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The Analytic Fixed Point Function in the Disk

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Abstract. Let φ be analytic in the unit disk $\mathbb{D}$ and let $\varphi(\mathbb{D}) \subset \mathbb{D}$, $\varphi(0) \neq 0$. Then $w = z/\varphi(z)$ has an analytic inverse $z = f(w)$, $w \in \mathbb{D}$, the fixed point function. Here $f(\mathbb{D})$ is a starlike domain and various results suggest that $f(\mathbb{D})$ might even be hyperbolically convex. We study the derivative and the coefficients of f, in particular their asymptotic behaviour. In the case that φ is the generating function of a random variable, several functions related to f have probabilistic interpretations.

Keywords. Fixed point function, byperbolically convex, coefficients, asymptotic behaviour, probability generating function, large deviations, branching process.

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1. Introduction and overview

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$ the unit circle. Throughout the paper we consider the function

$$(1.1) \qquad \varphi\colon \mathbb{D} \to \mathbb{D} \text{ analytic}, \qquad a := \varphi(0) \neq 0.$$

We shall see (Theorem 2.1) that

$$(1.2) \qquad F := \{z \in \mathbb{D} : \, |z| < |\varphi(z)|\}$$

is a starlike domain. The conformal map

$$(1.3) \qquad f\colon \mathbb{D} \to F, \qquad f(0) = 0, \; f'(0) = a$$

of $\mathbb{D}$ onto F satisfies (Section 3) the functional equation

$$(1.4) \qquad w\varphi(f(w)) = f(w) \qquad \text{for } w \in \mathbb{D}.$$

Hence $f(w)$ is a fixed point of the function $w\varphi$. Since $|f(w)| < 1$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$ this fixed point is unique and attractive and the iterates of $w\varphi$ converge to $f(w)$; see [CoMa95, Th. 2.51]. For that reason f will be called the *fixed point function*

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of φ. It follows from (1.4) that $z = f(w)$ is the inverse function of $w = z/\varphi(z)$. Throughout the paper f will denote this fixed point function and F will denote the domain $f(\mathbb{D})$ defined in (1.2).

In Sections 2 and 3, we develop the basic properties of F and f. We show for example that f is Hölder-continuous in $\overline{\mathbb{D}}$ and that F is bounded by analytic curves in $\mathbb{D}$ and the closed set $\{\zeta \in \mathbb{T} : |\varphi(\zeta)| = 1, |\varphi'(\zeta)| \le 1\}$. In Section 4 we study the behaviour of f at points $\omega \in \mathbb{T}$ with $\zeta = f(\omega) \in \mathbb{T}$. If $|\varphi'(\zeta)| < 1$ then F is tangential to $\mathbb{T}$ at ζ. If $|\varphi'(\zeta)| = 1$ then, under some regularity conditions, the domain F has either an corner of angle $\pi/2$ at ζ or a symmetric corner of angle $\pi/3$.

In Section 5 we consider coefficient problems. We write

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1.5}$$

and define $a_{n,k}$ as the coefficients of $\varphi(z)^n$. The Bürmann-Lagrange formula shows that

$$f(w) = \sum_{n=1}^{\infty} \frac{a_{n,n-1}}{n} w^n, \qquad w\frac{f'(w)}{f(w)} = \sum_{n=0}^{\infty} a_{n,n} w^n. \tag{1.6}$$

In Sections 6 and 7, we study the "probabilistic case" that $a_k = P(X = k)$ for some random variable X with values in $\mathbb{N}_0$. Then $a_{n,k} = P(S_n = k)$ and the functions in (1.6) have probabilistic interpretations. We derive some estimates for $a_{n,n}$ most of which are known from probability theory. Our function-theoretic proofs use the results in the previous sections. The behaviour of $a_{n,n}$ depends on whether $E(X) < 1$, $E(X) = 1$ or $1 < E(X) \le \infty$ as is to be expected from large deviations theory, see the discussion at the end of Section 7.

In Section 8 we consider some open problems, in particular whether the domain $F \subset \mathbb{D}$ is hyperbolically convex. This question started us on the present investigation. One fact (Theorem 3.1) in favour is that, for estimates of the function values $f(w)$, we encounter the same extremal function k_a as in the theory of hyperbolically convex functions [MaMi94].

2. The domain F

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic and $\varphi(0) \ne 0$. If $z_0, z \in \mathbb{D}$ then

$$\left| \frac{\varphi(z) - \varphi(z_0)}{1 - \overline{\varphi(z_0)}\varphi(z)} \right| \le \left| \frac{z - z_0}{1 - \overline{z_0} z} \right|. \tag{2.1}$$

Subtracting the square of both sides from 1 we obtain

$$\left| \frac{\varphi(z) - \varphi(z_0)}{z - z_0} \right|^2 \le \frac{1 - |\varphi(z_0)|^2}{1 - |z_0|^2} \, \frac{1 - |\varphi(z)|^2}{1 - |z|^2}. \tag{2.2}$$

Theorem 2.1. *The set F defined by* (1.2) *is a starlike domain and we have*

$$|\varphi(z) - \varphi(z_0)| < |z - z_0| \qquad \text{for } z \in F,\ z_0 \in \overline{F} \cap \mathbb{D}, \tag{2.3}$$

$$|\varphi'(z)| < 1 \qquad \text{for } z \in F. \tag{2.4}$$

Proof. The inequality (2.3) follows at once from (2.2) because $|\varphi(z)| > |z|$ and $|\varphi(z_0)| \geq |z_0|$ by (1.2), and (2.4) follows from (2.2) for $z_0 = z$.

Let $z \in F$ and $0 < t < 1$. If $\varphi(tz) \in F$ then

$$t\,\frac{d}{dt}\log\left|\frac{tz}{\varphi(tz)}\right| = \operatorname{Re}\left(1 - tz\frac{\varphi'(tz)}{\varphi(tz)}\right) \geq 1 - |\varphi'(tz)| > 0 \tag{2.5}$$

by (1.2) and (2.4). Now $\varphi(0) \neq 0$ and therefore $0 \in F$ by (1.2). Hence it follows from (2.5) by a standard argument from the theory of differential equations that $|tz/\varphi(tz)| < 1$ and thus $\varphi(tz) \in F$. Hence F is starlike with respect to 0 and thus connected. ■

Theorem 2.2. *If $\overline{F} \subset \mathbb{D}$ then ∂F is an analytic Jordan curve. If $\overline{F} \cap \mathbb{T} \neq \emptyset$ then $\mathbb{D} \cap \partial F$ is the union of open analytic arcs C with $\overline{C} \setminus C \subset \mathbb{T}$ and furthermore*

$$\mathbb{T} \cap \partial F = \{\zeta \in \mathbb{T} :\ |\varphi(\zeta)| = 1,\ |\varphi'(\zeta)| \leq 1\}. \tag{2.6}$$

Here $\varphi(\zeta)$ is the angular limit and $\varphi'(\zeta)$ the angular derivative. By the Julia-Wolff Lemma [Po92, Prop. 4.13], the angular derivative $\varphi'(\zeta)$ always exists and does not vanish if $\varphi(\zeta) \in \mathbb{T}$ but may be infinite in general.

Proof. Let C be a maximal connected component of $\mathbb{D} \cap \partial F$. By (1.2) and (2.4), we have

$$\frac{\partial}{\partial z}\log\left|\frac{z}{\varphi(z)}\right| = \frac{1}{2z}\left(1 - z\frac{\varphi'(z)}{\varphi(z)}\right) \neq 0 \qquad \text{for } z \in C. \tag{2.7}$$

Since C is maximal it follows that either C is an analytic Jordan curve in $\mathbb{D}$, or C is an analytic open Jordan arc with $\overline{C} \setminus C \subset \mathbb{T}$.

Now we prove (2.6). First let $\zeta \in \mathbb{T} \cap \partial F$. Then there are $z_n \in F$ with $z_n \to \zeta$ as $n \to \infty$. Hence $1 > |\varphi(z_n)| > |z_n| \to |\zeta| = 1$ and it follows [CoMa95, Th. 2.44] that $|\varphi(\zeta)| = 1$ and

$$|\varphi'(\zeta)| \leq \liminf_{n\to\infty} \frac{1 - |\varphi(z_n)|}{1 - |z_n|} \leq 1.$$

Conversely, let $|\varphi(\zeta)| = 1$ and $|\varphi'(\zeta)| \leq 1$. Then it follows from the Julia-Wolff Lemma [Po92, Prop. 4.13] that, for $0 \leq r < 1$,

$$\frac{1 - |\varphi(r\zeta)|}{1 + |\varphi(r\zeta)|} \leq \frac{|\varphi(\zeta) - \varphi(r\zeta)|^2}{1 - |\varphi(r\zeta)|^2} \leq |\varphi'(\zeta)|\,\frac{|\zeta - r\zeta|^2}{1 - r^2} \leq \frac{1-r}{1+r}$$

and thus $|\varphi(r\zeta)| \geq r$ and $r\zeta \in \overline{F}$. Since F is starlike and $\mathbb{D} \cap \partial F$ consists of analytic arcs we conclude that $r\zeta \in F$ and therefore $\zeta \in \partial F$. ■

3. The fixed point function f

Let F be the starlike domain defined by (1.2) and let f be the conformal map of $\mathbb{D}$ onto F normalized by $f(0) = 0$ and $\arg f'(0) = \arg a$ where $a = \varphi(0) \neq 0$. By (1.2) the function $h := f/(\varphi \circ f)$ is analytic in $\mathbb{D}$ and satisfies $h(\mathbb{D}) \subset \mathbb{D}$. Let $|w_n| \to 1$ as $n \to \infty$. In the case that $|f(w_n)| \to 1$ we have

$$1 \geq |h(w_n)| = \left|\frac{f(w_n)}{\varphi(f(w_n))}\right| \geq |f(w_n)| \to 1 \qquad \text{as } n \to \infty;$$

in the other case we may assume that $w_n \to \omega \in \mathbb{T}$, $f(w_n) \to \zeta \in \mathbb{D}$. Then f is analytic in ω by Theorem 2.2 and $|h(\omega)| = 1$. Hence we have $|h(w)| \to 1$ as $|w| \to 1$. It follows that h is a finite Blaschke product. Since

$$h(0) = 0, \quad h'(0) = \frac{f'(0)}{a} > 0, \quad h(w) \neq 0 \qquad \text{for } 0 < |w| < 1$$

by our normalization of f, we deduce that $h(w) \equiv w$.

Since $h(w) = f(w)/\varphi(f(w))$ it follows that

$$w\varphi(f(w)) = f(w) \qquad \text{for } w \in \mathbb{D} \tag{3.1}$$

so that $f(w)$ is a fixed point of the function $w\varphi$. Since $w\varphi$ has at most one fixed point in $\mathbb{D}$ [CoMa95, p. 50], it follows that f is uniquely determined by (3.1) together with the property that $f(\mathbb{D}) \subset \mathbb{D}$. We call f the *fixed point function* of φ. It follows from (3.1) that

$$z = f(w) \qquad \text{is the inverse function of} \qquad w = \frac{z}{\varphi(z)}. \tag{3.2}$$

Differentiating (3.1) we obtain $\varphi(z) + w\varphi'(z)f'(w) = f'(w)$. Since $\varphi(0) = a$ it follows that $f'(0) = a$ and furthermore that

$$w\frac{f'(w)}{f(w)} = \frac{1}{1 - w\,\varphi'(f(w))} = \frac{1}{1 - z\frac{\varphi'(z)}{\varphi(z)}}. \tag{3.3}$$

Since $|z| < |\varphi(z)|$ for $z \in F$ we deduce, by Theorem 2.1, that

$$\operatorname{Re} \frac{wf'(w)}{f(w)} > \frac{1}{2} \qquad \text{for } w \in \mathbb{D}. \tag{3.4}$$

Let $w \in \mathbb{D}$. We define the function φ_w by $\varphi_w(z) = w\varphi(z)$; this is not an elliptic transformation of $\mathbb{D}$ onto $\mathbb{D}$ because $|w| < 1$. Hence the iterates $\varphi_w^n = \varphi_w \circ \ldots \circ \varphi_w$ satisfy [CoMa95, Th. 2.51]

$$\varphi_w^n(z) \to f(w) \qquad \text{as } n \to \infty \qquad \text{locally uniformly in } z \in \mathbb{D}$$

and it follows that the functions $f_n(w) := \varphi_w^n(w)$ satisfy

$$f_{n+1}(w) = w\varphi(f_n(w)), \qquad f_n(w) \to f(w) \qquad \text{as } n \to \infty. \tag{3.5}$$

This is a recursive formula to compute $f(w)$.

Example 3.1 ([MaMi94, p. 85], [MePo00, p. 374])**.** Let $a \in \mathbb{C},\ a \neq 0$ and

$$\varphi_a(z) = \frac{z+a}{1+\overline{a}z} \qquad z \in \mathbb{D}. \tag{3.6}$$

Then $|z| < |\varphi(z)| < 1$ holds if and only if $|z(1+\overline{a}z)|^2 < |z+a|^2$ and it follows that

$$F_a = \left\{ z \in \mathbb{D} : \left| z + a^{-1} \right| > |a|^{-1} \sqrt{1-|a|^2} \right\}.$$

This is a lens-shaped domain. The normalized conformal map of $\mathbb{D}$ onto F_a is

$$k_a(w) := \frac{2aw}{1-w+\sqrt{(1-w)^2+4|a|^2w}} = \frac{\sqrt{(1-w)^2+4|a|^2w}-1+w}{2\overline{a}}. \tag{3.7}$$

Let $I_a = \{e^{it} : |t| \leq \pi/2 + \arcsin|a|\}$. Then

$$\operatorname{Re}\frac{k_a(w)}{aw} = \frac{1}{1+\sqrt{1-|a|^2}}, \quad \operatorname{Re}\frac{w\,k_a'(w)}{k_a(w)} = \frac{1}{2} \qquad \text{for } w \in I_\alpha \tag{3.8}$$

and $|k_a(w)| = 1$ for $w \in \mathbb{T} \setminus I_a$. See also Example 5.1.

Theorem 3.1. *Let $f(w) = aw + \cdots$ be the conformal map of $\mathbb{D}$ onto F. Then*

$$\varphi(f(w)) = \frac{f(w)}{w} \quad \textit{is subordinate to} \quad \frac{k_a(w)}{w}, \tag{3.9}$$

$$|k_{|a|}(-|w|)| \leq |f(w)| \leq k_{|a|}(|w|) \qquad \textit{for } w \in \mathbb{D}, \tag{3.10}$$

and furthermore, with $\alpha = \arccos|a|$,

$$\operatorname{area} f(\mathbb{D}) \leq \operatorname{area} k_a(\mathbb{D}) = \pi - \alpha + \tan\alpha - \left(\frac{\pi}{2} - \alpha\right)\tan^2\alpha. \tag{3.11}$$

Proof. (a) We consider the functions

$$q(w) := \frac{f(w)}{w}, \quad g(w) := \frac{q(w)-a}{(1-\overline{a}\,q(w))q(w)}, \qquad w \in \mathbb{D}. \tag{3.12}$$

We put $z_0 = 0$ and $z = f(w)$ into (2.1). Since $q(0) = a$ and $\varphi(z) = q(w)$ by (3.1), we obtain that

$$|g(w)| \leq \frac{|z|}{|q(w)|} = \left| \frac{f(w)}{q(w)} \right| = |w| \qquad \text{for } w \in \mathbb{D}. \tag{3.13}$$

It follows from (3.12) that $gq - \overline{a}gq^2 = q - a$ and thus

$$q = -\frac{1-g}{2\overline{a}g} \pm \frac{1}{2\overline{a}g}\sqrt{(1-g)^2+4|a|^2g} = \frac{2a}{1-g \pm \sqrt{(1-g)^2+4|a|^2g}}.$$

Since $g(0) = 0$ we have to choose the plus-sign. Hence

$$\frac{f(w)}{w} = \frac{k_a(g(w))}{g(w)}, \quad |g(w)| \leq |w| \qquad \text{for } w \in \mathbb{D}$$

by (3.7), (3.12) and (3.13). Hence (3.9) is true by the definition of subordination.

(b) Since $\varphi(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(0) = a$, we have

$$\frac{|a| - |z|}{1 - |az|} \leq |\varphi(z)| \leq \frac{|a| + |z|}{1 + |az|} \qquad \text{for } |z| < 1.$$

We put $z = f(w)$. Since $\varphi(z) = f(w)/w$ we obtain

$$|f(w)|^2 + \frac{1 - |w|}{|a|}\,|f(w)| \leq |w| \leq \frac{1 + |w|}{|a|}|f(w)| - |f(w)|^2,$$

which implies (3.10) in view of (3.7).

(c) Let $A(r) = \text{area}\{f(w) : |w| < r\}$ for $0 < r < 1$. Then

$$\begin{aligned} 2A(r) &= \int_0^{2\pi} \operatorname{Re}\Big(re^{it} f'(re^{it})\overline{f(re^{it})}\Big)\, dt \\ &= \int_0^{2\pi} \operatorname{Re}\Big(re^{it}\frac{f'}{f} - \frac{1}{2}\Big)\, |f|^2\, dt + \frac{1}{2}\int_0^{2\pi} |f|^2\, dt. \end{aligned} \tag{3.14}$$

Hence we obtain from (3.4) and $|f| \leq 1$ that

$$2A(r) \leq \int_0^{2\pi} \operatorname{Re}\Big(re^{it}\frac{f'}{f} - \frac{1}{2}\Big)\, dt + \frac{r^2}{2}\int_0^{2\pi} \left|\frac{f(re^{it})}{re^{it}}\right|^2 dt.$$

The first integral has the value 1/2 and the second integral can be estimated by subordination [Hi73, Th. 18.5.3]; see (3.9). Hence

$$2A(r) \leq \frac{1}{2} + \frac{1}{2}\int_0^{2\pi} \left|k_a(re^{it}\right|^2 dt. \tag{3.15}$$

Now we put $f = k_a$ in (3.14) and let $r \to 1$. Using (3.8) for $w \in I_\alpha$ and $|k_a(w)| = 1$ for $w \in \mathbb{T} \setminus I_a$ we obtain

$$\begin{aligned} 2\,\text{area}\, k_a(\mathbb{D}) &= \int_0^{2\pi} \operatorname{Re}\Big(e^{it}\frac{k_a'}{k_a} - \frac{1}{2}\Big)\, dt + \frac{1}{2}\int_0^{2\pi} \left|k_a(e^{it})\right|^2 dt \\ &= \frac{1}{2} + \frac{1}{2}\int_0^{2\pi} \left|k_a(e^{it})\right|^2 dt \end{aligned}$$

and we conclude from (3.15) that $A(r) \leq \text{area}\, k_a(\mathbb{D})$. The area of $k_a(\mathbb{D}) = F_a$ is obtained by a geometrical calculation. This proves (3.11). ■

Theorem 3.2. *The fixed point function f has a continuous extension to $\overline{\mathbb{D}}$ and moreover we have*

$$|f(w_1) - f(w_2)| \leq c_1|w_1 - w_2|^{1/3}, \tag{3.16}$$

$$|f'(w)| \leq c_2(1 - |w|)^{-2/3}, \tag{3.17}$$

where c_1 and c_2 depend only on φ.

It will follow from Theorem 4.3 and Example 4.2 that c_1 and c_2 can be arbitrarily large for suitable φ.

Proof. Let $b = \varphi'(0)/(1-|a|^2)$. We assume first that φ is not a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ so that $|b| < 1$. The function

$$\psi(\zeta, z) = \frac{\varphi(\zeta) - \varphi(z)}{1 - \overline{\varphi(z)}\varphi(\zeta)} \frac{1 - \bar{z}\zeta}{\zeta - z}, \qquad \zeta, z \in \mathbb{D}, \tag{3.18}$$

satisfies $\psi(\zeta, z)| < 1$ and is analytic in $\zeta \in \mathbb{D}$. Hence

$$|\psi(\zeta, z)| \le \frac{|\psi(0,z)| + |\zeta|}{1 + |\psi(0,z)||\zeta|} \qquad \text{for } \zeta, z \in \mathbb{D}. \tag{3.19}$$

Furthermore $\chi(z) := z^{-1}(\varphi(z) - a)/(1 - \bar{a}\varphi(z))$ is analytic in $\mathbb{D}$ and satisfies $\chi(\mathbb{D}) \subset \mathbb{D}$, $\chi(0) = b$. Hence we see from (3.18) that

$$|\psi(0,z)| = \left| \frac{\varphi(z) - a}{z(1 - a\,\overline{\varphi(z)})} \right| = |\chi(z)| \le \frac{|b| + |z|}{1 + |bz|}$$

for $z \in \mathbb{D}$ and therefore

$$1 - |\psi(0,z)| \ge \frac{(1-|b|)(1-|z|)}{1 + |bz|} > \frac{1 - |b|}{2}(1 - |z|). \tag{3.20}$$

Now let $z \in F$. We put $\zeta = z$ in (3.18). Since $1 - |\varphi(z)|^2 < 1 - |z|^2$ we obtain by (3.19) that

$$1 - |\varphi'(z)| \ge 1 - |\psi(z,z)| \ge \frac{(1 - |\psi(0,z)|)(1-|z|)}{1 + |\psi(0,z)||z|} > \frac{1-|b|}{4}(1-|z|)^2 \tag{3.21}$$

in view of (3.20). We put $z = f(w)$ and deduce from (3.3) that

$$|f'(w)| \le \frac{\left|\frac{f(w)}{w}\right|}{1 - |\varphi'(f(w))|} < \frac{4}{1-|b|}\frac{1}{(1-|f(w)|)^2}.$$

Since $|f(w)| < 1$ we also have

$$|f'(w)|^2 \le \left(\frac{1 - |f(w)|^2}{1 - |w|^2}\right)^2 \le 4\left(\frac{1 - |f(w)|}{1-|w|}\right)^2.$$

We multiply these two inequalities and take the third root to obtain

$$|f'(w)| < \left(\frac{16}{1-|b|}\right)^{1/3}(1-|w|)^{-2/3} \qquad \text{for } w \in \mathbb{D}. \tag{3.22}$$

If φ is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ then φ is a rotation of the function in (3.7) and a direct computation shows that even $|f'(w)| < c_3(1-|w|)^{-1/2}$.

Hence (3.17) holds, and (3.16) is a well-known consequence of (3.17). It follows that f has a continuous extension to $\overline{\mathbb{D}}$. ∎

4. The local boundary behaviour

Now we study the behaviour of the fixed point function f at a point $\omega \in \mathbb{T}$ where $\zeta = f(w) \in \mathbb{T}$ and investigate the shape of $F = f(\mathbb{D})$ near ζ. Replacing $\varphi(z)$ by $\overline{\zeta}\omega\varphi(\zeta z)$ and $f(w)$ by $\overline{\zeta}f(\omega w)$, we may assume that $\omega = \zeta = 1$.

Since $\varphi(1) = 1$ the angular derivative $\varphi'(1)$ is real and positive by the Julia-Wolff Lemma [Po92, p. 82] and it follows from Theorem 2.2 that $0 < \varphi'(1) \le 1$. First we consider the case $\varphi'(1) < 1$.

Theorem 4.1. *Let $\varphi(1) = 1$ and $\varphi'(1) < 1$. Then the angular derivative satisfies $1 < f'(1) < \infty$ and the domain $F = f(\mathbb{D})$ is tangential to $\mathbb{T}$ at 1. If moreover $\varphi'(z) \to \varphi'(1)$ as $z \to 1$, $z \in \mathbb{D}$ then $f'(w) \to f'(1)$ as $w \to 1$, $w \in \mathbb{D}$.*

Proof. Since $[0, 1) \subset F$ we see that $\Gamma = \{f^{-1}(x) : 0 \le x < 1\} \subset \mathbb{D}$ is a curve ending at 1. By (3.3) we have

$$f'(f^{-1}(x)) = \frac{x}{f^{-1}(x)\left(1 - \frac{x\varphi'(x)}{\varphi(x)}\right)} \to \frac{1}{1 - \varphi'(1)} < \infty.$$

Now $\log f'$ is a Bloch function because f is univalent [Po92, Prop. 4.1] and we have $f'(w) \to 1/(1-\varphi'(1))$ as $w \to 1$, $w \in \Gamma$. Hence it follows from the Theorem of Lehto and Virtanen [LeVi57] [Po92, Th. 4.3] that $f'(w) \to 1/(1 - \varphi'(1))$ as $w \to 1$ in every Stolz angle at 1.

Since f is univalent and $F \subset \mathbb{D}$, it follows [Po92, p. 80] from $f'(1) < \infty$ that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\{w \in \mathbb{D} : |1 - w| < \delta, |\arg(1 - w)| < \frac{\pi}{2} - \varepsilon\right\} \subset F$$

so that F is tangential to $\partial\mathbb{D}$ at 1. The last assertion is obvious because of equation (3.3). ∎

The case $\varphi'(1) = 1$ is more complicated and we have to make additional regularity assumptions. There are two subcases dealt with in the next two theorems.

Theorem 4.2. *Let $\varphi(1) = \varphi'(1) = 1$ and suppose that*

$$\varphi''(z) \to b \in \mathbb{C} \qquad \text{as } z \to 1,\ z \in \mathbb{D} \tag{4.1}$$

with $b \ne 0$. Then $b = |b|e^{i\beta}$ with $-\pi/2 \le \beta \le \pi/2$ and

$$f'(w) \sim [2b(1 - w)]^{-1/2} \qquad \text{as } w \to 1,\ w \in \mathbb{D}. \tag{4.2}$$

Furthermore, ∂F in a neighbourhood of 1 consists of two Jordan arcs $C^{\pm}$ ending at 1 with tangents of angles $\vartheta^{\pm}$ where

$$\vartheta^{+} = -\frac{\beta}{2} + \frac{\pi}{4}, \qquad \vartheta^{-} = -\frac{\beta}{2} - \frac{\pi}{4}, \qquad \vartheta^{+} - \vartheta^{-} = \frac{\pi}{2}. \tag{4.3}$$

See Theorem 6.1 and Example 6.2 for some information about the case that $\varphi''(x) \to +\infty$ as $x \to 1$.

Proof. Since $\varphi(1) = \varphi'(1) = 1$ it follows from (4.1) by integration that

$$\varphi(z) - z\varphi'(z) = -b(z-1) + o(|z-1|) \qquad \text{as } z \to 1,\ z \in \mathbb{D}.$$

Since $b \neq 0$ we therefore obtain from (3.3) with $z = f(w)$ that

$$f'(w) = \left(\frac{1}{b} + o(1)\right) \frac{1}{1 - f(w)} \qquad \text{as } w \to 1,\ w \in \mathbb{D}$$

and thus

$$(1 - f(w))^2 = 2 \int_w^1 (1 - f(s)) f'(s)\, ds = \left(\frac{2}{b} + o(1)\right)(1 - w).$$

This implies (4.2). Furthermore it follows that

$$\arg(1 - f(w)) = -\frac{\beta}{2} + \frac{1}{2}\arg(1 - w) + o(1) \qquad \text{as } w \to 1,\ w \in \mathbb{D}, \tag{4.4}$$

which implies the last assertion of the theorem. Since $|\arg(1 - f(w))| < \pi/2$ we conclude from (4.4) that $|\beta| \leq \pi/2$. ■

Theorem 4.3. *Let $\varphi(1) = \varphi'(1) = 1$ and suppose that (4.1) holds with $b = 0$ and furthermore that*

$$\varphi'''(z) \to c \in \mathbb{C} \qquad \text{as } z \to 1,\ z \in \mathbb{D}. \tag{4.5}$$

Then $c \in \mathbb{R}$, $c < 0$ and

$$f'(w) \sim \left(\frac{2}{9}|c|^{-1}\right)^{-1/3} (1 - w)^{-2/3} \qquad \text{as } w \to 1,\ w \in \mathbb{D}. \tag{4.6}$$

Furthermore ∂F in a neighbourhood of 1 consists of two Jordan arcs $C^{\pm}$ ending at 1 with tangents of angles $\vartheta^{\pm}$ where

$$\vartheta^{\pm} = \pm\frac{\pi}{3}, \qquad \vartheta^{+} - \vartheta^{-} = \frac{\pi}{3}. \tag{4.7}$$

The value $c = \varphi'''(1)$ can be arbitrarily small as Example 4.2 will show. We do not know what happens if $b = 0$ and (4.5) does not hold.

Proof. Since $\varphi(1) = \varphi'(1) = 1$ it follows from (4.1) with $b = 0$ and from (4.5) by integration that

$$\varphi(z) = z + \frac{1}{6}c(z-1)^3 + o(|z-1|^3) \qquad \text{as } z \to 1,\ z \in \mathbb{D}. \tag{4.8}$$

For small $t \in \mathbb{R}$ we put $z = (1 - t^4)e^{it}$ and obtain

$$1 + \frac{8}{3}(\operatorname{Im} c)t^3 + o(t^3) = |\varphi(z)|^2 < 1.$$

Since t may have both positive and negative values, it follows that $\operatorname{Im} c = 0$ and thus $c \in \mathbb{R}$. Furthermore

$$\varphi'(z) = 1 + \frac{c}{2}(z-1)^2 + o(|z-1|^2) \qquad \text{as } z \to 1,\ z \in \mathbb{D} \tag{4.9}$$

and therefore

$$-\frac{c}{2} = \lim_{x\to 1} \frac{\mathrm{Re}(1-\varphi'(x))}{(1-x)^2} \geq \liminf_{x\to 1} \frac{1-|\varphi'(x)|}{(1-x)^2}.$$

Now φ is not a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$ because $\varphi''(1)=0$. Hence the limes inferior is positive by (3.21) and therefore $c<0$.

By (4.8) and (4.9) we have

$$\varphi(z) - z\varphi'(z) = -\frac{c}{2}(1-z)^2(1+o(1))$$

and we thus obtain from (3.3) that

$$f'(w) = \left(\frac{2}{|c|} + o(1)\right)\frac{1}{(1-f(w))^2} \qquad \text{as } w\to 1,\ w\in\mathbb{D} \tag{4.10}$$

because $c<0$. Since $f(1)=1$ we deduce that

$$(1-f(w))^3 = 3\int_w^1 (1-f(s))^2 f'(s)\,ds = \left(\frac{6}{|c|}+o(1)\right)(1-w).$$

Using again (4.10) we thus obtain (4.6). The final assertion follows by taking the argument in the last expression. ∎

Example 4.1. We consider a *finite Blaschke product* φ of order $m\geq 1$ with $\varphi(0)\neq 0$, that is,

$$\varphi(z) = e^{i\tau}\prod_{k=1}^{m} \frac{z-z_k}{1-\overline{z_k}z}, \qquad z\in\mathbb{D}, \tag{4.11}$$

where $0<|z_k|<1$ for $k=1,\dots,m$. Logarithmic differentiation gives

$$\frac{\varphi'(z)}{\varphi(z)} = \sum_{k=1}^{m}\left(\frac{1}{z-z_k} + \frac{\overline{z_k}}{1-\overline{z_k}z}\right) = \sum_k \frac{1-|z_k|^2}{(z-z_k)(1-\overline{z_k}z)}. \tag{4.12}$$

If $\zeta\in\mathbb{T}$ we obtain that

$$\frac{\zeta\varphi'(\zeta)}{\varphi(\zeta)} = \sum_k \frac{1-|z_k|^2}{|\zeta-z_k|^2} = \mathrm{Re}\sum_k \frac{\zeta+z_k}{\zeta-z_k},$$

$$\mathrm{Re}\,\frac{\zeta^2\varphi''(\zeta)}{\varphi(\zeta)} - \mathrm{Re}\left(\frac{\zeta\varphi'(\zeta)}{\varphi(\zeta)}\right)^2 = -\,\mathrm{Re}\sum_k \frac{\zeta+z_k}{\zeta-z_k}.$$

Now suppose that $|\varphi'(\zeta)|=1$. Then $\zeta\,\varphi'(\zeta)/\varphi(\zeta)=1$ and thus

$$\mathrm{Re}\sum_k \frac{\zeta+z_k}{\zeta-z_k} = 1, \qquad \mathrm{Re}\,\frac{\zeta^2\varphi''(\zeta)}{\varphi(\zeta)} = 0.$$

Hence it follows from Theorems 4.2 and 4.3 that there are only two possibilities for the corners of F.

(i) $\varphi''(\zeta)\neq 0$, $\mathrm{Re}(\zeta^2\varphi''(\zeta)/\varphi(\zeta))=0$: a corner of angle $\pi/2$ with one side on $\mathbb{T}$.

(ii) $\varphi''(\zeta) = 0$: a symmetric corner of angle $\pi/3$.

In order to understand this, we study the behaviour of the Blaschke product (4.11) in $\hat{\mathbb{C}}$. Let $*$ denote reflection in $\mathbb{T}$ and let G_ν be the components of $\mathbb{D} \setminus \overline{F}$. Since $|\varphi(1/z)| = 1/|\varphi(z)|$ we see that

$$E := \{z \in \hat{\mathbb{C}} : |\varphi(z)| = |z|\} = \mathbb{T} \cup \partial F \cup \partial F^*.$$

The components of $\hat{\mathbb{C}} \setminus E$ are F, F^* and the domains G_ν and G_ν^*. Since $|\varphi(z)/z| = 1$ for $z \in \partial G_\nu$, it follows from the Minimum Principle that each G_ν contains at least one z_k. The arcs of $E = \{\log \varphi(z) - \log z \in i\mathbb{R}\}$ are the trajectories of the differential $i(\varphi'(z)/\varphi(z) - 1/z)\,dz$. Now trajectories meet only at the zeros and poles of the differential and the angles between trajectories at these zeros have the form π/n with $n = 2, 3, \ldots$. This explains why (i) (where $n = 2$) and (ii) (where $n = 3$) are the only possible cases.

Example 4.2. We consider the special case

$$\varphi(z) = \frac{z + x_1}{1 + x_1 z} \frac{z + x_2}{1 + x_2 z}, \qquad 0 < x_1 < x_2 < 1,$$

of the Blaschke product (4.11) where $3x_1x_2 + x_1 + x_2 = 1$. Starting from (4.12) we obtain $\varphi(1) = \varphi'(1) = 1$, $\varphi''(1) = 0$ and

$$\varphi'''(1) = -2 + \frac{2(1 - {x_1}^3)}{(1 + x_1)^3} + \frac{2(1 - {x_2}^3)}{(1 + x_2)^3}.$$

If $x_1 \to 0$ and $x_2 \to 1$ this expression tends to 0. Hence $\varphi'''(1)$ can be arbitrarily small if $\varphi'(1) = 1$ and $\varphi''(1) = 0$.

5. The Bürmann-Lagrange formula and the coefficients

We recall the statement of the *Bürmann-Lagrange formula*: let φ and ψ be analytic in a neighbourhood of 0 and $\varphi(0) \neq 0$. If $z = f(w)$ is the local inverse function of $w = z/\varphi(z)$, then [PoSz25, p. 125]

$$\psi(f(w)) = \psi(0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} \left(\frac{d}{d\zeta}\right)^{n-1} [\psi'(\zeta)\varphi(\zeta)^n] \bigg|_{\zeta=0}. \tag{5.1}$$

This remarkable formula is easily proved by the Cauchy integral representation of the coefficients. It follows from (5.1) that

$$\psi'(f(w))f'(w) = \sum_{n=0}^{\infty} \frac{w^n}{n!} \left(\frac{d}{d\zeta}\right)^{n} \left[\psi'(\zeta)\varphi(\zeta)^{n+1}\right] \bigg|_{\zeta=0}. \tag{5.2}$$

The notation in (5.1) is precisely our notation introduced in Section 1. We write

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 = a = \varphi(0) \tag{5.3}$$

and define the numbers $a_{n,k}$, $k, n \in \mathbb{Z}$, by

$$\varphi(z)^n = \sum_{k=0}^{\infty} a_{n,k} z^k. \tag{5.4}$$

In particular we have $a_{1,k} = a_k$ and $a_{n,0} = a^n$.

Now let ψ be defined by $\psi(0) = 0$ and $\psi'(z) = \varphi(z)^p$ where $p \in \mathbb{Z}$. Since $\varphi(f(w)) = f(w)/w$ we obtain from (5.2) and (5.4) that

$$f(w)^p\, f'(w) = \sum_{n=0}^{\infty} a_{n+p+1,n} w^{n+p} = \sum_{n=p}^{\infty} a_{n+1,n-p} w^n. \tag{5.5}$$

For $p = m - 1$, $m = 1, 2, \ldots$, it follows by integration that

$$f(w)^m = \sum_{n=m}^{\infty} \frac{m}{n} a_{n,n-m} w^n. \tag{5.6}$$

The recursion formula (3.5) shows that the coefficients $n^{-1} a_{n,n-1}$ of f belong to the ring $\mathbb{Z}[a_0, a_1, a_2, \ldots]$.

If we choose $p = -1$ then we can write (5.5) as

$$\frac{w f'(w)}{f(w)} = \sum_{n=0}^{\infty} a_{n,n} w^n; \tag{5.7}$$

see (3.3) for the significance of this function. If we divide by w and integrate we obtain

$$\log \frac{f(w)}{w} = \log a + \sum_{n=1}^{\infty} \frac{a_{n,n}}{n} w^n. \tag{5.8}$$

Example 5.1. As in Example 3.1, we consider $\varphi(z) = (z + a)/(1 + \overline{a}z)$ where $0 < |a| < 1$. Then $\varphi(z)^n = (z + a)^n (1 + \overline{a}z)^{-n}$ for $n \in \mathbb{N}$ and therefore, by (5.4),

$$a_{n,k} = a^{n-k} \sum_{j=\max(k-n,0)}^{k} (-1)^j \binom{n}{k-j} \binom{n+j-1}{n-1} |a|^{2j}.$$

Example 5.2. Let $0 < \beta < \infty$ and

$$\varphi(z) = \exp\left(-\beta\, \frac{1-z}{1+z}\right), \qquad z \in \mathbb{D}. \tag{5.9}$$

Then $\varphi(\mathbb{D}) \subset \mathbb{D}$. If $n \in \mathbb{N}$ then

$$\varphi(z)^n = e^{-\beta n} \exp\left(\frac{2\beta n z}{1+z}\right) = e^{-\beta n} \sum_{k=0}^{\infty} (-1)^k L_k^{(-1)}(2\beta n) z^k,$$

where the $L_k^{(\alpha)}(x)$ are the generalized Laguerre polynomials [MaObSo66, p. 242]. Since $L_k^{(-1)}(x) = -k^{-1}x\, L_{k-1}^{(1)}(x)$ it follows that, for $k \geq 1$,

$$a_{n,k} = (-1)^k \frac{2\beta n}{n} e^{-\beta n} L_{k-1}^{(1)}(2\beta n). \tag{5.10}$$

We see from (5.9) that $|\varphi(e^{it})| = 1$ and $|\varphi'(e^{it})| = \beta/(1+\cos t)$ for $-\pi < t < \pi$ whereas $\varphi(-1) = 0$. Hence it follows from Theorem 2.2 that $\overline{F} \subset \mathbb{D}$ for $\beta > 2$ and $\mathbb{T} \cap \partial F = \{e^{it} : \cos t \geq \beta - 1\}$ for $\beta \leq 2$. Now we consider the case $\beta = 2$ where $\mathbb{T} \cap \partial F = \{1\}$ and moreover $\varphi'(1) = 1$, $\varphi''(1) = 0$. Using the asymptotic formula of Erdelyi [MaObSo66, p. 245] we obtain from (5.10) that

$$a_{n,n+j} \sim \frac{(2/3)^{2/3}}{\Gamma(2/3)} n^{-1/3} + \mathcal{O}\big(n^{-2/3}\big) \tag{5.11}$$

for fixed $j \in \mathbb{Z}$ and $n \to \infty$. In the special case $j = 0$ the last term can be replaced by $\mathcal{O}(n^{-3/4})$. The order of growth is in accordance with Theorem 4.3.

The following special but frequently occurring case is related to Theorem 4.2.

Theorem 5.1. *Suppose that φ is analytic in the closed unit disk $\overline{\mathbb{D}}$ and that $\mathbb{T} \cap \partial F = \{\zeta\}$, $|\varphi'(\zeta)| = 1$ and $\varphi''(\zeta) \neq 0$. Then*

$$a_{n,n} = \left(\frac{\varphi(\zeta)}{2\pi\zeta^2\varphi''(\zeta)} \right)^{1/2} \left(\frac{\varphi(\zeta)}{\zeta} \right)^n n^{-1/2} + \mathcal{O}\left(n^{-3/2}\right). \tag{5.12}$$

Proof. We may assume that $\zeta = \varphi(\zeta) = 1$ and therefore $\varphi'(1) = 1$; see Section 4. Writing $b = \varphi''(1)$ we have

$$w = \frac{z}{\varphi(z)} = 1 - \frac{b}{2}(z-1)^2 + \cdots \qquad \text{near } z = 1.$$

Hence the inverse function has the form

$$z = f(w) = 1 + \sum_{k=1}^{\infty} d_k (w-1)^{k/2} \qquad \text{near } w = 1,$$

where $d_1 = \sqrt{-2/b}$. It follows that, for some $\rho > 0$,

$$g(w) := \frac{wf'(w)}{f(w)} = \frac{d_1}{2}(w-1)^{-1/2} + \sum_{k=0}^{\infty} b_k (w-1)^{k/2} \qquad \text{for } |w-1| \leq \rho. \tag{5.13}$$

Since $\mathbb{T} \cap \partial F = \{1\}$ the function f is analytic in $\overline{\mathbb{D}} \setminus \{1\}$. Hence, by (5.13), there is $r \in (1, 1+\rho)$ such that g is single-valued and analytic in $\{|w| \leq r\} \setminus [1, r]$. On the two sides $S^{\pm}$ of $[1, r]$ the values differ according to (5.13). We integrate along the curve consisting of S^-, S^+ and $\{|w| = r\}$ and we obtain by (5.7) that

$$a_{n,n} = \frac{1}{2\pi i} \int_{|w|=r} \frac{g(w)}{w^{n+1}}\, dw + \frac{1}{2\pi i} \int_{S^+ \cup S^-} \frac{g(w)}{w^{n+1}}\, dw. \tag{5.14}$$

Since g is bounded on $\{|w| = r\}$ and since $r > 1$, we have

$$\int_{|w|=r} \frac{g(w)}{w^{n+1}}\, dw = \mathcal{O}\left(\frac{1}{r^n}\right) \qquad \text{as } n \to \infty. \tag{5.15}$$

We obtain from (5.13) that

$$\left| g(u) - \frac{d_1}{2}(u-1)^{-1/2} - b_0 \right| \le c_1 (u-1)^{1/2} \qquad \text{for } 1 \le u \le r;$$

here c_1 and c_2 are suitable constants. Since S^+ and S^- have opposite orientation the term b_0 cancels on integration. Hence we see that

$$\left| \int_{S^+ \cup S^-} g(w) \frac{dw}{w^{n+1}} - d_1 \int_1^r (u-1)^{-1/2} \frac{du}{u^{n+1}} \right| \le c_2 \int_1^r (u-1)^{1/2} \frac{du}{u^{n+1}}.$$

The substitution $u = 1/(1-x)$ yields

$$\int_1^r (u-1)^{\pm 1/2} \frac{du}{u^{n+1}} = \int_0^\delta x^{\pm 1/2} (1-x)^{n \mp 1/2 - 1}\, dx,$$

where $\delta = (r-1)/r$. This is an incomplete beta-integral. Hence the last integral equals [MaObSo66, p. 7]

$$\frac{\Gamma(1 \pm \frac{1}{2})\Gamma(n \mp \frac{1}{2})}{\Gamma(n+1)} - \int_r^\infty (u-1)^{\pm 1/2} \frac{du}{u^{n+1}}$$

where the second term is $O(r^{-n})$. It follows that

$$\begin{aligned} \int_1^r (u-1)^{-1/2} \frac{du}{u^{n+1}} &= \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} + \mathcal{O}(r^{-n}) = \sqrt{\pi} n^{-1/2} + \mathcal{O}(n^{-3/2}), \\ \int_1^r (u-1)^{+1/2} \frac{du}{u^{n+1}} &= \mathcal{O}(n^{-3/2}). \end{aligned}$$

Hence it follows from (5.14) and (5.15) that

$$a_{n,n} = \frac{d_1}{2\pi i} n^{-1/2} + \mathcal{O}(n^{-3/2}) \qquad \text{as } n \to \infty$$

and we have $d_1/(2\pi i) = 1/\sqrt{2\pi b}$. ∎

6. The probabilistic case

Now we assume that our function φ has the form

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad a_0 = a > 0,\ a_k \ge 0 \qquad \text{and} \qquad \varphi(1) = 1. \tag{6.1}$$

We shall call this the *probabilistic case*. It is clear that $|\varphi(z)| < \varphi(1) = 1$. Now φ is the generating function of a random variable X with values in $\mathbb{N}_0$ and distribution

$$a_k = P(X = k) \qquad \text{for } k \in \mathbb{N}_0.$$

The mean value of the random variable X is

$$\mu := E(X) = \varphi'(1) = \sum_{k=1}^{\infty} k a_k \le +\infty. \tag{6.2}$$

Let $a_{n,k}$ be the coefficients of $\varphi(z)^n$ as in (5.4). Then [Fe68, p. 286]

$$a_{n,k} = P(X_1 + \cdots + X_n = k) = P(S_n = k), \tag{6.3}$$

where $X_1, \dots, X_n$ are independent random variables with $P(X_\nu = k) = a_k$ for all ν and k and where $S_n = X_1 + \cdots + X_n$. In particular, by (5.6) and (5.7) we have

$$f(w) = \sum_{n=1}^{\infty} \frac{1}{n} P(S_n = n-1) w^n, \tag{6.4}$$

$$w\frac{f'(w)}{f(w)} = \sum_{n=0}^{\infty} a_{n,n} w^n = \sum_{n=0}^{\infty} P(S_n = n) w^n. \tag{6.5}$$

Furthermore we claim that

$$w\varphi'(f(w)) = \sum_{n=1}^{\infty} P(S_n = n,\, S_k \ne k \text{ for } 1 \le k < n) w^n, \tag{6.6}$$

so that $w\,\varphi'(f(w))$ is the generating function of the probability that $S_k = k$ at $k = n$ for the first time. The identity (6.6) is true because, by (3.3),

$$w\frac{f'(w)}{f(w)} = \frac{1}{1 - w\varphi'(f(w))} \qquad \text{for } w \in \mathbb{D}, \tag{6.7}$$

which corresponds to formula (3.2) in [Fe68, p. 311]. Since the events in (6.6) are disjoint we have

$$\varphi'(f(1)) = P(\exists\, n > 0 : S_n = n). \tag{6.8}$$

The fixed point function can also be interpreted in terms of branching processes [Fe68, p. 293], [AtNe72]. Let Z_m denote the random number of individuals in the mth generation where $Z_0 = 1$. The individuals reproduce independently and the number of children of each is distributed as is X. Then [Fe68, p. 296]

$$\varphi^m(z) = \sum_{k=0}^{\infty} P(Z_m = k) z^k, \tag{6.9}$$

where φ^m denotes the mth iterate $\varphi \circ \cdots \circ \varphi$. Now $Y_m = Z_0 + Z_1 + \cdots + Z_m$ is the total number of individuals that ever lived up to and including the mth generation. Writing

$$f_m(z) = \sum_{n=1}^{\infty} P(Y_m = n) w^n$$

we have [Fe68, p. 298].

$$f_{m+1}(w) = w\varphi(f_m(w)) = \varphi_w(f_m(w)) \qquad \text{for } m \in \mathbb{N}_0,$$

where $\varphi_w(z) = w\varphi(z)$. Hence $f_m(w) = \varphi_w^m(w)$ and it follows from (3.5) that $f_m \to f$ as $m \to \infty$. The random variable

$$Y := \sum_{k=0}^{\infty} Z_k = \lim_{m\to\infty} Y_m \leq \infty$$

is the total number of all descendents over all generations and we have

$$f(w) = \sum_{n=1}^{\infty} P(Y = n) w^n, \tag{6.10}$$

hence $P(Y < \infty) = f(1)$ and $E(Y \mid Y < \infty) = f'(1)$.

Example 6.1 (compare [Fe68, p. 272] [Fe68, p. 299])**.** We consider the geometric distribution

$$\varphi(z) = \frac{q}{1 - pz}, \qquad q = 1 - p,\ 0 < p < 1,$$

so that $\mu = E(X) = p/q$. By (3.2) the function $z = f(w)$ is the solution of $w = z(1 - pz)/q$ with $f(0) = 0$ and thus

$$f(w) = \frac{2qw}{1 + \sqrt{1 - 4pqw}}, \qquad w\frac{f'(w)}{f(w)} = \frac{1}{2} + \frac{1}{2\sqrt{1 - 4pqw}}$$

and therefore, by (6.7),

$$w\varphi'(f(w)) = \frac{4pqw}{(1 + \sqrt{1 - 4pqw})^2} = \frac{p}{qw} f(w)^2.$$

The equation $\partial F = f(\mathbb{T})$ can be written as $|(2pz - 1)^2 - 1| = 4pq$, $z \in \overline{\mathbb{D}}$. This is one leaf of a Cassinian curve which becomes a lemniscate for $p = 1/2$.

The probabilistic interpretations (6.10) and (6.8) in this classical example are as follows: for $0 < p \leq 1/2$ we obtain

$$\begin{aligned} P(Y < \infty) &= f(1) = 1, \\ E(Y) &= f'(1) = \frac{q}{q - p} \leq \infty, \\ P(\exists n > 0 : S_n = n) &= \varphi'(f(1)) = \frac{p}{q}, \end{aligned}$$

whereas for $1/2 < p < 1$ we obtain

$$P(Y < \infty) = f(1) = \frac{q}{p},$$

$$E(Y \mid Y < \infty) = f'(1) = \frac{p}{p-q}$$

$$P(\exists n > 0 : S_n = n) = \varphi'(f(1)) = \frac{q}{p}.$$

Now we specialize some of our general results to the probabilistic case. Let again $\mu = E(X) = \varphi'(1)$. If $1 < \mu \le \infty$ then we have $f(1) < 1$ by Theorem 2.2. If $0 < \mu < 1$ then it follows from Theorem 2.2 and from (6.7) that

$$f(w) \to 1, \quad f'(w) \to \frac{1}{1-\mu} \qquad \text{as } w \to 1,\ w \in \mathbb{D}. \tag{6.11}$$

Let $\mu = 1$. Then the variance is $\sigma^2 = V(X) = 2\varphi''(1)$. First we assume that $\sigma < \infty$. Since the coefficients are non-negative it follows that φ'' is continuous at 1. Hence we obtain from Theorem 4.2 that

$$f(w) \to 1, \quad f'(w) \sim \frac{1}{\sigma}(1-w)^{-1/2} \qquad \text{as } w \to 1,\ w \in \mathbb{D}. \tag{6.12}$$

Furthermore we obtain from (4.3) with $\beta = 0$ that ∂F in a neighbourhood of 1 consists of two Jordan arcs symmetric to $\mathbb{R}$ with tangent angles $\pm\pi/4$.

Theorem 6.1. *If the variance is infinite and thus $\varphi''(1) = \infty$ then*

$$f'(w) = o\big((1-|w|)^{-1/2}\big) \qquad \text{as } |w| \to 1. \tag{6.13}$$

Proof. By Theorem 7.1 below we have $a_{n,n} = o(n^{-1/2})$. Hence it follows from (6.5) that

$$|f'(w)| \le \sum_{n=0}^{\infty} a_{n,n}|w|^n = \sum_{n=0}^{\infty} o(n^{-1/2}|w|^n) = o\big((1-|w|)^{-1/2}\big)\,.$$

■

It is not clear whether ∂F always has tangents at 1 if $\varphi''(1) = \infty$. In any case the angle can be any number between $\pi/2$ and π as the following example shows.

Example 6.2. Let $1 < \alpha < 2$. The function

$$\begin{aligned}\varphi(z) &= z + \frac{1}{2\alpha}(1-z)^{\alpha} + \frac{1}{4}(1-z)^2 \\ &= \left(\frac{1}{4} + \frac{1}{2\alpha}\right) + \frac{\alpha}{4}z^2 + \sum_{k=3}^{\infty} \frac{(\alpha-1)(2-\alpha)\cdots(k-1-\alpha)}{2k!} z^k\end{aligned}$$

has non-negative coefficients. It satisfies $\varphi(1) = 1$ and

$$\varphi'(z) = \frac{1}{2} + \frac{z}{2} - \frac{1}{2}(1-z)^{\alpha-1}, \qquad \varphi''(z) = \frac{1}{2} + \frac{\alpha-1}{2}(1-z)^{\alpha-2}.$$

It follows that $\mu = \varphi'(1) = 1$ and $\varphi''(1) = \infty$ and thus $\sigma = \infty$. Furthermore

$$\left|\frac{\varphi(z)}{z}\right|^2 = 1 + \frac{1}{\alpha}\operatorname{Re}((1-z)^\alpha) + \mathcal{O}(|1-z|^2) \qquad \text{as } z \to 1,\ z \in \mathbb{D}.$$

If $z \in \partial F$ then $|\varphi(z)| = |z|$ and it follows that

$$\cos(\alpha \arg(1-z)) = \mathcal{O}(|1-z|^{2-\alpha}) \to 0 \qquad \text{as } z \to 1,\ z \in \partial F.$$

Hence $\arg(1-z) \to \pm\pi/(2\alpha)$ so that F has a symmetric corner of opening angle $\pi/\alpha \in (\pi/2, \pi)$ at 1.

We need the following theorem in Section 7. Let $1 < \varphi'(1) \le \infty$. Then the function

$$x\varphi'(x) - \varphi(x) = -a + \sum_{k=2}^{\infty}(k-1)a_k x^k, \qquad 0 \le x \le 1, \tag{6.14}$$

is increasing from $-a < 0$ to $\varphi'(1) - 1 > 0$. Hence there is a unique $r \in (0,1)$ such that

$$r\varphi'(r) = \varphi(r). \tag{6.15}$$

Theorem 6.2. *Let* $1 < \varphi'(1) \le \infty$ *and let* $r \in (0,1)$ *be defined by* (6.15). *Then* $r > \varphi(r)$ *and the function* $\varphi(z)/z$ *is univalent in* $\{|z| < r\}$. *Furthermore* f *has an analytic continuation to* $\{|w| < r/\varphi(r)\}$ *with values in* $\mathbb{D}$.

Proof. Let $|z| < r$. Then we see (compare (6.14)) that

$$|z\,\varphi'(z) - \varphi(z) + a| \le \sum_{k=2}^{\infty}(k-1)a_k r^k = r\varphi'(r) - \varphi(r) + a = a$$

by (6.15). It follows that the function

$$\psi(z) = \frac{z\varphi'(z) - \varphi(z) + a}{z^2} = a_2 + \cdots, \qquad |z| < r, \tag{6.16}$$

is analytic and bounded by a. If $|z_1| < r$, $|z_2| < r$ and $z_1 \ne z_2$ then

$$\begin{aligned} \left|\frac{\varphi(z_2)}{z_2} - \frac{\varphi(z_1)}{z_1}\right| &= \left|\frac{a(z_1 - z_2)}{z_1 z_2} + \int_{z_1}^{z_2} \frac{d}{dz}\frac{\varphi(z) - a}{z}\,dz\right| \\ &\ge \frac{a|z_1 - z_2|}{|z_1 z_2|} - \int_{z_1}^{z_2} |\psi(z)|\,|dz| > 0. \end{aligned} \tag{6.17}$$

Hence $\varphi(z)/z$ is univalent in $\{|z| < r\}$.

If $r < x < 1$ then $(\varphi(x)/x)' = (x\varphi'(x) - \varphi(x))/x^2 > 0$ by (6.14) and (6.15), and since $\varphi(1) = 1$ we deduce that $\varphi(r)/r < 1$. We have just shown that $w = z/\varphi(z)$ is univalent in $|z| < r$ and thus has a well-defined inverse function which is an analytic continuation of our function f. Since

$$|w| = \left|\frac{z}{\varphi(z)}\right| \ge \frac{|z|}{\varphi(|z|)} \to \frac{r}{\varphi(r)} \qquad \text{as } |z| \to r,$$

this continuation is valid at least in $\{|w| < r/\varphi(r)\}$. ∎

7. The coefficients in the probabilistic case

We assume again that (6.1) holds and define the $a_{n,k}$ by (5.4). The random variable X or its generating function φ is called *periodic* if there exists $q \geq 2$ such that

$$a_k = 0 \qquad \text{for } k \not\equiv 0 \mod q. \tag{7.1}$$

Otherwise φ is called *aperiodic*. If φ is periodic and q is maximal in (7.1) then

$$\varphi^*(z) = \varphi(z^{1/q}) = \sum_{j=0}^{\infty} a_{qj} z^j \tag{7.2}$$

is an aperiodic generating function. This way it is easy to reduce the periodic case to the aperiodic case.

Lemma 7.1. *If $0 < a < 1$ and φ is aperiodic then $|\varphi(z)| < 1$ for $z \in \overline{\mathbb{D}}$, $z \neq 1$.*

Proof. Suppose that $|\varphi(e^{it})| = 1$ for some t with $0 < t < 2\pi$ and let $k > 0$ be such that $a_k > 0$. Then

$$1 = |\varphi(e^{it})| \leq |a + a_k e^{ikt}| + \sum_{j \neq 0,k} a_j \leq 1.$$

Since $a > 0$ and $a_k > 0$ it follows that $t = 2\pi\nu/k$ for some $\nu \in \mathbb{Z}$, $\nu \neq 0$. Hence we can write $t = \pm 2\pi p/q$ with relative prime p and q. It follows that k is divisible by a fixed q whenever $a_k > 0$. Hence (7.1) holds. ∎

Theorem 7.1. *Let φ be aperiodic and write $a_{n,k} = P(S_n = k)$. Let the variance satisfy $0 < \sigma^2 \leq \infty$. Then*

$$\limsup_{n\to\infty} \sqrt{n} \sum_{k=0}^{\infty} a_{n,k}^2 \leq \frac{1}{2\sqrt{\pi}\sigma}, \tag{7.3}$$

$$\limsup_{n\to\infty} \sqrt{n} \sup_k a_{n,k} \leq \frac{1}{\sqrt{2\pi}\sigma}. \tag{7.4}$$

Proof. For $\varepsilon > 0$ there exists $\delta > 0$ such that $|\sin\tau| \geq (1-\varepsilon)|\tau|$ for $|\tau| \leq \delta$. Let $m \in \mathbb{N}$ and $s_m = a_0 + \ldots + a_m$. Then

$$\begin{aligned} \left|\sum_{k=0}^{m} a_k e^{ikt}\right|^2 &= \left|s_m - \sum_{k=1}^{m} a_k(1-e^{ikt})\right|^2 \\ &= {s_m}^2 - 4s_m \sum_{k=1}^{m} a_k \left(\sin\frac{kt}{2}\right)^2 + \left|\sum_{k=1}^{m} a_k(1-e^{ikt})\right|^2. \end{aligned}$$

Since $|1-e^{ikt}| \le k|t|$ this is not larger than

$$s_m{}^2 - (1-\varepsilon)^2 s_m \sum_{k=1}^{m} k^2 a_k t^2 + \left(\sum_{k=1}^{m} k a_k\right)^2 t^2$$

for $|t| \le \delta/m$. Since $\sqrt{x+y} \le \sqrt{x} + y/(2\sqrt{x})$ for $x > 0$, $x+y \ge 0$, it follows that

$$|\varphi(e^{it})| \le s_m - \frac{(1-\varepsilon)^2}{2} \sum_{1}^{m} k^2 a_k t^2 + \frac{1}{2s_m}\left(\sum_{1}^{m} k a_k\right)^2 t^2 + \sum_{m+1}^{\infty} a_k = 1 - c_m{}^2 t^2 \tag{7.5}$$

for $|t| \le \delta/m$ where

$$c_m{}^2 = \frac{(1-\varepsilon)^2}{2} \sum_{1}^{m} k^2 a_k - \frac{1}{2s_m}\left(\sum_{1}^{m} k a_k\right)^2,$$

which is positive for small ε and large m, see (7.7) below. Let η_m denote the minimum of δ/m and $1/c_m$. Since φ is aperiodic it follows from Lemma 7.1 that

$$|\varphi(e^{it})| \le b_m = b_m(\varepsilon) < 1 \qquad \text{for } \eta_m \le |t| \le m. \tag{7.6}$$

By Parseval's formula we obtain from (7.5) and (7.6) that

$$\begin{aligned}\sum_{k=0}^{\infty} a_{n,k}{}^2 &= \frac{1}{\pi}\int_0^{\eta_m} |\varphi(e^{it})|^{2n}\, dt + \frac{1}{\pi}\int_{\eta_m}^{\pi} |\varphi(e^{it})|^{2n}\, dt \\ &\le \frac{1}{\pi}\int_0^{\eta_m} \left(1 - c_m{}^2 t^2\right)^{2n}\, dt + b_m{}^{2n}.\end{aligned}$$

We substitute $\tau = c_m{}^2 t^2$ in the last integral. Then $0 < \tau \le 1$ so that this integral is bounded by

$$\frac{1}{2\pi c_m}\int_0^1 (1-\tau)^{2n}\tau^{-1/2}\, d\tau = \frac{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)}{2\pi c_m \Gamma\left(2n+\frac{3}{2}\right)} \sim \frac{(2n)^{-1/2}}{2\sqrt{\pi} c_m}$$

as $n \to \infty$. Hence we conclude from (7.7) that

$$\limsup_{n\to\infty} \sqrt{n} \sum_{k=0}^{\infty} a_{n,k}{}^2 \le \frac{1}{2\sqrt{2\pi} c_m}.$$

Now let $m \to \infty$. Then

$$\begin{aligned} c_m{}^2 &\to \frac{(1-\varepsilon)^2}{2} \sum_{k=1}^{\infty} k^2 a_k - \frac{1}{2}\left(\sum_{k=1}^{\infty} k a_k\right)^2 \\ &= \frac{(1-\varepsilon)^2}{2}\sigma^2 - \left(\varepsilon - \frac{\varepsilon^2}{2}\right)\left(\sum_{k=1}^{\infty} k a_k\right)^2 \end{aligned} \tag{7.7}$$

if $\sigma < \infty$ and $c_m \to \infty$ if $\sigma = \infty$. Hence (7.3) holds.

To prove (7.4) we assume that $n = 2\nu + 1$; the case $n = 2\nu$ is slightly simpler. Since $\varphi(z)^{2\nu+1} = \varphi(z)^{\nu}\,\varphi(z)^{\nu+1}$ it follows from the definition (5.4) of $a_{n,k}$ that

$$a_{2\nu+1,k} = \sum_{j=0}^{k} a_{\nu,j}a_{\nu+1,k-j} \le \frac{1}{2}\sum_{j=0}^{\infty} {a_{\nu,j}}^2 + \frac{1}{2}\sum_{j=0}^{\infty} {a_{\nu+1,j}}^2.$$

Hence we deduce from (7.3) that

$$\limsup_{\nu\to\infty} \sqrt{2\nu+1}\, \sup_k a_{2\nu+1,k} \le \frac{\sqrt{2}}{2\sqrt{\pi}\sigma} = \frac{1}{\sqrt{2\pi}\sigma}.$$

■

Now let $\mu = \varphi'(1) = E(X)$. We only consider the coefficients

$$a_{n,n} = P(X_1 + \cdots + X_n = n) = P(S_n = n)$$

of $wf'(w)/f(w)$; see (6.3).

First we consider the case $\mu = 1$. Then $E(S_n) = n$ and $a_{n,n}$ is the probability to be precisely in equilibrium at time n. Theorem 7.1 shows that $a_{n,n} = \mathcal{O}(n^{-1/2})$ and the order of growth is sharp. Much more than (7.4) can be said if we assume that the variance σ^2 is finite. Then [Pe75, p. 187]

$$a_{n,n+j} \sim \frac{1}{\sqrt{2\pi n}\sigma} \qquad \text{as } n \to \infty \quad \text{for fixed } j \in \mathbb{Z}.$$

It follows that for $\mu = 1$ the inequality (7.4) can always be replaced by

$$\sqrt{n}\sup_k a_{n,k} \to \frac{1}{\sqrt{2\pi}\sigma} \qquad \text{as } n \to \infty.$$

Now we consider the case $1 < \mu \le \infty$. Then S_n tends to be much larger than n at time n.

Theorem 7.2. *Suppose that φ is aperiodic and that $\mu = \varphi'(1) > 1$. If $r \in (0,1)$ is the unique solution of* (6.15)*, then*

$$(7.8)\quad a_{n,n} = \sqrt{\frac{\varphi(r)}{2\pi r^2\varphi''(r)}}\left(\frac{\varphi(r)}{r}\right)^n n^{-1/2} + \mathcal{O}\left(\left(\frac{\varphi(r)}{r}\right)^n n^{-3/2}\right) \qquad \textit{as } n \to \infty.$$

Note that $\varphi(r)/r < 1$ by Theorem 6.2. Thus we have an exponential decay of $P(S_n = n)$ without any finiteness conditions, see (7.13) below.

Proof. We have $\rho := \varphi(r)/r < 1$. We consider the functions

$$(7.9)\qquad \varphi^*(z) := \frac{\varphi(rz)}{\varphi(r)}, \quad z \in \mathbb{D}, \qquad f^*(w) := \frac{1}{r} f\left(\frac{w}{\rho}\right) \quad w \in \mathbb{D};$$

the second function is analytic by Theorem 6.2. It follows from (1.4) that

$$w\varphi^*(f^*(w)) = \frac{w}{\varphi(r)}\varphi\left(f\left(\frac{w}{\rho}\right)\right) = r^{-1} f\left(\frac{w}{\rho}\right) = f^*(w).$$

Hence f^* is the fixed point function of φ^*. Now φ^* is analytic in $\overline{\mathbb{D}}$ because $r < 1$ and it satisfies

$$\varphi^*(1) = 1, \qquad \varphi^{*\prime}(1) = \frac{r\varphi'(r)}{\varphi(r)} = 1, \qquad \varphi^{*\prime\prime}(1) = \frac{r^2\varphi''(r)}{\varphi(r)}$$

by (6.15). Since φ^* is aperiodic it follows from Lemma 7.1 that $|\varphi^*(z)| < 1$ for $z \in \overline{\mathbb{D}}$, $z \neq 1$. Hence $\mathbb{T} \cap \partial f^*(\mathbb{D}) = \{1\}$ and we obtain from Theorem 5.1 that

$$a^*_{n,n} = \sqrt{\frac{\varphi(r)}{2\pi r^2\varphi''(r)}}\, n^{-1/2} + \mathcal{O}(n^{-3/2}) \qquad \text{as } n \to \infty. \tag{7.10}$$

By (5.4) and (7.9) we have

$$\varphi(z)^n = \sum_{k=0}^{\infty} a^*_{n,k}\varphi(r)^n \left(\frac{z}{r}\right)^k.$$

Hence (7.8) follows from (7.10). ∎

Finally we discuss the connection with the theory of *large deviations*; see for e.g. [DeZe93]. For $x > 0$ we define $\varphi(x) = \infty$ whenever the power series of φ does not converge. Then

$$R = \sup\{x > 0 : \varphi(x) < \infty\} \leq +\infty \tag{7.11}$$

is the radius of convergence of this power series. Specializing [DeZe93, Def. 2.2.2] we define

$$\rho := \inf_{0<x<\infty} \frac{\varphi(x)}{x} = \exp\left(\inf_{t\leq R} \left(\log E(e^{tX}) - t\right)\right) \leq 1. \tag{7.12}$$

It follows from the upper limit in Cramér's Theorem on large deviations [DeZe93, Th. 2.2.3] that

$$a_{n,n} = \mathcal{O}((\rho+\varepsilon)^n) \qquad \text{as } n \to \infty \quad \text{for every } \varepsilon > 0. \tag{7.13}$$

Let again $\mu = E(X)$. If $1 < \mu \leq \infty$ then $\rho = \varphi(r)/r < 1$ where $r \in (0,1)$ is the unique solution of $r\,\varphi'(r) = r$; compare (6.11). Hence we get an exponential decay as in Theorem 7.2.

Now let $\mu < 1$. If $R = 1$ then $\rho = 1$ by (7.12) so that (7.13) is trivial. Therefore we suppose that $1 < R \leq \infty$. This corresponds to the assumption that $E(e^{\lambda X}) < \infty$ for λ near 0 as, for example, in the Gärtner-Ellis Theorem [DeZe93, Th. 2.3.6]. Since $x\,\varphi'(x) - \varphi(x)$ is increasing in $(0, R)$ and is $\mu - 1 < 0$ for $x = 1$, there are two alternatives.

(a) There exists $r \in (1, R)$ satisfying (6.15): then $\rho = \varphi(r)/r$ by (7.12) and $\rho < 1$ because $\varphi(1) = 1$. As in (7.9) we consider the function

$$\varphi^*(z) = \frac{\varphi(rz)}{\varphi(r)} = \sum_{k=0}^{\infty} a_k \frac{r^k}{\varphi(r)} z^k$$

which is analytic in $\overline{\mathbb{D}}$. Hence we obtain from Theorem 5.1 that (7.8) holds, that is

$$a_{n,n} = c\rho^n n^{-1/2} + \mathcal{O}\big(\rho^n n^{-3/2}\big) \qquad \text{as } n \to \infty,\ c > 0.$$

(b) We have $x\varphi'(x) - \varphi(x) < 0$ for $1 \le x < R$: then $R < \infty$ and $(\varphi(x)/x)' < 0$. Therefore we have $\rho = \varphi(R)/R < \varphi(1) = 1$. Now we consider the function $\varphi^*(z) = \varphi(Rz)/\varphi(R)$, $z \in \mathbb{D}$, which satisfies $\varphi^{*\prime}(1) \le 1$. Applying Theorem 7.1 to φ^* we obtain that

$$a_{n,n} = \mathcal{O}\left(\rho^n n^{-1/2}\right) \qquad \text{as } n \to \infty,$$

where $\mathcal{O}$ can be replaced by o if $\varphi''(R) = \infty$ and thus $\varphi^{*\prime\prime}(1) = \infty$.

8. Hyperbolic convexity and other problems

The domain $G \subset \mathbb{D}$ is called *hyperbolically convex* (h-convex) if, for every pair $z_1, z_2 \in G$, the hyperbolic line segment between z_1 and z_2 also lies in G. A conformal map g of $\mathbb{D}$ into $\mathbb{D}$ is called h-convex if $g(\mathbb{D})$ is h-convex. These maps were studied in [MaMi94] and later e.g. in [MePo98] and [MePo00].

The following theorem is an easy consequence of a result of V.Jørgensen [Jø56].

Theorem 8.1. *If the function $\varphi(z)/z$ is univalent in $\mathbb{D}$ then the fixed point function f is hyperbolically convex.*

Proof. The meromorphic function $\chi(z) = \varphi(z)/z$, $z \in \mathbb{D}$, satisfies

$$\limsup_{|z|\to 1} |\chi(z)| = \limsup_{|z|\to 1} |\varphi(z)| \le 1.$$

Since χ is univalent and $\chi(0) = \infty$, it follows that the disk $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ lies in $\chi(\mathbb{D})$. Using again that χ is univalent we conclude from Jørgensen's Theorem [Jø56] and from (1.2) that the preimage $\chi^{-1}(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}) = \{z \in \mathbb{D} : |\chi(z)| > 1\} = F = f(\mathbb{D})$ is h-convex. ∎

Theorem 8.2. *If the coefficients a_k of φ satisfy*

$$\sum_{k=2}^{\infty} (k-1)|a_k| \le |a_0| = |a| \tag{8.1}$$

then f is hyperbolically convex. In particular this holds in the probabilistic case (6.1) *if $\varphi'(1) = E(X) \le 1$.*

Proof. We argue as in the proof of Theorem 6.2 but now with $r = 1$. By (8.1) the function ψ defined by (6.16) is bounded by $|a|$, so that $\varphi(z)/z$ is univalent in $\mathbb{D}$. Hence f is h-convex by Theorem 8.1. In the probabilistic case we have

$$\sum_{k=2}^{\infty} (k-1)a_k = \varphi'(1) - \varphi(1) + a \le a.$$

∎

In general the function $\varphi(z)/z$ is not univalent in $\mathbb{D}$, for instance if φ has more than one zero. This raises the following

Problem 1. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic and $\varphi(0) \neq 0$. Is it always true that the domain $F = \{z \in \mathbb{D} : |z| < |\varphi(z)|\}$ is hyperbolically convex?

There are some indications that may point in this direction. By (3.4) we have

$$\operatorname{Re} \frac{wf'(w)}{f(w)} > \frac{1}{2} \qquad \text{for } w \in \mathbb{D}.$$

This inequality holds [MePo00, Th. 2.1] for all h-convex functions f with $f(0) = 0$ and, together with the hyperbolic invariance of h-convexity, plays an important role in the theory of h-convex functions. But there does not seem to be any hyperbolic invariance in our present situation.

The function k_a of Example 3.1 is also the extremal function for various problems on h-convex functions [MaMi94]. It follows from (3.9) in Theorem 3.1 that

$$\operatorname{Re} \frac{f(w)}{aw} \geq \frac{1}{1 + \sqrt{1 - |a|^2}} \qquad \text{for } w \in \mathbb{D}$$

and this inequality is conjectured [MePo98] to be true for h-convex functions.

The boundary of any h-convex domain has length at most π^2; see [BF83]. It follows that the derivative belongs to the Hardy space H^1. Thus the following question is a weaker form of Problem 1.

Problem 2. Let f map $\mathbb{D}$ conformally onto F with $f(0) = 0$. Does ∂F have length at most π^2 or does at least $f' \in H^1$ hold?

The coefficients of any function in H^1 tend to 0. Hence we see from (5.7) that $f' \in H^1$ implies $a_{n,n} \to 0$ as $n \to \infty$. But we do not even know whether this is generally true. However, Theorems 3.2, 4.3 and 5.1 suggest the following question.

Problem 3. Is it true that $a_{n,n} = \mathcal{O}(n^{-1/3})$ as $n \to \infty$?

By Theorem 7.1 this is true, even with $\mathcal{O}(n^{-1/2})$, in the probabilistic case. Furthermore, in Example 5.2 with $\beta = 2$, we have precisely the order $O(n^{-1/3})$.

A *quasicircle* is a Jordan curve J such that

$$\operatorname{diam} J(z_1, z_2) \leq c|z_1 - z_2| \qquad \text{for } z_1, z_2 \in J,$$

where $J(z_1, z_2)$ is the smaller arc of J between z_1 and z_2 and c is a constant. In particular J does not have cusps. In all our examples and in the regular situations in Section 4, there are no cusps.

Problem 4. Is ∂F a quasicircle?

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