a limit ordinal then certainly for $\beta < \alpha$

we have by the induction hypothesis,

$$\aleph_\beta = \aleph_\beta^{L(V_{\omega+\beta})}$$

which implies $\aleph_\alpha = \aleph_\alpha^{L(V_{\omega+\alpha})}$. By the induction hypothesis we also have $|V_{\omega+\alpha}| = \aleph_\alpha$. So in V there is a subset of $V_{\omega+\alpha} \times V_{\omega+\alpha}$ which is a well ordering of $V_{\omega+\alpha}$ of order type $\aleph_\alpha$. Since $P(V_{\omega+\alpha} \times V_{\omega+\alpha}) \in L(V_{\omega+\alpha+1})$,

$$L(V_{\omega+\alpha+1}) \models |V_{\omega+\alpha}| = \aleph_\alpha$$

If $\alpha = \beta + 1$ then since by the induction hypothesis we have that $\aleph_\beta = \aleph_\beta^{L(V_{\omega+\beta})}$ and $L(V_{\omega+\alpha}) \models |V_{\omega+\beta}| = \aleph_\beta$, all order types of ordinals less than $\aleph_\alpha$ are incoded by subsets of $V_{\omega+\beta} \times V_{\omega+\beta}$. Since $P(V_{\omega+\beta} \times V_{\omega+\beta}) \in L(V_{\omega+\alpha})$, this implies $\aleph_\alpha = \aleph_\alpha^{L(V_{\omega+\alpha})}$. Let P be the partial order of all one to one maps from initial segments of $\aleph_\alpha$ into $V_{\omega+\alpha}$. P is $\aleph_\beta$ closed and P is a definable element of $L(V_{\omega+\alpha})$. By $IFS^-(\forall L(V_\alpha))$, there exists a $G \subseteq P$ which is $L(V_{\omega+\alpha})$ generic. $\bigcup G$ is a bijection from $\aleph_\alpha$ onto $V_{\omega+\alpha}$. Since there is a subset of $V_{\omega+\alpha} \times V_{\omega+\alpha}$ which is a well ordering $V_{\omega+\alpha}$ of order type $\aleph_\alpha$ and $P(V_{\omega+\alpha} \times V_{\omega+\alpha}) \in L(V_{\omega+\alpha+1})$, $V_{\omega+\alpha}$ has size $\aleph_\alpha$ in $L(V_{\omega+\alpha+1})$.

Corollary 14. $IFS^-(\forall L(V_\alpha)) \vdash GCH$

Theorem 15. $IFS(\forall L(V_\alpha)) \rightarrow \aleph_\alpha = \aleph_\alpha^{L(V_{\omega+\alpha})} \wedge L(V_{\omega+\alpha+1}) \models |V_{\omega+\alpha}| = \aleph_\alpha$

PROOF Exactly the same as for $IFS^-(\forall L(V_\alpha))$.

Corollary 16. $IFS(\forall L(V_\alpha)) \vdash IFS^-(\forall L(V_\alpha))$.

Theorem 17. $IFS^-(\forall L(V_\alpha))$ implies that for every regular cardinal $\aleph_\alpha$,

$$L(V_{\omega+\alpha+1}) - L(V_{\omega+\alpha}) \neq \emptyset$$

PROOF Suppose not. Let $\aleph_\alpha$ be the least regular cardinal such that

$$L(V_{\omega+\alpha+1}) = L(V_{\omega+\alpha})$$

Note that α is definable in $L(V_{\omega+\alpha})$ either as the least β such that $V = L(V_{\omega+\beta})$ or as the least β such that $V = L(V_{\omega+\beta-1})$. Let P be the set of bijections between subsets of $\aleph_{\alpha+1}$ of size less than $\aleph_\alpha$ into subsets of $\aleph_\alpha$. P is a definable element of $L(V_{\omega+\alpha+1})$ and since $L(V_{\omega+\alpha+1}) = L(V_{\omega+\alpha})$, P is a definable element of $L(V_{\omega+\alpha})$. As $\aleph_\alpha$ is regular, P is $< \aleph_\alpha$ closed.

By $IFS^-(\forall L(V_\alpha))$, there is a $L(V_{\omega+\alpha})$ generic subset G of P in V. $\bigcup G$ is a bijection from $\aleph_{\alpha+1}$ to $\aleph_\alpha$ a contradiction.

Theorem 18. $CIFS$ implies $IFS^-(\forall L(V_\alpha))$.
PROOF Let $\aleph_\alpha$ be regular and P definable in $L(V_{\omega+\alpha})$ such that P is $\aleph_\beta$ closed for every $\beta < \alpha$. Since P is definable in $L(V_{\omega+\alpha})$ so is the collection $\mathcal{D}$ of dense subsets of P in $L(V_{\omega+\alpha})$ so $|\mathcal{D}| \leq \aleph_\alpha$. List $\mathcal{D}$ as $\{D_\zeta \mid \zeta < \aleph_\alpha\}$. Now by induction on $\zeta < \aleph_\alpha$, by the $< \aleph_\alpha$ closedness of P we can build a sequence $\langle p_\zeta \mid \zeta < \aleph_\alpha\rangle$ such that $p_\zeta \in D_\zeta$. Let G be the filter generated by the $\langle p_\zeta \mid \zeta < \aleph_\alpha\rangle$. Now let $\aleph_\alpha$ be singular and P definable in $L(V_{\omega+\alpha})$ such that P is $\aleph_\beta$ closed for every $\beta < \alpha$. Since $\aleph_\alpha$ is singular, P is also $\aleph_\alpha$ closed. P is definable in $L(V_{\omega+\alpha+1})$ and so is the set $\mathcal{D}$ of dense subsets of P in $L(V_{\omega+\alpha+1})$, so $|\mathcal{D}| \leq \aleph_{\alpha+1}$. As before we can build an $L(V_{\omega+\alpha+1})$ generic subset of P.

Theorem 19. $CIFS \rightarrow$ for each regular cardinal $\aleph_\alpha$, every set definable in $L(V_{\omega+\alpha})$ has size a most $\aleph_\alpha$ in $L(V_{\omega+\alpha+1})$.
PROOF Let x be definable in $L(V_{\omega+\alpha})$ where $\aleph_\alpha$ is regular. By theorem 17 α is definable in $L(V_{\omega+\alpha})$. Let γ be the least ordinal greater than $\aleph_\alpha$ such that $x \in L_\gamma(V_{\omega+\alpha})$. Since γ is definable in $L(V_{\omega+\alpha})$, γ has size $\aleph_\alpha$. Therefore $(L_\gamma(V_{\omega+\alpha}), \in)$ is isomorphic to a model $(V_{\omega+\alpha}, E)$ where E is a subset of $V_{\omega+\alpha} \times V_{\omega+\alpha}$. Since the Mostowski Collapsing Theorem holds in $L(V_{\omega+\alpha+1})$, $(V_{\alpha+\omega}, E)$ is isomorphic to $(L(V_{\omega+\alpha}), \in)$ in $L(V_{\omega+\alpha+1})$ and therefore x can have size at most $|V_{\omega+\alpha}| = \aleph_\alpha$ in $L(V_{\omega+\alpha+1})$.

Picture of the Universe under $ZFC + CIFS$

Under $CIFS$, for every regular cardinal $\aleph_\alpha$, all definable elements of $L(V_{\omega+\alpha})$ have size at most $\aleph_\alpha$ in $L(V_{\omega+\alpha+1})$, forcing $L(V_{\omega+\alpha+1}) - L(V_{\omega+\alpha})$ large.

SOME PARTING PHILOSOPHICAL REMARKS

The conventional view of the history of set theory says that Godel in 1938 proved that the consistency of ZF implies the consistency of ZFC and of $ZFC+GCH$, and that Cohen with the invention of forcing proved that $Con(ZF)$ implies $Con(ZF+\neg AC)$ and $Con(ZFC+\neg GCH)$, but if $IFS(L)$ is correct, a better way to state the history would be to say that Godel discovered L and Cohen discovered that there are many generic extensions of L.

The author believes that not all transitive models of ZFC are created equal and that set theorists should make more active use of this fact, while they should place less emphasis on relative consistency results. Some Formalists may object to the Platonistic slant of this exposition, but a Formalist can always play the game of pretending to be a Platonist. Finally, the author thinks it is ironic that although mathematics and especially mathematical logic is an art noted for its precise and formalized reasoning, it seems that in order to solve problems at the frontiers of logic's foundations we must tackle questions of an esthetic nature of the kind addressed in this article.

REFERENCES

1. C. C. Chang and J. Keisler, *Model Theory*, North Holland Publishing Co.

2. M. Foreman, *Potent Axioms*, Transactions of the A.M.S., vol 294 (1986) pp 1-27.

3. C. Freiling, Axioms of Symmetry: Throwing Darts at the Real Line, this Journal, vol. 51 (1988) pp 190-200.

4. [Jech1] T. Jech, *Multiple Forcing*, Cambridge University Press.

5. [Jech2] T. Jech, *Set Theory*, Academic Press.

6. [Kunen] K. Kunen, *Set Theory*, Studies in Logic and the Foundations of Mathematics, vol 102 (1980), Elsevier Science Publishing Company, Amsterdam.

7. S. Mac Lane, *Is Mathias an Ontologist?*, in Set Theory of the Continuum, H. Judah, W. Just, and H. Woodin editors, Springer Verlag (1992) pp 119-122

8. [BA I] P. Maddy, *Believing the Axioms I*, this Journal vol 53 (1988) pp 481-511.

9. [BA II] P. Maddy, *Believing the Axioms II*, this Journal vol 54 (1988) pp 736-764.

10. A. R. D. Mathias, *What is Mac Lane Missing?*, in Set Theory of the Continuum, H. Judah, W. Just, and H. Woodin editors, Springer Verlag (1992) pp 113-118

11. R. Penrose, *The Emperors New Mind*, Oxford University Press, Oxford (1989)