Computational Methods and Function Theory
Volume 12 (2012), No. 1, 201–212

On the Boundary Behaviour of Polymorphic Functions

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(Communicated by Stephan Ruscheweyh)

Abstract. Let Φ be a Fuchsian group acting on the unit disk $\mathbb{D}$. This group may be infinitely generated and may have elliptic elements. A non-constant function f meromorphic in $\mathbb{D}$ is called Φ-*polymorphic* if, for every $\varphi \in \Phi$, there is a Moebius transformation γ such that $f \circ \varphi = \gamma \circ f$. This induces a homomorphism f^* of Φ into $\mathrm{PSL}(2, \mathbb{C})$. The image group $\Gamma := f^*(\Phi)$ need not be discrete. In this paper we study function-theoretic properties of a polymorphic function f and its relation to the limit sets and fixed points of Φ and the image group Γ. This has some consequences for degenerate Kleinian groups.

Keywords. Polymorphic function, normal function, boundary behaviour, Fuchsian group, Kleinian group.

2000 MSC. 30F35, 30D45, 30F40.

1. Introduction

Let Φ be a Fuchsian group acting on the unit disk $\mathbb{D}$. This group may be infinitely generated and may have elliptic elements. A non-constant function f meromorphic in $\mathbb{D}$ is called Φ-*polymorphic* if, for every $\varphi \in \Phi$, there is a Moebius transformation γ such that $f \circ \varphi = \gamma \circ f$. Defining $f^*(\varphi) := \gamma$ we obtain a homomorphism

$$(1.1) \qquad f^* \colon \Phi \to \mathrm{PSL}(2, \mathbb{C}), \qquad f \circ \varphi = f^*(\varphi) \circ f.$$

The image group $\Gamma := f^*(\Phi)$ need not be discrete. We will however often assume that Γ is non-elementary. We use the term elementary in the wide sense, that is, including the elliptic groups [1, p. 84].

We use the classical name "polymorphic function" [9, 10, 25]. In the context of the Riemann surface $\mathbb{D}/\Gamma$, the polymorphic function is called the developing map of a projective structure, or of a branched projective structure if f is not locally

Received February 22, 2011, in revised form October 22, 2011.
Published online January 21, 2012.
The first author was partially supported by COLCIENCIAS, project 1118-521-28160.

ISSN 1617-9447/\$ 2.50

univalent, see [8, 11, 7, 2]. Families of homomorphisms (1.1) are for example considered in [28, 30, 15].

In this paper we study function-theoretic properties of a polymorphic function f and its relation to the limit sets and fixed points of Φ and the image group Γ.

A function f meromorphic in $\mathbb{D}$ is called *normal* [12] if the spherical derivative

$$f^{\#} = \frac{|f'|}{1+|f|^2}$$

satisfies

$$\sup_{z\in\mathbb{D}}(1-|z|^2)f^{\#}(z) < \infty. \tag{1.2}$$

Every meromorphic function omitting three values in $\widehat{\mathbb{C}}$ is normal. Now let f be Φ-polymorphic and let $\Gamma = f^*(\Phi)$ be non-elementary. Then [19], [14, Th. 8]

$$f \text{ is normal} \qquad \Leftrightarrow \qquad \operatorname{cap}(\widehat{\mathbb{C}} \setminus f(\mathbb{D})) > 0. \tag{1.3}$$

If f is normal then there exists a component U of $\widehat{\mathbb{C}} \setminus L(\Gamma)$ such that $\partial U = L(\Gamma)$ and $f(\mathbb{D}) \subset U$ where $L(\Gamma)$ is the limit set of Γ.

The *cluster set* $C(\zeta)$ at a point $\zeta \in \mathbb{T} = \partial\mathbb{D}$ is the connected compact set $C(\zeta)$ of all $\omega \in \widehat{\mathbb{C}}$ such that

$$\text{there are } z_k \in \mathbb{D} \text{ with } z_k \to \zeta, \quad f(z_k) \to \omega \text{ as } k \to \infty. \tag{1.4}$$

In Section 2 we study the cluster sets for $\zeta \in L(\Phi)$ and also the angular cluster set on the angular limit set of Φ. We assume that f is normal because $C(\zeta) = \widehat{\mathbb{C}}$ if f is not normal.

In Section 3 we consider the question what more can be said if the Fuchsian group Φ is of convergence or accessible type, in particular if f is supposed to be univalent. We also investigate the problem whether the polymorphic function f is uniquely determined by the induced homomorphism f^*.

Finally we characterize the continuity of f at a hyperbolic fixed point of Φ. For the case that f is univalent we apply this result to the study of the limit set $L(\Gamma)$ of degenerate Kleinian groups, that is $L(\Gamma)$ does not separate $\widehat{\mathbb{C}}$.

2. Normal functions and limit sets

First we formulate the definitions of $L(\Phi)$ and $L(\Gamma)$ in the forms that we need below. The limit set $L(\Phi)$ of the Fuchsian group Φ consists of all $\zeta \in \mathbb{T}$ for which there are $\varphi_n \in \Phi$ such that

$$\varphi_n(z) \to \zeta \qquad \text{as } n \to \infty \qquad \text{for } z \in \mathbb{D}. \tag{2.1}$$

If the image group $\Gamma = f^*(\Phi)$ is non-elementary then $L(\Gamma)$ is the closure of the fixed points of all loxodromic and parabolic $\gamma \in \Gamma$, see [1, p. 97, p. 104].

Theorem 2.1. *Let f be a normal Φ-polymorphic function and let $\Gamma = f^*(\Phi)$ be non-elementary. Then*

(i) *for every $\zeta \in L(\Phi)$ there is $\omega \in L(\Gamma)$ such that $\omega \in C(\zeta)$,*
(ii) *for every $\omega \in L(\Gamma)$ there is $\zeta \in L(\Phi)$ such that $\omega \in C(\zeta)$.*

The first statement follows from the classical definition [1, Thm. 5.3.9] of $L(\Gamma)$ if f^* is an isomorphism onto Γ and Γ is discrete.

Proof. (i). Let $\zeta \in L(\Phi)$. Then there are $\varphi_n \in \Phi$ satisfying (2.1). Let $p_n^\pm$ be the fixed points of $\gamma_n := f^*(\varphi_n)$ and μ_n its multiplier. First we assume that $p_n^+ \neq p_n^-$. The case $p_n^+ = p_n^-$ is handled as in Case 1 below. Then

$$\frac{\gamma_n(w) - p_n^+}{\gamma_n(w) - p_n^-} = \mu_n \frac{w - p_n^+}{w - p_n^-}, \qquad |\mu_n| \leq 1 \tag{2.2}$$

if p_n^+ is the attractive fixed point of γ_n and $p_n^\pm \neq \infty$.

Taking a subsequence we may assume that

$$p_n^\pm \to p^\pm, \quad \mu_n \to \mu \qquad \text{as } n \to \infty. \tag{2.3}$$

By a rotation of $\widehat{\mathbb{C}}$ we may assume that $p^\pm \neq \infty$. We set

$$g_n := f \circ \varphi_n = \gamma_n \circ f, \qquad h_n := f \circ \varphi_n^{-1} = \gamma_n^{-1} \circ f. \tag{2.4}$$

Then the spherical derivatives satisfy

$$(1 - |z|^2) g_n^\#(z) = (1 - |\varphi_n(z)|^2) f^\#(\varphi_n(z)) \leq M, \qquad z \in \mathbb{D},$$

for some $M < \infty$ because f is normal, see (1.2), similarly for $h_n^\#(z)$. It follows that (g_n) and (h_n) are normal sequences. Taking again a subsequence we may also assume that

$$g_n \to g, \quad h_n \to h \qquad \text{as } n \to \infty. \tag{2.5}$$

It follows from (1.4), (2.1) and (2.4) that

$$g(z) \in C(\zeta) \qquad \text{for } z \in \mathbb{D},\ \zeta \in L(\Phi). \tag{2.6}$$

Case 1. First we consider the case that $p^+ \neq p^-$. We put (2.4) into (2.2) and then let $n \to \infty$. Using (2.3) and (2.5) we obtain

$$\frac{g - p^+}{g - p^-} = \mu \frac{f - p^+}{f - p^-}, \qquad \frac{h - p^+}{h - p^-} = \frac{1}{\mu} \frac{f - p^+}{f - p^-}. \tag{2.7}$$

If $\mu = 0$ then γ_n is loxodromic for large n. We deduce from the first identity (2.7) that $g(z) = p^+$ for all $z \in \mathbb{D}$ with $f(z) \neq p^-$. Hence $p^+ \in C(\zeta)$ by (2.6) and $p^+ \in L(\Gamma)$ by (2.3) and the definition of $L(\Gamma)$. This proves (i) in Case 1 if $\mu = 0$.

Now let $\mu \neq 0$. Taking the limit $n \to \infty$ in (2.2) we obtain a Moebius transformation α defined by

$$\frac{\alpha(w) - p^+}{\alpha(w) - p^-} = \mu \frac{w - p^+}{w - p^-}.$$

With $w = f(z)$ we deduce from (2.7) and (2.4) that $g = \alpha \circ f$ and $h = \alpha^{-1} \circ f$. Writing $F = f(\mathbb{D})$ we see from (2.4) that $g(\mathbb{D}) \subset \overline{F}$, $h(\mathbb{D}) \subset \overline{F}$ and therefore $\alpha(F) = g(\mathbb{D}) \subset F$, $\alpha^{-1}(F) = h(\mathbb{D}) \subset F$. Hence $\alpha(F) \subset F \subset \alpha(F)$ so that, if $p^+ \neq p^-$,

$$g(\mathbb{D}) = f(\mathbb{D}). \tag{2.8}$$

Case 2. $p^+ = p^- =: p$. We rewrite (2.2) and obtain as in (1) that

$$\frac{1}{g_n - p_n^-} = \frac{\mu_n}{f - p_n^-} + \lambda_n, \qquad \lambda_n := \frac{1 - \mu_n}{p_n^+ - p_n^-}. \tag{2.9}$$

Taking another subsequence we may also assume that $\lambda_n \to \lambda \in \widehat{\mathbb{C}}$. Note that $|\mu_n| \leq 1$. If $\lambda = \infty$ then, taking the limit in (2.9), we obtain from (2.3) and (2.5) that $g(z) = p$ for all $z \in \mathbb{D}$ with $f(z) \neq p$. Hence $p \in C(\zeta)$ by (2.6) and $p \in L(\Gamma)$ by (2.3). If $p_n^+ = p_n^-$ then (2.9) holds with $\mu_n = 1$ and a different λ_n. This proves (i) in Case 2 if $\lambda = \infty$.

Now let $\lambda \neq \infty$. Since $p_n^+ - p_n^- \to 0$ it follows from (2.9) that $\mu = 1$. Now

$$\frac{1}{\alpha(w) - p} = \frac{1}{w - p} + \lambda$$

defines a Moebius transformation α. Hence we obtain from (2.9), (2.4) and (2.5) that $g = \alpha \circ f$ and $h = \alpha^{-1} \circ f$. As in Case 1 we conclude that (2.8) holds also if $p^+ = p^-$.

Now we have handled both cases. In Case 1 with $\mu \neq 0$ and in Case 2 with $\lambda \neq \infty$ we obtain from (2.6) and (2.8) that

$$f(\mathbb{D}) = g(\mathbb{D}) \subset C(\zeta) = \overline{C(\zeta)}. \tag{2.10}$$

Since Γ is non-elementary there is a loxodromic $\gamma \in \Gamma$, say $\gamma = f^*(\psi)$ with $\psi \in \Phi$. Let $\eta \in \mathbb{T}$ be a fixed point of ψ. Then the fixed point $\omega := f(\eta)$ (in the sense of an angular limit) of γ satisfies $\omega \in L(\Gamma)$ by the definition of $L(\Gamma)$. Furthermore we have

$$\omega = \lim_{k\to\infty} \gamma^k(f(0)) = \lim_{k\to\infty} f(\psi^k(0)) \in \overline{f(\mathbb{D})} \subset C(\zeta) \tag{2.11}$$

by (2.10). This proves (i) in the remaining Cases, i.e. Case 1 with $\mu \neq 0$ and Case 2 with $\lambda \neq \infty$.

(ii) Let $\omega \in L(\Gamma)$. There are loxodromic fixed points p_k of $\gamma_k = f^*(\varphi_k)$ that satisfy $p_k \to \omega$ as $k \to \infty$. We may assume that the fixed points ζ_k of φ_k converge to some ζ in the closed set $L(\Phi)$. We choose n_k so large that

$$\varphi_k^{n_k}(0) \to \zeta, \quad f(\varphi_k^{n_k}(0)) \to \omega \qquad \text{as } k \to \infty.$$

Then we have $\omega \in C(\zeta)$ by (1.4). ■

A Stolz angle at $\zeta \in \mathbb{T}$ is an open triangle Δ in $\mathbb{D}$ such that $\overline{\Delta} \cap \mathbb{T} = \{\zeta\}$. The *angular cluster set* $C_{\text{ang}}(\zeta)$ of the function f at $\zeta \in \mathbb{T}$ consists of all $\omega \in \widehat{\mathbb{C}}$ such that, for some Stolz angle Δ at ζ, there are $z_n \in \Delta$ with $z_n \to \zeta$, $f(z_n) \to \omega$ as $n \to \infty$. The *angular limit set* $L_{\text{ang}}(\Phi)$ of Φ consists of all ζ such that, for some Stolz angle Δ at ζ, there are $\varphi_n \in \Phi$ with $\varphi_n(0) \in \Delta$, $\varphi_n(0) \to \zeta$ as $n \to \infty$.

Theorem 2.2. *Let f be normal and Φ-polymorphic and let $\Gamma = f^*(\Phi)$ be non-elementary. Then*

$$C_{\text{ang}}(\zeta) \cap L(\Gamma) \neq \emptyset \qquad \textit{for } \zeta \in L_{\text{ang}}(\Phi). \tag{2.12}$$

If $\zeta \in L_{\text{ang}}(\Phi)$ and the angular limit $f(\zeta)$ exists then $f(\zeta) \in L(\Gamma)$.

Proof. The proof is quite similar to the part (i) of the proof of Theorem 2.1. Let $\zeta \in L_{\text{ang}}(\Phi)$. Instead of (i) we have to show that

(i') there is $\omega \in L(\Gamma)$ with $\omega \in C_{\text{ang}}(\zeta)$.

There are $\varphi_n \in \Phi$ with $\varphi_n(0) \to \zeta$ and $\varphi_n(0) \in \Delta$ for some Stolz angle Δ at ζ. For given $z \in \mathbb{D}$, each $\varphi_n(z)$ has a fixed hyperbolic distance from $\varphi_n(0)$. Hence there is a Stolz angle Δ' at ζ such that $\varphi_n(z) \in \Delta'$. Defining g_n as in (2.4) we deduce that

$$g(z) = \lim_{n\to\infty} f(\varphi_n(z)) \in C_{\text{ang}}(\zeta) \text{ for } z \in \mathbb{D}. \tag{2.13}$$

Now we argue as before with (2.6) replaced by (2.13). In Case 1 with $\mu = 0$ and Case 2 with $\lambda = \infty$ the proof of (i) is the same as the proof of Theorem 2.1 (i). In Cases 1 with $\mu \neq 0$ and in Case 2 with $\lambda \neq \infty$ we again construct $\omega \in L(\Gamma)$ as in (2.11). We have

$$f(\mathbb{D}) = g(\mathbb{D}) \subset C_{\text{ang}}(\zeta)$$

instead of (2.10). As in (2.11) we see that $\omega \in C_{\text{ang}}(\zeta)$. This proves (i').

Now suppose that the angular limit $f(\zeta)$ exists. Then $C_{\text{ang}}(\zeta) = \{f(\zeta)\}$ so that $f(\zeta) \in L(\Gamma)$ by (2.12). ■

In both theorems we assumed that f is normal. Now we show that this assumption cannot be omitted.

Proposition 2.3. *Let f be Φ-polymorphic and Γ non-elementary. If f is not normal then $C(\zeta) = \widehat{\mathbb{C}}$ for all $\zeta \in L(\Phi)$ and $C_{\text{ang}}(\zeta) = \widehat{\mathbb{C}}$ for all $\zeta \in L_{\text{ang}}(\Phi)$.*

Proof. It follows from the characterization (1.3) that $f(\mathbb{D}) = \widehat{\mathbb{C}}$. Hence there exists a disk $D_0 = \{|z| < r\}$ such that $f(D_0) = \widehat{\mathbb{C}}$. If $\zeta \in L(\Phi)$ and $\varphi_n(0) \to \zeta$ as $n \to \infty$ then we conclude that

$$f(\varphi_n(D_0)) = \gamma_n(f(D_0)) = \gamma_n(\widehat{\mathbb{C}}) = \widehat{\mathbb{C}}$$

so that $C(\zeta) = \widehat{\mathbb{C}}$. The proof of $C_{\text{ang}}(\zeta) = \widehat{\mathbb{C}}$ for $\zeta \in L_{\text{ang}}(\Phi)$ is similar. ■

3. Types of Fuchsian groups

The angular limit set $L_{\text{ang}}(\Phi)$ was defined in Section 2. A *horodisk* at $\zeta \in \mathbb{T}$ is a disk $H \subset \mathbb{D}$ with $\overline{H} \cap \mathbb{T} = \{\zeta\}$. The *horocyclic limit set* $L_{\text{hor}}(\Phi)$ consists of all $\zeta \in \mathbb{T}$ such that, for some H at ζ, there are $\varphi_n \in \Phi$ with $\varphi_n(0) \in H$ and $\varphi_n(0) \to \zeta$ as $n \to \infty$.

The Fuchsian group Φ is of *convergence type* if

$$\sum_{\varphi \in \Phi} (1 - |\varphi(0)|) < \infty,$$

otherwise of divergence type. There is a sharp distinction: If Φ is of convergence type then $\operatorname{mes} L_{\text{ang}}(\Phi) = 0$, if Φ is of divergence type then $\operatorname{mes} L_{\text{ang}}(\Phi) = 2\pi$. The group Φ is of convergence type if and only if the Riemann surface $\mathbb{D}/\Phi$ has a Green's function.

The Fuchsian group Φ is of *accessible type* if there is a measurable set $E \subset \mathbb{T}$ such that

$$\operatorname{mes} E > 0, \qquad \varphi(E) \cap E = \emptyset \qquad \text{for } \varphi \in \Phi,\ \varphi \neq \operatorname{id}.$$

This is true if and only if $\operatorname{mes} L_{\text{hor}}(\Phi) < 2\pi$. The group Φ is of *fully accessible type* if moreover

$$\sum_{\varphi \in \Phi} \operatorname{mes} \varphi(E) = 2\pi.$$

This is true if and only if mes $L_{\text{hor}}(\Phi) = 0$. See [24, 26]. See [29] for the interpretation in terms of ergodic theory.

Since $\operatorname{mes} L_{\text{ang}}(\Phi) \leq \operatorname{mes} L_{\text{hor}}(\Phi)$ every Fuchsian group of divergence type is of non-accessible type. Here is a group Φ of convergence type that is of non-accessible type: Let $A \subset \mathbb{R}$ be the classical Cantor set and let g be a universal covering map of $\mathbb{D}$ onto $\widehat{\mathbb{C}} \setminus A$. Then

$$\Phi = \{\varphi \colon g \circ \varphi = g\}$$

is a Fuchsian group that is of convergence type because $\operatorname{cap} A > 0$ and is of non-accessible type because $\operatorname{mes} A = 0$, see [26, p. 257].

Theorem 3.1. *Let Φ be not of accessible type. If f is univalent and Φ-polymorphic and Γ is non-elementary then*

$$\partial f(\mathbb{D}) = L(\Gamma).$$

Every univalent function is normal. The inclusion $L(\Gamma) \subset \partial f(\mathbb{D})$ holds for all Φ and all normal polymorphic functions [14, Cor. 3].

We need two results from the theory of univalent functions. The first is rather surprising and is due to Nagel, Rudin and Shapiro [20].

Lemma 3.2. *If f is univalent in $\mathbb{D}$ then there is a set $E \subset \mathbb{T}$ with* $\operatorname{mes} E = 2\pi$ *such that*

$$f(\zeta) = \lim_{z\to\zeta, z\in H} f(z) \qquad \text{exists for } \zeta \in E \tag{3.1}$$

where H is any horodisk at ζ.

The next lemma is rather standard and serves to pass from "almost all" to "all". See [6, Thm. 5.7] for the proof.

Lemma 3.3. *Let f be univalent in $\mathbb{D}$ and let $A \subset \mathbb{T}$ have measure 2π. If all angular limits $f(\zeta)$ $(\zeta \in A)$ exist and lie in the compact set B then $\partial f(\mathbb{D}) \subset B$.*

Proof of Theorem 3.1. Since Φ is not of accessible type we have

$$\operatorname{mes} L_{\text{hor}}(\Phi) = 2\pi.$$

If E is the set of Lemma 3.2 then $A := E \cap L_{\text{hor}}(\Phi)$ has measure 2π. Let $\zeta \in A$. Since $\zeta \in L_{\text{hor}}(\Phi)$ there exist $\varphi_n \in \Phi$ with $\varphi_n(0) \to \zeta$, $\varphi_n(0) \in H$ for some horodisk H at ζ. Since $\zeta \in E$ it now follows from (3.1) that $f(\varphi_n(0)) \to f(\zeta)$ as $n \to \infty$. The group Γ is discrete because f is univalent. Hence [1, Thm. 5.3.9]

$$f(\zeta) = \lim_{n\to\infty} f(\varphi_n(0)) = \lim_{n\to\infty} \gamma_n(f(0)) \in L(\Gamma),$$

and the result follows from Lemma 3.3. ∎

Now we turn to the problem whether f is uniquely determined by the induced homomorphism f^* defined in (1.1). Poincaré proved that this is true if $\mathbb{D}/\Phi$ is compact and f is locally univalent, and [9, Thm. 15] extended this result to functions with few ramifications.

Theorem 3.4. *Let f_j, $j = 1, 2$, be Φ-polymorphic and let*

$$f_1^* = f_2^*. \tag{3.2}$$

Suppose that $\Gamma := f_1^*(\Phi) = f_2^*(\Phi)$ is non-elementary.

(i) *If Φ is of divergence type and the f_j are normal then $f_1 = f_2$.*
(ii) *If Φ is not of fully accessible type and the f_j are univalent then $f_1 = f_2$.*

Proof. (i). By (1.3) the compact sets $\widehat{\mathbb{C}} \setminus f_j(\mathbb{D})$ have positive logarithmic capacity. Hence [22, p. 209] the functions f_j have angular limits almost everywhere and it follows from [25, Thm. 5] that $f_1 = f_2$.

(ii). Since Φ is not fully accessible we have $\operatorname{mes} L_{\text{hor}}(\Phi) > 0$. Hence, by [20, Lem. 3.2], there is a set $E \subset L_{\text{hor}}(\Phi)$ with $\operatorname{mes} E > 0$ such that, for $j = 1, 2$ and $\zeta \in E$,

$$f_j(z) \to f_j(\zeta) \qquad \text{as } z \to \zeta,\ z \in H, \tag{3.3}$$

where H is a horodisk at ζ. Furthermore, for $\zeta \in E$, there are $\varphi_n \in \Phi$ such that $\varphi_n(0) \in H$ and $\varphi_n(0) \to \zeta$ as $n \to \infty$.

It follows from (3.2) that $\gamma_n := f_1(\varphi_n) = f_2(\varphi_n)$. Passing to a subsequence we may assume that $\gamma_n(w) \to \gamma(w)$ as $n \to \infty$ for all w with one exception. Hence γ is a Moebius transformation or a constant. Since $\varphi_n(0) \in H$ it follows from (3.3) that, for $j = 1, 2$ and $\zeta \in E$,

$$f_j(\zeta) = \lim_{n\to\infty} f_j(\varphi_n(z)) = \lim_{n\to\infty} \gamma_n(f_j(z)) = \gamma(f_j(z))$$

for all $z \in \mathbb{D}$ with at most countably many exceptions. We conclude that γ is a constant. Hence $f_1(\zeta) = f_2(\zeta)$ for $\zeta \in E$, and since $\operatorname{mes} E > 0$ the Privalov Uniqueness Theorem [27, Cor. 6.14] implies that $f_1 = f_2$. ∎

4. Continuity at the fixed points

In this section, we study the behaviour at a single hyperbolic fixed point. Let φ be a Moebius transformation of $\mathbb{D}$ onto itself with attractive fixed point $\zeta^+ \in \mathbb{T}$ and repulsive fixed point $\zeta^- \neq \zeta^+$. We assume that f is meromorphic and normal and satisfies

$$f \circ \varphi = \gamma \circ f, \qquad \gamma \in \mathrm{PSL}(2, \mathbb{C}) \tag{4.1}$$

If γ is loxodromic or parabolic then the angular limits $f(\zeta^\pm)$ exist and lie in $\partial f(\mathbb{D})$; if γ is elliptic then the angular limits do not exist, see e.g. [14, Thm. 1].

We restrict ourselves to the case that γ is loxodromic with fixed points

$$p^+ = f(\zeta^+), \qquad p^- = f(\zeta^-). \tag{4.2}$$

We will use that, as $n \to \infty$,

$$\varphi^n(z) \to \zeta^+ \quad \text{for } z \neq \zeta^-, \qquad \gamma^{-n}(w) \to p^- \quad \text{for } w \neq p^+. \tag{4.3}$$

The *one-sided cluster sets* $C_>(\zeta)$ and $C_<(\zeta)$ consist of all points $\omega \in \widehat{\mathbb{C}}$ such that there exist $z_k \in \mathbb{D}$ with

$$\arg z_k \gtrless \arg \zeta, \qquad z_k \to \zeta, \quad f(z_k) \to \omega \qquad \text{as } k \to \infty.$$

The Collingwood Symmetry Theorem [5] [27, Prop. 2.21] states that

$$\{\zeta \in \mathbb{T} \colon C_<(\zeta) \neq C_>(\zeta)\}$$

is countable for every function f in $\mathbb{D}$. Let $T_>$ and $T_<$ be the arcs of $\mathbb{T} \setminus \{\zeta^+, \zeta^-\}$ with $\arg z \gtrless \arg \zeta^+$ near ζ^+.

Theorem 4.1. *Let f be normal in $\mathbb{D}$ and satisfy* (4.1) *where φ is hyperbolic and γ is loxodromic. Then the following three conditions are equivalent.*

(i) $C_>(\zeta^+) = \{p^+\}$,
(ii) $p^- \notin C_>(\zeta^+)$,
(iii) $p^- \notin C(\eta)$ *for* $\eta \in T_>$.

The corresponding assertion holds with $>$ *replaced by* $<$.

Condition (i) says that f is continuous at ζ^+ from the right or left respectively. It follows that

$$f \text{ is continuous at } \zeta^+ \quad \Leftrightarrow \quad p^- \notin C_>(\zeta^+) \quad \Leftrightarrow \quad p^- \notin C(\eta),\, \eta \neq \zeta^\pm.$$

Proof. (i) $\Rightarrow$ (ii). This is trivial because $p^+ \neq p^-$.

(ii) $\Rightarrow$ (iii). Suppose that $p^- \in C(\eta)$ for some $\eta \in T_>$. Then $|\eta - \zeta^\pm| > r > 0$. Let $\epsilon > 0$ and $\delta \in (0, r)$ be given. By (4.3) we can find $n \in \mathbb{N}$ such that

$$|\varphi^n(z) - \zeta^+| < \delta \qquad \text{for } |z - \eta| < r. \tag{4.4}$$

Furthermore there are points $z_k \in \mathbb{D}$ with $z_k \to \eta$ and $f(z_k) \to p^-$ as $k \to \infty$, hence $\gamma^{-n}(f(z_k)) \to p^-$ by (4.3). If k is sufficiently large then $z_k^* := \varphi^n(z_k)$ satisfies $|z_k^* - \zeta^+| < \delta$ because of (4.4), furthermore $\arg z_k^* > \arg \zeta^+$ because $\eta \in T_> = \varphi^n(T_>)$. For large k the spherical distance satisfies

$$d^\#(f(z_k^*), p^-) = d^\#(\gamma^n(f(z_k)), \gamma^n(p^-)) < \epsilon.$$

It follows that $p^- \in C_>(\zeta^+)$ which contradicts (ii).

(iii) $\Rightarrow$ (i). Suppose that $C_>(\zeta^+) \neq \{p^+\}$. Then there are $z_k \in \mathbb{D}$ and $\omega \in \widehat{\mathbb{C}}$ such that

$$\arg z_k > \arg \zeta^+, \qquad z_k \to \zeta^+, \quad f(z_k) \to \omega \neq p^+ \qquad \text{as } k \to \infty. \tag{4.5}$$

Let F be a fundamental domain of the cyclic group $\langle \varphi \rangle$ with $\zeta^\pm \notin \overline{F}$. There are $n_k \in \mathbb{Z}$ such that

$$z_k^* := \varphi^{-n_k}(z_k) \in F \qquad (k \in \mathbb{N}). \tag{4.6}$$

Passing to a subsequence we may assume that $z_k^* \to \eta \in \overline{F}$ and thus $\eta \neq \zeta^\pm$. Since $z_k \to \zeta^+$ it follows from (4.3) that $|n_k| \to \infty$. Hence, by (4.3), (4.5) and (4.6),

$$f(z_k^*) = f(\varphi^{-n_k}(z_k)) = \gamma^{-n_k}(f(z_k)) \to p^-.$$

Since f is normal it is not possible that $\eta \in \mathbb{D}$. Hence $\eta \in \mathbb{T} \setminus \{\zeta^+, \zeta^-\}$ and thus $\eta \in T_>$ by (4.5). It follows that $p^- \in C_>(\eta)$, which contradicts (iii). ∎

Example. Let F be the complement of

$$E := [0, +\infty] \cup \bigcup_{n \in \mathbb{Z}} [2^n i, +\infty + 2^n i]$$

and let f map $\mathbb{D}$ conformally onto F such that $f(1) = 0$ and $f(-1) = \infty$. Let $\eta_n \in \mathbb{T}$ be the preimage of $+\infty$ between the horizontal halflines $[2^n i, +\infty + 2^n i]$ and $[2^{n+1} i, +\infty + 2^{n+1} i]$. Since E is invariant under $\gamma(w) = 2w$ the same is true of F. Hence there is a Moebius transformation φ that satisfies $\varphi(\mathbb{D}) = \mathbb{D}$ and $f \circ \varphi = \gamma \circ f$. We have

$$p^+ = 0, \qquad C_>(1) = [0, +\infty], \qquad C_<(1) = \{0\},$$

so that f is continuous from the left but not from the right. Furthermore $p^- = \infty \in [0, +\infty]$ and $p^- = f(\eta_n)$ for $n \in \mathbb{Z}$.

Now we turn to the special case that f is univalent in $\mathbb{D}$. The impression $I(\zeta)$ of the prime end corresponding to $\zeta \in \mathbb{T}$ is the cluster set $C(\zeta)$ defined in (1.4), see e.g. [27, Thm. 2.16]. We say [3, p. 646] that p is a *terminal point* of $\partial f(\mathbb{D})$ if the univalent function f is continuous at $\zeta \in \mathbb{T}$ and if

$$p = f(\zeta), \qquad p \notin I(\eta) \quad \text{for } \eta \in \mathbb{T} \setminus \{\zeta\} \tag{4.7}$$

and if there are $\zeta_n^{\lessgtr} \in \mathbb{T}$ with $\zeta_n^{\lessgtr} \to \zeta$ as $n \to \infty$ such that

$$\arg \zeta_n^< < \arg \zeta < \arg \zeta_n^>, \qquad I(\zeta_n^<) \cap I(\zeta_n^>) \neq \emptyset.$$

Corollary 4.2. *Let f be univalent and Φ-polymorphic and suppose that $\widehat{\mathbb{C}} \setminus f(\mathbb{D})$ is connected. If f is continuous at both fixed points $\zeta^\pm$ of $\varphi \in \Phi$ then $p^\pm = f(\zeta^\pm)$ are terminal points of $\partial f(\mathbb{D})$.*

Proof. Since $I(\zeta^\pm) = \{f(\zeta^\pm)\}$ it follows from Theorem 4.1 that (4.7) holds with $\zeta = \zeta^\mp$, $p = f(\zeta^\mp)$. Hence the prime end $\widehat{f}(\zeta^\mp)$ with $I(\zeta^\mp) = p^\mp$ is non-intersecting. Since $\widehat{\mathbb{C}} \setminus f(\mathbb{D})$ is connected, by [3, Cor. 4] the only alternative that remains is that $p^\mp$ is a terminal point. ■

We need some definitions of continuum theory [21]. The continuum X is *unicoherent* if $A_1 \cap A_2$ is connected for any two continua $A_k \subset X$. The continuum X is *decomposable* if there exist two continua A_k such that

$$X = A_1 \cup A_2, \qquad A_k \neq X \quad k = 1, 2, \tag{4.8}$$

otherwise X is *indecomposable*.

If X is decomposable and unicoherent then there is a continuum $E \subset X$ that separates X, that is

$$X \setminus E = B_1 \cup B_2, \qquad B_1 \cap \overline{B_2} = \overline{B_1} \cap B_2 = \emptyset, \qquad B_1 \neq \emptyset,\ B_2 \neq \emptyset. \tag{4.9}$$

Indeed, it follows from (4.8) that $E := A_1 \cap A_2$ is a continuum, and $B_k := A_k \setminus E$ satisfies (4.9) because, for instance

$$B_1 \cap \overline{B_2} \subset (A_1 \setminus E) \cap \overline{A_2} = (A_1 \setminus (A_1 \cap A_2)) \cap A_2 = \emptyset.$$

On the other hand, an indecomposable continuum is not separated by any subcontinuum.

Theorem 4.3. *Let f be univalent and Φ-polymorphic and let $\Gamma = f^*(\Phi)$ be non-elementary. If $\widehat{\mathbb{C}} \setminus L(\Gamma)$ is connected and $L(\Gamma)$ is connected and decomposable, then there are infinitely many hyperbolic $\varphi \in \Phi$ such that f is continuous at both fixed points of φ.*

Proof. By assumption $L(\Gamma)$ is a continuum. Since its complement is connected, $L(\Gamma)$ is unicoherent by a theorem of plane topology [23, Thm. 11.5]. Hence there is a continuum E satisfying (4.9) with $X = L(\Gamma)$. Since Γ is non-elementary there is a loxodromic $\gamma \in \Gamma$ with fixed points $p^+ \in B_1$ and $p^- \in B_2$, see [1, Thm. 5.3.8].

We have $\gamma = f^*(\varphi)$ for some hyperbolic $\varphi \in \Phi$ with fixed points $\zeta^\pm \in \mathbb{T}$. By [4, Thm. 2] there are only three cases:

(a) $C(\zeta^+) \subset E$: This implies $p^+ = f(\zeta^+) \in E$ in contradiction to (4.9).
(b) $C(\zeta^+) \subset B_1 \cup E$, $f(\zeta^+) \in \overline{B_1}$: Since $p^- = f(\zeta^-) \in B_2$ it follows from (4.9) that $p^- \notin C(\zeta^+)$. Hence f is continuous at ζ^+ by Theorem 4.1.
(c) $f(\zeta^+) \in \overline{B_1} \cap \overline{B_2}$: Now $f(\zeta^+) \in E$ which contradicts $p^+ \in B_1$.

The same argument shows that the function f is continuous at ζ^-. Now let $\psi \in \Phi$. Then $\chi = \psi \circ \varphi \circ \psi^{-1}$ has the fixed points $\psi(\zeta^\pm)$ where f is continuous due to $f \circ \psi = f^*(\psi) \circ f$. ■

The Kleinian group Γ is called *degenerate* if $\widehat{\mathbb{C}} \setminus L(\Gamma)$ is connected and simply connected. We now assume that f is univalent and Φ-polymorphic and that $\Gamma = f^*(\Phi)$ is degenerate and $f(\mathbb{D}) = \widehat{\mathbb{C}} \setminus L(\Gamma)$.

Maham Mj has recently proved in some preprints [16, 17, 18]: If Γ is finitely generated and has no accidental parabolics then $L(\Gamma)$ is locally connected and thus f is continuous on $\mathbb{T}$.

In the opposite direction Matsuzaki [13, Cor. 18] has shown: If Γ is infinitely generated and $L(\Gamma)$ is indecomposable then f is nowhere continuous on $\mathbb{T}$.

The present Theorem 4.3 and Corollary 4.2 only say: If $L(\Gamma)$ is decomposable then there is $\varphi \in \Phi$ such that f is continuous at both fixed points $\zeta^\pm$ of φ and $f(\zeta^\pm)$ are terminal points of $L(\Gamma)$.

Acknowledgement. We wish to thank the referee for pointing out that Corollary 4.2 had not been correctly formulated.

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