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Analytic Families of Homomorphisms into $\mathrm{PSL}(2,\mathbb{C})$

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Abstract. We develop a systematic theory of families of homomorphisms

$$h_t\colon G \to \mathrm{PSL}(2,\mathbb{C}), \qquad t \in T,$$

that depend analytically on the complex parameter t. Important results have been obtained by Jørgensen, Riley and Sullivan.

The main new concept is the singular set S of parameter values t for which some Moebius transformation $h_t(x)$ becomes non-loxodromic. We study the structure and geometry of S and the behaviour of h_t in domains $V \subset T \setminus \overline{S}$, in particular the extension to values in $\partial V \cap \partial T$.

Keywords. Analytic family, homomorphism, Kleinian group, singular set, Moebius transformation, trace, non-elementary group.

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1. Introduction

Let T be a domain in $\mathbb{C}$ and let G be any countable group. A family $(h_t)_{t\in T}$ of homomorphisms

$$h_t\colon G \overset{\text{onto}}{\to} \Gamma_t \subset \mathrm{PSL}(2,\mathbb{C}), \qquad t \in T,$$

is called *analytic* if $h_t(x)$ depends analytically on t for each $x \in G$. The coefficients of $h_t(x) = (a_t z + b_t)/(c_t z + d_t)$ are only defined up to a factor ± 1 but $\mathrm{tr}^2\, h_t(x) = (a_t + d_t)^2$ is well-defined and analytic in $t \in T$, see Section 2.2.

Important examples come from polymorphic functions: Given a Fuchsian group Φ acting on the unit disk $\mathbb{D}$, a function f meromorphic in $\mathbb{D}$ is called *polymorphic* if, for every $\varphi \in \Phi$, there is $\gamma \in \mathrm{PSL}(2,\mathbb{C})$ such that $f \circ \varphi = \gamma \circ f$. This induces a homomorphism $f^*\colon \Phi \to \mathrm{PSL}(2,\mathbb{C})$ defined by $f^*(x) = \gamma$. See for instance [14, 24]. More often polymorphic functions are called *deformations* of Φ

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or developing maps of projective structures on the Riemann surface $\mathbb{D}/\Phi$, see for instance [13, 18, 7, 23].

In a classical case, there is an analytic function g in $\mathbb{D}$ that satisfies $(g\circ\varphi)\varphi'^2 = g$ for $\varphi \in \Phi$. Then the quotients f_t of two independent solutions of the differential equation

$$w'' + t\,g(z)\,w = 0, \qquad t \in \mathbb{C},$$

normalized by $f_t(0) = 0$, $f_t'(0) = 1$ and $f_t''(0) = 0$, are polymorphic functions that are furthermore meromorphic in t. Then

$$f_t^*\colon \Phi \to \mathrm{PSL}(2,\mathbb{C}), \qquad t \in \mathbb{C},$$

form an analytic family $(f_t^*)_{t\in\mathbb{C}}$ of homomorphisms.

Let $(h_t)_{t\in T}$ be an analytic family of homomorphisms. We call $s \in T$ a *critical value* if there exists $x \in G$ such that $h_s(x) = \mathrm{id}$ but $h_t(x) \not\equiv \mathrm{id}$. The following fundamental results were proved by Troels Jørgensen [15], Robert Riley [26] and Dennis Sullivan [27] in terms of isomorphisms (see Section 2.3) instead of critical values.

Theorem 1.1 (Jørgensen, Riley, Sullivan)**.** *We suppose that $\Gamma_t = h_t(G)$ is not elementary for all $t \in T$ and that $\mathrm{tr}^2\ h_t(x)$ is non-constant for some $x \in G$. If W denotes the set of $t \in T$ for which Γ_t is not discrete, then*

(i) *W is open;*
(ii) *the critical values are dense in W;*
(iii) *some punctured neighbourhood of every critical value belongs to W;*
(iv) *if G is finitely generated then Γ_t has a non-empty domain of discontinuity for every $t \in T \setminus \overline{W}$.*

Part (i) is due to Jørgensen [15, Thm. 1]. Parts (ii) and (iii) were stated by Riley [26, Thm. 1] for finitely generated subgroups of $\mathrm{SL}(2,\mathbb{C})$. His proofs are however valid in the general situation. Part (iv) is due to Sullivan [27, p. 250] who proved (ii) at the same time as Riley. Sullivan's paper contains many other interesting results.

In the present paper, we give a systematic presentation of the theory in the general context. Some results are of course known in some form or other.

In Section 3.1 we introduce the main concept of this paper, the singular set S, a small and structured subset of T that contains the critical values. Theorem 3.3 gives geometric information about part (ii) of Theorem 1.1.

The example in Section 4 suggests that $\overline{S}$ is bounded by a highly complicated set. The groups generated by $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\t&1\end{pmatrix}$ has been studied in detail by many authors, for instance [22, 16, 2, 10, 12, 11].

In Section 5 we prove some results about domains V complementary to $\overline{S}$. In particular we study the question what happens if $\partial V \cap \partial T \neq \emptyset$, which is related to [15, Thm. 1]. At the end, we state two open problems about singular sets.

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2. Analytic families and critical values

2.1. We call $(h_t)_{t\in T}$ an *analytic family of homomorphisms* of G into the group PSL(2, ℂ) of (orientation preserving) Moebius transformations if

- T is a domain in $\mathbb{C}$,
- G is a countable group,
- $h_t\colon G \to \mathrm{PSL}(2,\mathbb{C})$ is a homomorphism for each $t \in T$,
- $h_t(x)$ depends analytically on t for each $x \in G$; this means that, for every $x \in G$ and $t_0 \in T$, there are domains $U_0 \subset \mathbb{C}$ and $V_0 \subset T$ with $t_0 \in V_0$ such that $h_t(x)(z)$ is analytic in $t \in V_0$ for $z \in U_0$.

We always write $\Gamma_t = h_t(G)$. The group G serves only to index the elements γ_t of Γ_t, no further structure of G is assumed.

Conversely, let $\Gamma_t\,(t \in T)$ be a group generated by countably many elements $\alpha_{t,n} \in \mathrm{PSL}(2,\mathbb{C})$ that depend analytically on t. Let G be freely generated by corresponding symbols x_n. Then [20, Thm. 8.04] the definition $h_t(x_n) = \alpha_{t,n}$ can be extended to homomorphisms $h_t\colon G \to \Gamma_t$. Thus we obtain an analytic family of homomorphisms.

2.2. Let $(h_t)_{t\in T}$ be an analytic family of homomorphisms. We often write

$$h_t(x)(z) = \frac{a_t(x)z + b_t(x)}{c_t(x)z + d_t(x)}, \qquad a_t d_t - b_t c_t = 1. \tag{2.1}$$

The complex coefficients are only determined up to a common sign $\pm$. We often write $a_t, \ldots, d_t$ instead of $a_t(x), \ldots, d_t(x)$.

Proposition 2.1. *Let $x \in G$ and $\gamma_t = h_t(x)$. With the notation* (2.1)*, all squares ${a_1}^2, \ldots, {d_t}^2$ and all products $a_t b_t, \ldots, c_t d_t$ are well-defined analytic functions of $t \in T$. In particular $\mathrm{tr}^2\,\gamma_t = (a_t + d_t)^2$ is analytic.*

Proof. It follows from (2.1) by differentiation that

$$\begin{aligned} {a_t}^2 z^2 + 2a_t b_t z + {b_t}^2 &= \frac{\gamma_t(z)^2}{\gamma_t'(z)}, \\ {c_t}^2 z^2 + 2c_t d_t z + {d_t}^2 &= \frac{1}{\gamma_t'(z)}, \\ a_t c_t z^2 + (a_t d_t + b_t c_t) z + b_t d_t &= \frac{\gamma_t(z)}{\gamma_t'(z)}. \end{aligned}$$

The functions on the right-hand sides are analytic in t. Hence all the coefficients of z^2, z and 1 are analytic in t. Finally we use $a_t d_t = 1 + b_t c_t$. ∎

Now we study the problem whether the coefficients $a_t, \ldots, d_t$ themselves are well-defined functions of t. We say that the homomorphisms

$$H_t\colon G \to \mathrm{SL}(2,\mathbb{C}), \qquad t \in T,$$

form an analytic family if the elements of the matrices

$$H_t(x) = \begin{pmatrix} a_t(x) & b_t(x) \\ c_t(x) & d_t(x) \end{pmatrix}, \qquad x \in G,$$

are analytic in $t \in T$. Let π be the projection $H_t \to h_t$, see (2.1). We say that h_t can be lifted to H_t if $\pi(H_t(x)) = h_t(x)$ for all $x \in G$.

Proposition 2.2. *Let T be simply connected. If h_s can be lifted to H_s for one $s \in T$ then $(h_t)_{t\in T}$ can be lifted to an analytic family $(H_t)_{t\in T}$.*

Proof. First we assume that $c_s \neq 0$. Since T is simply connected there is a well-defined analytic function w_t, $t \in T$, such that ${w_t}^2 = {c_t}^2$, $w_s = c_s$, see (2.1) and Proposition 2.1. For t with $c_t \neq 0$ we define

$$\tilde{a}_t = \frac{a_t c_t}{w_t}, \qquad \tilde{b}_t = \frac{b_t c_t}{w_t}, \qquad \tilde{c}_t = \frac{{c_t}^2}{w_t}, \qquad \tilde{d}_t = \frac{c_t d_t}{w_t}. \tag{2.2}$$

These functions are analytic by Proposition 2.1. They have an analytic continuation to the zeros of c_t because ${\tilde{a}_t}^2 = {a_t}^2$ etc. is finite. Now let $c_s = 0$. Then $a_s \neq 0$ by (2.1) and there is an analytic function w_t with ${w_t}^2 = {a_t}^2$, $w_s = a_s$. In this case we obtain analytic functions

$$\tilde{a}_t = \frac{{a_t}^2}{w_t}, \qquad \tilde{b}_t = \frac{a_t b_t}{w_t}, \qquad \tilde{c}_t = \frac{a_t c_t}{w_t}, \qquad \tilde{d}_t = \frac{a_t d_t}{w_t}. \tag{2.3}$$

We define maps $H_t \colon G \to \mathrm{SL}(2,\mathbb{C})$ by

$$H_t(x) = \begin{pmatrix} \tilde{a}_t(x) & \tilde{b}_t(x) \\ \tilde{c}_t(x) & \tilde{d}_t(x) \end{pmatrix}, \qquad x \in G.$$

If we form $\pi(H_t(x))$ then the factors c_t/w_t etc cancel and we obtain $h_t(x)$.

Now we show that this defines a homomorphism of G into $\mathrm{SL}(2,\mathbb{C})$. Let $x, y \in G$. Since h_t is a homomorphism we check that

$$H_t(xy)H_t(y)^{-1}H_t(x)^{-1} = J_t = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{2.4}$$

Since $\tilde{c}_s = c_s$ or $\tilde{a}_s = a_s$, it follows from (2.2) or (2.3) that $\tilde{a}_s = a_s, \ldots, \tilde{d}_s = d_s$. Since H_s can be lifted to a homomorphism we obtain from (2.4) that $J_s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and since J_t is analytic it follows that $J_t \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. ∎

The assumption that T is simply connected cannot be omitted. To see this let $T = \mathbb{C} \setminus \{0\}$, $G = \langle x \rangle$ and $h_t(x^n)(z) = 4^n t z$. Then

$$H_t(x^n) = \begin{pmatrix} 2^n\sqrt{t} & 0 \\ 0 & 2^{-n}/\sqrt{t} \end{pmatrix}$$

is a lift for every single t but does not give a well-defined function.

2.3. Let $(h_t)_{t\in T}$ be an analytic family of homomorphisms into PSL$(2,\mathbb{C})$. We call $s \in T$ a *critical value* if there exists $x \in G$ such that

$$h_s(x) = \mathrm{id}, \quad h_t(x) \not\equiv \mathrm{id}, \qquad t \in T. \tag{2.5}$$

If we were to allow $h_t(x) \equiv \mathrm{id}$ then every value in T would be critical. Since G is countable there are only countably many critical values. Theorem 1.1 shows the importance of this concept.

Now we show how the abstract group G can be replaced by one of the groups $\Gamma_t = h_t(G)$. Let $u \in T$ be not critical. For $t \in T$ we define

$$g_{u,t}(\gamma) = h_t(x) \qquad \text{if} \qquad \gamma = h_u(x), \quad x \in G. \tag{2.6}$$

We have to show that $g_{u,t}(\gamma)$ is well-defined. Suppose that $\gamma = h_u(y)$ for another $y \in G$. Since h_t is a homomorphism it follows that $h_u(x\,y^{-1}) = \mathrm{id}$, and since u is not critical we conclude from (2.5) that $h_t(x\,y^{-1}) \equiv \mathrm{id}$ so that $h_t(x) \equiv h_t(y)$.

Proposition 2.3. *Let $u \in T$ be not critical. Then* (2.6) *defines an analytic family $(g_{u,t})_{t\in T}$ of homomorphisms*

$$g_{u,t}\colon \Gamma_u \overset{\text{onto}}{\to} \Gamma_t, \qquad t \in T, \ \textit{with } g_{u,u} = \mathrm{id}\,.$$

This is an isomorphism if and only if t is not critical.

Proof. Let $\alpha \in \Gamma_u$. Then $\alpha = h_u(x)$ for some $x \in G$ which we keep fixed. Then $g_{u,t}(\alpha) = h_t(x)$ is analytic in t. If $\beta = h_u(y) \in \Gamma_u$ then

$$g_{u,t}(\alpha \circ \beta) = h_t(xy) = h_t(x) \circ h_t(y) = g_{u,t}(\alpha) \circ g_{u,t}(\beta).$$

Hence $g_{u,t}$ is a homomorphism and it is easy to see that $g_{u,t}(\Gamma_u) = \Gamma_t$.

Suppose now that $g_{u,s}$ is not injective. Then $s \neq u$ and there is $\alpha = h_u(x) \in \Gamma_u$ with $\alpha \neq \mathrm{id}$ but $g_{u,s}(\alpha) = \mathrm{id}$. Thus we have

$$h_s(x) = g_{u,s}(\alpha) = \mathrm{id}, \qquad h_u(x) \neq \mathrm{id}$$

so that s is critical by (2.5). Conversely let s be critical. By (2.5) there exist x and t such that $h_s(x) = \mathrm{id}$ and $h_t(x) \neq \mathrm{id}$. Hence, by (2.6),

$$g_{u,s}(\alpha) = h_s(x) = \mathrm{id}, \qquad g_{u,t}(\alpha) = h_t(x) \neq \mathrm{id}$$

so that $\alpha \neq \mathrm{id}$. Thus $g_{u,s}$ is not injective. ■

As we have just shown, we can replace G by Γ_u for most $u \in T$ without changing the groups Γ_t and the critical values. The defining relation (2.5) is simply replaced by $g_{u,s}(\alpha) = \mathrm{id}$, $\alpha \neq \mathrm{id}$. However any information about G gets lost, for instance that G is a Fuchsian group in the case of polymorphic functions, see Section 1.

2.4. Elementary groups are not rich enough to allow certain constructions involving commutators $[x,y]$. We include the purely elliptic groups among the elementary groups [3, p. 84].

Proposition 2.4. *Suppose that not all groups* $\Gamma_t = h_t(G)$, $t \in T$, *are elementary and let* s *be a critical value. Then there are* $x, y \in G$ *and* $t \in T$ *such that*

$$h_s(x) = \mathrm{id}, \qquad \mathrm{tr}^2\, h_t(x) \neq 4, \tag{2.7}$$

$$\mathrm{tr}[h_t(x), h_t(y)] \neq 2. \tag{2.8}$$

Proof. By (2.5) there exists $x_0 \in G$ such that $h_s(x_0) = \mathrm{id}$ and $h_t(x_0) \not\equiv \mathrm{id}$. By assumption there is $u \in T$ such that Γ_u is not elementary. Hence [3, Thm. 5.1.3] there are $y_k \in G$, $k = 1, 2, 3$, such that the $h_u(y_k)$ do not share a fixed point. By continuity the same is true for t close to u. Since $h_t(x_0)$ is analytic and non-constant we may, by a slight change of u, assume that $h_u(x_0) \neq \mathrm{id}$. Hence $h_u(x_0)$ can share a fixed point with at most two $h_u(y_k)$. We conclude that $h_u(x_0)$ and $h_u(y_j)$ for some j have no fixed point in common, which implies [3, Thm. 4.3.5] that the commutator $x = [x_0, y_j]$ satisfies

$$\mathrm{tr}\ h_u(x) = \mathrm{tr}[h_u(x_0),\ h_u(y_j)] \neq 2. \tag{2.9}$$

Furthermore we have $h_s(x) = [h_s(x_0),\ h_s(y_j)] = \mathrm{id}$. Now we apply the above argument to x instead of x_0 and we see that (2.8) holds for some $y \in G$ and some $t \in T$. We see from (2.9) that $\mathrm{tr}\, h_v(x) \not\equiv 2$, $v \in T$. Since $h_s(x_0) = \mathrm{id}$ it follows that $\mathrm{tr}^2\, h_v(x) \not\equiv 4$. Changing t slightly we conclude that (2.7) holds. ∎

3. The singular set

3.1. Let $(h_t)_{t \in T}$ be an analytic family of homomorphisms h_t of G onto Γ_t. We define a partition $G = G' \cup G''$ by

$$G' = \{x \in G\colon h_t(x) \text{ is loxodromic for some } t \in T\}, \tag{3.1}$$

$$G'' = \{x \in G\colon h_t(x) \text{ is elliptic, parabolic or} = \mathrm{id} \text{ for all } t \in T\}. \tag{3.2}$$

We use the term loxodromic in the wider sense including hyperbolic. Since $h_t(x)$ is loxodromic if and only if $\mathrm{tr}^2\, h_t(x) \notin [0,4]$, we obtain from Proposition 2.1 that

$$x \in G'' \quad \Leftrightarrow \quad \mathrm{tr}^2\, h_t(x) \text{ is constant and } \in [0,4]. \tag{3.3}$$

For $x \in G'$ we define

$$S(x) = \{t \in T\colon\ \mathrm{tr}^2\, h_t(x) \in [0,4]\}; \tag{3.4}$$

if we were to allow $x \in G''$ then this set would be $= T$. Our most important concept is the *singular set* defined by

$$S = \bigcup_{x \in G'} S(x) \subset T. \tag{3.5}$$

Note that the partition $G = G' \cup G''$ depends on the family.

Proposition 3.1. *Let $x \in G'$ and $S(x) \neq \emptyset$. Then $S(x)$ is the union of countably many analytic arcs, and if A is a component of $S(x)$ then, for large n,*

$$\{\operatorname{tr}^2 h_t(x^n) \colon t \in A\} = [0,4]. \tag{3.6}$$

Proof. Since $x \notin G''$ and $S(x) \neq \emptyset$, it follows from (3.3) that $g(t) = \operatorname{tr}^2 h_t(x)$ is analytic and non-constant in $t \in T$. Hence $S(x) = g^{-1}([0,4])$ consists of countably many (closed, half-open or open) analytic Jordan arcs that meet only at the zeros of g'.

If μ_t is the multiplier of $h_t(x)$ then $g(t) = \mu_t + 2 + \mu_t^{-1}$. Hence μ_t covers an arc of $\mathbb{T} = \partial\mathbb{D}$ for $t \in A$. Therefore the multiplier μ_t^n of $h_t(x^n)$ covers all of $\mathbb{T}$ for large n so that (3.6) holds. ∎

Theorem 3.2. *The singular set S is the union of countably many analytic arcs and the critical values are dense in S. If Γ_t is non-elementary for some t then all critical values lie in S.*

It follows that the singular set has area measure 0 and is of first Baire category. Hence S is a subset of T that is small both in the metric and the topological sense.

Proof. The assertion that S is the countable union of analytic arcs follows at once from Proposition 3.1 and (3.5) because G is countable.

Now let $s \in S$. Then $s \in S(x)$ for some $x \in G'$ by (3.5). Let A be the component of $S(x)$ containing s. We choose n so large that (3.6) holds. Since $\operatorname{tr}^2 h_s(x) \in [0,4]$ there are relatively prime $l_k, m_k \in \mathbb{N}$ such that

$$4\cos^2\left(\frac{2\pi l_k}{m_k}\right) \to \operatorname{tr}^2 h_s(x), \qquad k \to \infty. \tag{3.7}$$

By (3.6) there are $t_k \in A$ such that,

$$\operatorname{tr}^2 h_{t_k}(x^n) = 4\cos^2\left(\frac{2\pi l_k}{m_k}\right).$$

Hence $h_{t_k}(x)$ is elliptic of order $n\,m_k$ and thus $h_t(x^{n\,m_k}) = \mathrm{id}$. Furthermore we have $h_{t_k}(x^{n\,m_k}) \not\equiv \mathrm{id}$ because $x \in G'$, see (3.3). It follows that the t_k are critical values. By (3.7) we may assume that $t_k \to s$ as $k \to \infty$. Hence the critical values are dense in S.

Finally let s be a critical value and assume that Γ_t is non-elementary for some t. By (2.7) in Proposition 2.4, there exists x such that $h_s(x) = \mathrm{id}$ and $x \in G'$, see (3.3). It follows that $x \in S(x) \subset S$. ∎

The assumption that not all groups Γ_t are elementary cannot be omitted as the following example shows. Let Γ_t be generated by $\alpha_t(z) = e^{2t}z$ and $\beta_t(z) = z+t-1$ for $t \in \mathbb{C}$. These groups are all elementary. Since $\operatorname{tr}^2\gamma_t = (e^{nt}+e^{-nt})^2$ with $n \in \mathbb{Z}$ for all $\gamma_t \in \Gamma_t$ we see that $S = \{\operatorname{Re} t = 0\}$. But $\beta_1 = \mathrm{id}$ whereas $\beta_t \neq \mathrm{id}$ for $t \neq 1$. Hence 1 is a critical value not in S.

3.2. Now we turn to the form of S near a critical value. Our result is closely connected to parts (ii) and (iii) of Theorem 1.1.

Theorem 3.3. *Suppose that $\Gamma_t = h_t(G)$ is non-elementary for some $t \in T$ and let s be a critical value. Then there exists $r > 0$ such that, for infinitely many n, there are Jordan arcs $C_{n,\nu} \subset S$, $\nu = 1, \ldots, n$, of the form*

$$C_{n,\nu}\colon t = s + \rho \exp(i\vartheta_{n,\nu}(\rho)), \qquad 0 \le \rho \le r \tag{3.8}$$

with $0 < \vartheta_{n,\nu}(\rho) - \vartheta_{n,\nu-1}(\rho) < 8/n$ where $\vartheta_{n,0} = \vartheta_{n,n} - 2\pi$.

Thus there is a dense star of arcs in S around each critical values. Hence every critical value is surrounded by a disk in $\overline{S}$. This was proved by R. Riley [26, Thm. 1] in a somewhat more special situation. Therefore Theorem 3.3. implies the following corollary.

Corollary 3.4 (Riley)**.** *If at least one group Γ_t is non-elementary then the critical values form a countable dense subset of the interior of the closure $\overline{S}$ of the singular set.*

Proof of Theorem 3.3. We use the same general method as Riley. But we stress the connection with the iteration of polynomials as Gehring and Martin [9] do in a different context.

Let s be critical. We determine $x \in G'$ and $y \in G$ as in Proposition 2.4 and define

$$y_0 = y, \quad y_{n+1} = [x, y_n] \qquad \text{for } n \in \mathbb{N}_0, \tag{3.9}$$

$$\lambda(t) = 4 - \operatorname{tr}^2 h_t(x), \qquad w_n(t) = \operatorname{tr} h_t(y_n) - 2. \tag{3.10}$$

By (2.7) and (2.8) there is a disk D_1 around s such that

$$\lambda(t) \neq 0, \quad w_1(t) = \operatorname{tr}[h_t(x), h_t(y)] - 2 \neq 0 \qquad \text{for } t \in D_1,\ t \neq s. \tag{3.11}$$

Since $h_s(x) = \mathrm{id}$ it follows from (3.9) and (3.10) that $\lambda(s) = w_1(s) = 0$.

Now we use the identity

$$\operatorname{tr}[\alpha, [\alpha, \beta]] = (\operatorname{tr}[\alpha, \beta] - 2)\left(2 - \operatorname{tr}^2 \alpha + \operatorname{tr}[\alpha, \beta]\right).$$

From (3.9) and (3.10) for n and for $n - 1$, we obtain

$$w_{n+1}(t) = \lambda(t) w_n(t) + w_n(t)^2 \qquad \text{for } n \ge 1. \tag{3.12}$$

With $p_t(w) = \lambda(t) w + w^2$ we can write (3.12) as the iteration sequence

$$w_{n+1}(t) = p_t(w_n(t)).$$

The Koenigs function f_t of the polynomial p_t is analytic near 0 and satisfies

$$f_t \circ p_t = \lambda(t) f_t, \qquad f_t(0) = 0, \qquad f_t'(0) = 1.$$

See [4, Thm. 6.3.3]; the proof shows that f_t depends analytically on t near 0. Hence there are disks D^* around 0 and $D_2 \subset D_1$ around s such that f_t is analytic and injective in D^* and depends analytically on $t \in D_2$, furthermore $w_1(D_2) \subset D^*$. Induction shows that

$$f_t(w_n(t)) = \lambda(t)^{n-1} f_t(w_1(t)). \tag{3.13}$$

We have $\lambda(s) = 0$. Let k be the order of the zero at s. It follows from (3.11) that w_1 has a zero of order $k + l$ with $l \geq 0$ at s. There is a disk $D_3 \subset D_2$ around s such that $z = \lambda(t)^{1/k}$ is analytic and injective in D_3. Let $t = \varphi(z)$ be the inverse function. By (3.13) we can write

$$f_{\varphi(z)}(w_n(\varphi(z))) = z^{kn+l} h(\varphi(z)), h(t) = \frac{f_t(w_1(t))}{\lambda(t)^{1+l/k}}.$$

This function h is analytic and $\neq 0$ in a disk $D_4 \subset D_3$. Hence there are analytic Jordan arcs $C^*_{n,\nu}$, $\nu = 1, \ldots, m = kn + l$, of the form (3.8) such that, for $\varphi(z) \in D_5 \subset D_4$,

$$f_{\varphi(z)}(w_n(\varphi(z))) \in f_{\varphi(z)}((-\infty, 0] \cap D^*) \qquad \text{for } z \in D^*_{n,\nu}$$

and $|\vartheta^*_{n,\nu} - \vartheta^*_{n,0} - 2\pi\nu/m| < 1/m$. The arcs $C_{n,\nu} = \varphi(C^*_{n,\nu})$ then satisfy the conditions of Theorem 3.3 where r is the radius of D_5. Furthermore we have $w_n(t) \in (-\infty, 0]$. Since $w_n(t)$ lies in the Fatou set of p_t which in turn lies in $\{|w| < 4\}$, we conclude from (3.10) that

$$\operatorname{tr} h_t(y_n) = w_n(t) + 2 \in (-2, 2].$$

This proves Theorem 3.3 with $m = kn + l$ instead of n. ∎

4. An example

Let Γ_t be generated by the hyperbolic transformation

$$\alpha(z) = \frac{5z + 4}{4z + 5}$$

with fixpoints ± 1 and the parabolic transformation $\beta_t(z) = z + t$ ($t \in \mathbb{C}$). If G is the free group $\langle x, y \rangle$ then $h_t(x) = \alpha$ and $h_t(y) = \beta_t$ generate an analytic homomorphism of G onto Γ_t. The domain outside the disks

$$\left\{ \left| z \pm \frac{5}{4} \right| \leq \frac{3}{4} \right\}$$

is a fundamental domain of $\langle \alpha \rangle$. Hence it follows from the Klein combination theorem [21, p. 139] that Γ_t is discrete if these disks lie inside

$$\left\{ -\frac{1}{2} |t| \leq \operatorname{Re}\left(\frac{|t|}{t} z \right) \leq \frac{1}{2} |t| \right\},$$

this is, if t lies outside the curve

$$B\colon \left(\frac{3}{2} + \frac{5}{2} |\cos \vartheta| \right) e^{i\vartheta}, \qquad -\pi \leq \vartheta \leq \pi.$$

But Γ_t is discrete in a much larger domain [12, Sec. 8].

Figure 1 shows the singular set S of $(h_t)_{t\in\mathbb{C}}$. As for all digital representations of continuous objects, the fine details depend on design decisions. The singular set is symmetric with respect to $\mathbb{R}$ and $i\mathbb{R}$ but this fact has not been used in the computation in order to have a check. For each pixel 600 random words of G of length up to 10000 were checked. The marks on the frame of the picture are 0.5 units apart.

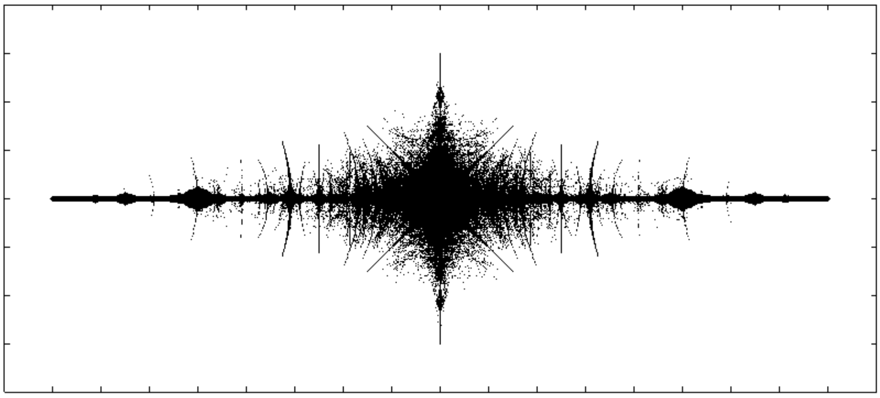

FIGURE 1. Singular set S of $(h_t)_{t\in\mathbb{C}}$.

Next we determine $S(w)$ for a few words w in G and indicate where $S(w)$ is in the figure.

(a) $w = xy$: $\operatorname{tr} h_t(w) = \dfrac{10}{3} + \dfrac{4}{3}t$,

$S(w) = [-4, -1]$ is part of the horizontal line.

(b) $w = [xyx, y]$: $\operatorname{tr} h_t(w) = 2 + \left(\dfrac{10}{9}t + \dfrac{16}{9}t^2\right)^2$,

$S(w)$ lies on the hyperbola $\operatorname{Re}\left(\frac{10}{9}t + \frac{4}{9}t^2\right) = 0$, the arc through $-\frac{5}{2}$ can be seen in the picture.

(c) $w = [x, y]$: $\operatorname{tr} h_t(w) = 2 + \left(\dfrac{4}{3}t\right)^2$,

$S(w) = \left[-\frac{3}{2}i, \frac{3}{2}i\right]$, the vertical line.

(d) $w = [[x, y], y]$: $\operatorname{tr} h_t(w) = 2 + \left(\dfrac{4}{3}t\right)^4$,

$S(w)$ is the cross in the center of Fig. 1 $\left[-\frac{3}{4} - \frac{3}{4}i, \frac{3}{4} + \frac{3}{4}i\right] \cup \left[-\frac{3}{4} + \frac{3}{4}i, \frac{3}{4} - \frac{3}{4}i\right]$.

Now we use Theorem 1.1 of Jørgensen, Riley and Sullivan. Since Γ_t is discrete for t outside the curve B, it follows from part (i) that Γ_t is discrete for t in the closed exterior of B. The case #6 in [17, Table 2] shows that Γ_t is also discrete for

$$t = \frac{3}{2} i \cos\Big(\frac{\pi}{m}\Big), \qquad m = 3, 4, \ldots,$$

these values lie in S by (c).

It follows from parts (ii) and (iii) of Theorem 1.1 that $\overline{S}$ lies in the closed interior of B. By (a) and (c) we have ± 4, $\pm 3i/2 \in S \cap B$. The figure suggests that the $S(w)$ with short words define the large scale form of S which is then filled out by the $S(w)$ with long words. It also suggests that $\overline{S}$ is bounded by a very complicated curve.

5. Domains without critical values

5.1. Let $(h_t)_{t \in T}$ be an analytic family of homomorphisms of G into $\mathrm{PSL}(2, \mathbb{C})$ and let S be its singular set defined in (3.5). If $x \in G$ and $h_t(x)$ is loxodromic, we denote its multiplier by $\mu_t(x)$ where $0 < |\mu_t(x)| < 1$, furthermore its attractive fixed point by $p_t^+(x)$, and its repulsive fixed point by $p_t^-(x)$. If $p_t^\pm(x) \neq \infty$ then

$$\frac{h_t(x)(z) - p_t^+(x)}{h_t(x)(z) - p_t^-(x)} = \mu_t(x) \, \frac{z - p_t^+(x)}{z - p_t^-(x)}. \tag{5.1}$$

Proposition 5.1. *Let V be a domain with $V \subset T \setminus S$ and let $x \in G'$. Then $h_t(x)$ is loxodromic for all $t \in V$, and $\mu_t(x)$ is analytic in V and $p_t^\pm(x)$ is meromorphic (possibly $\equiv \infty$) in V. If one of the equations*

$$p_t^+(x) = p_t^+(y), \quad p_t^+(x) = p_t^-(y), \quad p_t^-(x) = p_t^+(y), \quad p_t^-(x) = p_t^-(y) \tag{5.2}$$

holds for one $t \in V$ then it holds identically in V and $[x, y] \in G''$.

Proof. We write $\gamma_t = h_t(x)$ and use the notation (2.1). Note that $a_t, \ldots, d_t$ are defined only up to a common sign $\pm$. Since $x \in G'$ and $V \cap S = \emptyset$ it follows from (3.4) that $\mathrm{tr}^2 \alpha_t \notin [0, 4]$ so that α_t is loxodromic. Let e_t be the solution of

$$e_t + e_t^{-1} = a_t + d_t \in \mathbb{C} \setminus [-2, 2], \qquad t \in V, \tag{5.3}$$

with $|e_t| < 1$. Now $f(w) = w + w^{-1}$ is a conformal map of $\mathbb{D} \setminus \{0\}$ onto $\mathbb{C} \setminus [-2, 2]$. It follows from (5.3) that $e_t = f^{-1}(a_t + d_t)$, and since f is an odd function we obtain from Proposition 2.1 that $(a_t + d_t) e_t$ is well-defined and analytic in V.

If $c_t \equiv 0$ then $\gamma_t(z) = {a_t}^2 z + a_t b_t$ where $|a_t| \neq 1$. Say that $|a_t| < 1$. Then $p_t^+ = a_t b_t / (1 - {a_t}^2)$ and $p_t^- = \infty$. Both are well-defined meromorphic functions. Now let $c_t \not\equiv 0$. If $c_t \neq 0$ then we obtain from (5.3) that $p_t = \big(a_t - e_t^{\pm 1}\big)/c_t$ where p_t stands for $p_t^\pm$ or $p_t^\mp$. Since $a_t + d_t \neq 0$ by (5.3), we can write

$$p_t = \frac{(a_t + d_t) a_t - (a_t + d_t) e_t^{\pm 1}}{(a_t + d_t) c_t}$$

and Proposition 2.1 shows that p_t is well-defined and meromorphic with poles in the zeros of c_t. Now p_t stands for $p_t^\pm$ or $p_t^\mp$. Since γ_t is always loxodromic and V is connected it follows that we have the same case for all $t \in V$. Since $\mu_t + \mu_t^{-1} = \operatorname{tr}^2 \gamma_t - 2 \notin [-2, 2]$ we see that $\mu_t = f^{-1}(\operatorname{tr}^2 \gamma_t - 2)$ is analytic. ■

Now suppose that one of the equations (5.2) holds for $t = t_0$. Then [3, Thm. 4.3.5] we have

$$\operatorname{tr} h_t([x, y]) = \operatorname{tr}[h_t(x), h_t(y)] = 2 \tag{5.4}$$

for $t = t_0$. Since $t_0 \notin S$ we conclude from (3.4) that $[x, y] \in G'' = G \setminus G'$. Then (5.4) holds for all $t \in T$ by (3.3) so that [3, Thm. 4.3.5] $h_t(x)$ and $h_t(y)$ share a fixed point which is always attractive or repulsive by continuity.

Corollary 5.2. *Let V be a domain with $V \subset T \setminus S$. Given any $y \in G'$ there exists $\lambda_t \in \mathrm{PSL}(2, \mathbb{C})$ depending analytically on t such that*

$$g_t = \lambda_t \circ h_t \circ \lambda_t^{-1} \colon G \to \mathrm{PSL}(2, \mathbb{C}), \qquad t \in V,$$

satisfies $g_t(y)(z) = \mu_t z$ for $t \in V$.

This is an immediate consequence of Proposition 5.1 because

$$\lambda_t(z) = \frac{z - p^+(y)}{z - p^-(y)}, \qquad t \in V,$$

depends analytically on t. The corollary shows that, restricting t to a domain without critical values, we may assume that a given loxodromic element $h_t(y)$ is always normalized.

Proposition 5.3. *If $t \in T \setminus S$ then*

$$\operatorname{dist}(t,\, S \cup \partial T) \le \inf_{x \in G'} 2 \left| \frac{\mu_t(x)}{\dot{\mu}_t(x)} \right| \log \frac{1}{|\mu_t(x)|} \tag{5.5}$$

where $\dot{\mu}_t$ is the derivative with respect to t.

Proof. Let $r = \operatorname{dist}(t, S \cup \partial T)$ and let $x \in G'$. The function

$$f(z) = -\log \mu_{t+rz}(x), \qquad z \in \mathbb{D}, \tag{5.6}$$

is analytic because $V = \{t + rz \colon z \in \mathbb{D}\} \subset T \setminus S$ and satisfies $\operatorname{Re} f(z) > 0$ because $|\mu_t| < 1$. These two properties imply $|f'(0)| \le 2 \operatorname{Re} f(0)$ and it follows from (5.6) that

$$r \left| \frac{\dot{\mu}_t(x)}{\mu_t(x)} \right| \le 2 \log |\mu_t(x)|^{-1}$$

■

5.2. Now we consider the problem of the extension of h_t to points of $\partial V \cap \partial T$ where V is a simply connected domain in $T \setminus S$. The formulation is complicated because we do not want to make strong assumptions about ∂V.

We say that C is an arc ending at $s \in \partial V$ if C is a Jordan arc in $V \cup \{s\}$ with endpoint s. If an arc ends at $s \in \partial V$ then s is called an *accessible boundary point*. Let A denote the set of accessible boundary points of ∂V. We say that $s \in \partial V$ is *cut point* if $\partial V \setminus \{s\}$ is not connected; every cut point is accessible by the Plane Separation Theorem [25, Thm. 2.10].

Now let φ be a conformal map of $\mathbb{D}$ onto V. If the arc C ends at $s \in \partial V$ then $\varphi^{-1}(C)$ ends at a point $\zeta \in \mathbb{T} = \partial\mathbb{D}$ and the radial limit $\varphi(\zeta)$ exists [25, Thm. 2.8]. By definition the set $E \subset A$ has *harmonic measure* 0 if $\varphi^{-1}(E)$ has measure 0 on $\mathbb{T}$, see [1] [8, p. 39]. The set $\partial V \setminus A$ has harmonic measure 0.

Theorem 5.4. *Let V be a simply connected domain in $T \setminus S$ such that ∂V has no cut points and let A be the set of accessible boundary points. We assume that, for every $s \in A$, there are $x_1, x_2 \in G'$ with $[x_1, x_2] \in G'$ such that*

$$\lim_{t\to s,\, t\in V} h_t(x_j) \quad \textit{exists} \ \in \mathrm{PSL}(2,\mathbb{C}) \setminus \{\mathrm{id}\}, \qquad j = 1, 2. \tag{5.7}$$

Then there is a set $E \subset A$ of harmonic measure 0 such that, for every $s \in A \setminus E$ and $x \in G$,

$$h_s(x) \colon = \lim_{t\to s,\, t\in C} h_t(x) \quad \textit{exists} \ \in \mathrm{PSL}(2,\mathbb{C}) \tag{5.8}$$

for some arc C ending at s, and this limit is the same for all arcs C ending at s for which it exists.

Proof. (a) Let $s \in A$ and let C be an arc ending at s. When we talk about limits we mean finite limits as $t \to s$ along C. We assume that

$$\lim_{t\to s,\, t\in C} \mathrm{tr}^2\, h_t(x) \quad \text{exists for all } x \in G \tag{5.9}$$

and that there are $x_1, x_2 \in G'$ with $[x_1, x_2] \in G'$ satisfying (5.7). Since $V \cap S = \emptyset$ we may, by Corollary 5.2, assume that $h_t(x_1)$ has the matrix $\left(\begin{smallmatrix} q_t & 0 \\ 0 & 1/q_t \end{smallmatrix}\right)$. Let $a_t^*, \ldots$ be the coefficients of $h_t(x_2)$ and $a_t, \ldots$ those of $h_t(x)$; we choose all signs in a continuous manner on C. Then

$$\mathrm{tr}\, h_t(x) = a_t + d_t, \qquad \mathrm{tr}\, h_t(x_1\, x) = q_t a_t + q_t^{-1} d_t, \tag{5.10}$$

$$\mathrm{tr}\, h_t(x_2\, x) = a_t^* a_t + d_t^* d_t + e_t, \qquad e_t = c_t^* b_t + b_t^* c_t. \tag{5.11}$$

We obtain from (5.7) that $a_t^*, \ldots, d_t^*$ and q_t have limits with $\lim q_t \neq \pm 1$. Hence (5.9) and (5.10) show that

$$a_t = \frac{q_t\, \mathrm{tr}\, h_t(x_1 x) - \mathrm{tr}\, h_t(x)}{{q_t}^2 - 1}$$

has a limit, similarly d_t. Furthermore

$$\operatorname{tr} h_t([x_1, x_2]) = 2 - (q_t - q^{-1})^2 b_t^* c_t^*$$

has a limit $\neq 2$ because $[x, x_2] \in G'$. We conclude that $b_t^* c_t^*$ has a limit $\neq 0$. Now it follows from (5.11) that e_t has a limit and therefore also

$${c_t}^2 - \frac{e_t}{b_t^*} c_t = \frac{c_t^*}{b_t^*}(a_t d_t - 1).$$

Hence the limit set of c_t as $t \to s$, $t \in C$ contains at most two points, and since it is connected we conclude that c_t has a limit, similarly b_t. Thus we have shown that $h_t(x)$ has a limit in $\mathrm{PSL}(2, \mathbb{C})$ as $t \to s$, $t \in C$ for every $x \in G$. This limit depends only on the value in (5.9) and the limits of $a_t^*, \ldots, d_t^*$.

(b) Now we consider a conformal map φ of $\mathbb{D}$ onto V. Then the radial limit $\varphi(\zeta)$ exists for almost all $\zeta \in \mathbb{T}$ [25, Thm. 2.9]. Let $x \in G$. The function

$$f_z(x) = \operatorname{tr}^2 h_{\varphi(z)}(x), \qquad z \in \mathbb{D}, \tag{5.12}$$

is analytic. If $x \in G'$ then $f_z(x) \notin [0, 4]$ for $z \in \mathbb{D}$ because $V \cap S = \emptyset$, if $x \in G''$ then $f_z(x)$ is constant by (3.3). Hence $f_z(x)$ is of bounded characteristic [6, p. 16] so that the radial limit $f_\zeta(x)$ exists for almost all $\zeta \in \mathbb{T}$. Since the group G is countable we conclude that there is a set $F \subset \mathbb{T}$ of measure 0 such that all functions $\varphi(z)$ and $f_z(x)$ with $x \in G$ have radial limits on $\mathbb{T} \setminus F$. The set $E = \varphi(F)$ of radial limits has, by definition, harmonic measure 0.

First we apply part (a) for $s = \varphi(\zeta)$, $\zeta \in \mathbb{T} \setminus F$ to the arc $C = \varphi([0, \zeta])$. Then $s \in A$ and (5.9) is satisfied by the construction of F. We conclude that (5.8) holds for this arc C ending at s. We define $h_s(x)$ as this limit and define $f_\zeta(x)$ by (5.12).

Now suppose that the limit in (5.8) exists for another arc C ending at s. Then $\varphi^{-1}(C)$ ends at some point $\zeta' \in \mathbb{T}$. Since ∂V has no cut points if follows [25, Thm. 2.10] that $\zeta = \zeta'$. We apply part (a) to this arc C. Then $f_z(x)$ has a limit as $z \to \zeta$, $z \in \varphi^{-1}(C)$. Since $f_z(x)$ ($z \in \mathbb{D}$) is a normal function it follows [19] that

$$\operatorname{tr}^2 h_t(x) = f_z(x) \to f_\zeta(x) \qquad \text{as } z \to \zeta,\ t \to s,\ t \in C,$$

see (5.12). By (5.7) the limit for $h_t(x_j)$ is independent of the choice of the arc C ending at s. Hence the last statement of part (a) shows that $h_t(x) \to h_s(x)$ as $t \to s$, $t \in C$. ■

The assumption (5.7) serves to stabilize $h_t(G)$ against wild conjugations and cannot be omitted as the following rather general example shows. Suppose that $\overline{\mathbb{D}} \subset T \setminus S$ and let $(h_t)_{t \in T}$ be any analytic family of homomorphisms. We consider the lacunary series

$$\psi(t) = \sum_{k=0}^{\infty} t^{2^k}, \qquad t \in \mathbb{D},$$

and define $f_t(z) = z + \psi(t)$. Then $h_t^* = f_t \circ h_t \circ f_t^{-1}$ $(t \in \mathbb{D})$ is an analytic family. Let $x \in G$ and $s \in \mathbb{T}$. If $c_s \neq 0$ then $a_t^* = a_t - c_t\,\psi(t)$ has no finite limit along any arc C ending at s [5]. Hence $(h_t^*)_{t\in\mathbb{D}}$ cannot be extended to s.

Under the assumptions of Theorem 5.4 the family $(h_t)_{t\in T}$ can be extended to $(h_t)_{t\in T\cup(\partial V\setminus E)}$ where E has harmonic measure 0 relative to V. The extension is continuous in the restricted sense of the theorem. It easily follows that (5.8) that h_t is a homomorphism also for $t \in \partial V \setminus E$.

Now we suppose that $\Gamma_u = h_u(G)$ is not elementary for some $u \in T$. We consider the homomorphisms

$$f_{u,t}\colon \Gamma_u \overset{\text{onto}}{\to} \Gamma_t, \qquad t \in V, \qquad g_{u,u} = \mathrm{id}$$

of Proposition 2.3, which are isomorphisms because $t \in V$ is not critical. The groups Γ_t are discrete by Theorem 1.1. Since $h_t(x) \to h_s(x) \in \mathrm{PSL}(2,\mathbb{C})$ as $t \to s,\ t \in C$, it follows from the theorem of T. Jørgensen [15, Thm. 1] that Γ_s is discrete also for $s \in \partial V \setminus E$ and furthermore that $g_{u,s}$ is an isomorphism.

5.3. The sets arising in the theory of iteration of rational functions have a very complicated structure [4]. Note that the closure of our singular set S corresponds to the Mandelbrot set whereas the limit sets of Γ_t correspond to the Julia sets.

Problem 1. Is it generically true that the boundary of $\overline{S}$ is non-rectifiable or even of Hausdorff dimension > 1 ?

This is suggested by the figure in Section 4, which also motivated the formulation of Theorem 5.4 in terms of accessible points. A computer picture of David Wright [12, Fig. 6] however suggests that $\partial\overline{S}$ is a nice curve for the example mentioned in the introduction. The exclusion of cut points leads to the following question.

Problem 2. Can $\overline{S}$ have cut points?

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