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Polynomials on the Cauchy circle

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Summary. Let P be a complex polynomial of degree n with $P(0) = 1$ and Cauchy radius 1 about the origin. We discuss the order of magnitude of the minimal number $N = N(\epsilon_n, n)$ such that

$$\min_{1\leq k\leq N} |P(\mathrm{e}^{2\pi \mathrm{i}k/N})| \leq 1 - \epsilon_n .$$

Previous estimates of $N = O(n^{3/2})$ are improved to $N = O(n \log n)$. Some other related properties of these polynomials are also exhibited.

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1. Introduction and statement of the results

For $n \in \mathbb{N}$ let $\mathscr{P}_n$ denote the set of complex polynomials of degree $\leq n$. For $P(z) = \sum_{k=0}^{n} a_k z^k \in \mathscr{P}_n$, non-constant with $a_0 \neq 0$, the positive solution ρ_0 of the equation

$$|a_0| = \sum_{k=1}^{n} |a_k| \rho^k$$

is called the *Cauchy radius* of P at $z = 0$. For the problems to be discussed in this note we can deal with the following normalized situation: let $\mathscr{P}_n^{\mathrm{c}}$ consist of the polynomials

$$P(z) = 1 + \sum_{k=1}^{n} a_k z^k, \qquad \sum_{k=1}^{n} |a_k| = 1,$$

i.e. $P(0) = 1$ and with Cauchy radius 1. Clearly, if $P \in \mathscr{P}_n^{\mathrm{c}}$ then $P \neq 0$ in the unit disk $\mathbb{D} := \{z : |z| < 1\}$, and hence

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$$S_P \;:=\; \min_{|z|=1} |P(z)| \;<\; 1.$$

In fact, in [5] it was shown that

$$S_P \le \sqrt{1-\frac{1}{n}}, \tag{1}$$

and the polynomials

$$F_n(z) \;:=\; 1+\frac{2}{n(n+1)}\sum_{k=1}^{n}(n-k+1)\,z^k \;\in \mathscr{P}_n^{\mathrm{c}}, \quad n\in\mathbb{N},$$

which satisfy $S_{F_n} \ge 1-\frac{1}{n}$ show that this is of the right order (for a detailed discussion of the still open problem to determine

$$\Gamma_n \;:=\; \sup_{P\in\mathscr{P}_n^{\mathrm{c}}} \; S_P$$

compare [7, 8]). The study of polynomials on their Cauchy circles is of importance in the context of numerical rootfinders for algebraic polynomials using the *methods of descent* (compare Henrici’s book [3, vol 1., †6.14] for the basics of this method, and Kellenberger [4], Ruscheweyh [5] for specific cases). The complexity of these rootfinders depend very much on good upper estimates for the following numbers $N = N(\epsilon_n, n)$, for $\epsilon_n \ge 0$:

$$N(\epsilon_n, n) \;:=\; \min\left\{\nu\in\mathbb{N} \,:\, \forall P\in\mathscr{P}_n^{\mathrm{c}} \;\; \exists k\in\mathbb{N} \quad \text{s.t.} \quad |P(\mathrm{e}^{2\pi \mathrm{i}k/\nu})| \;\le\; 1-\epsilon_n\right\}.$$

In [5] we obtained

$$N\left(\frac{\lambda}{2n}, n\right) \le \pi n\sqrt{\frac{n+1}{2-2\lambda}}, \quad 0\le\lambda\le 1,$$

which was improved to

$$N\left(\frac{\lambda}{2n}, n\right) \le \pi(n+1)\sqrt{\frac{n+1}{6-6\lambda}}, \quad 0\le\lambda\le 1. \tag{2}$$

in [6]. On the other hand, the examples $P(z) = 1 + z^n$ show that

$$N(\epsilon_n, n) \ge 3n+1, \quad \epsilon_n > 0,\; n\in\mathbb{N}. \tag{3}$$

It has been conjectured [5] that equality holds in (3), at least for ϵ_n small enough. This is also consistent with the so-called Brickman conjecture, compare [5, 2]. So far, however, we cannot even show that $N(\epsilon_n, n)$ behaves linearly in n for ϵ_n small. In the present note we tackle this problem from two different sides. Our main result is

Theorem 1. *For $n\in\mathbb{N}$ we have*

$$N\left(\frac{1}{16n}, n\right) \;\le\; n\,\max\{9.47,\, \log(n) + \log(\log(n)) + 4\}. \tag{4}$$

Table 1. Best available upper bounds for $N(\frac{1}{16n}, n)$, $n = 2, \ldots, 199$

n	0	1	2	3	4	5	6	7	8	9
0	–	–	8	11	16	21	26	32	38	44
10	51	57	65	73	80	88	97	105	114	123
20	132	142	152	162	172	182	193	204	215	226
30	234	243	252	261	269	278	287	296	305	313
40	322	331	340	349	358	367	376	385	394	403
50	412	422	431	440	449	458	467	477	486	495
60	504	514	523	532	542	551	560	570	579	588
70	598	607	617	626	636	645	655	664	674	683
80	693	702	712	721	731	740	750	760	769	779
90	788	798	808	817	827	837	846	856	866	876
100	885	895	905	915	924	934	944	954	963	973
110	983	993	1003	1013	1022	1032	1042	1052	1062	1072
120	1082	1092	1102	1111	1121	1131	1141	1151	1161	1171
130	1181	1191	1201	1211	1221	1231	1241	1251	1261	1271
140	1281	1291	1301	1311	1321	1331	1341	1352	1362	1372
150	1382	1392	1402	1412	1422	1432	1442	1453	1463	1473
160	1483	1493	1503	1514	1524	1534	1544	1554	1564	1575
170	1585	1595	1605	1616	1626	1636	1646	1657	1667	1677
180	1687	1698	1708	1718	1728	1739	1749	1759	1770	1780
190	1790	1800	1811	1821	1831	1842	1852	1862	1873	1883

Note that for large n (4) is much better than (2). In fact, the estimate (4) can be considerably improved for every single (small) n. For instance, the number $N(\frac{1}{1600}, 100)$ is bounded by 1392 according to (2), by 1014 according to (4), but by 885 if we replace the asymptotic expression (4) with the precise result for $n = 100$ in the method described in the proof of Theorem 1. Since for practical purposes (rootfinders) only polynomials of modest degree (less than 200, say) are of interest, we supply in Table 1 the best upper bounds for $N(\frac{1}{16n}, n)$, $n \leq 199$, so far available. Note that the choice of $\epsilon_n = \frac{1}{16n}$ is of no particular importance in this context, and made only for practical reasons in the proof of Theorem 1.

The methods of descent are aiming at the zeros of the polynomials in $\mathscr{P}_n^{\mathrm{c}}$, and they do so by stepping from one point (the origin) to another one (on the Cauchy circle), where the polynomial has a smaller absolute value (and then repeating the procedure after re-normalization). It is therefore of interest to obtain more information on the distance of the Cauchy circle to the next zero of the polynomial, and of what descent is possible/can be expected. The following estimate is known, and easily established: if z_0 is the zero of least modulus of $P \in \mathscr{P}_n^{\mathrm{c}}$, then we have the sharp bounds

$$1 \leq |z_0| \leq \frac{1}{2^{1/n} - 1} \approx \frac{n}{\log 2}. \tag{5}$$

Theorem 2 says that this modulus is much smaller, if the polynomial takes ”large” values in sufficiently many roots of unity. We could phrase this as ”if the descent is bad then the zero is close”.

Theorem 2. *For $n, N \in \mathbb{N}$ let $Z(N, n)$ denote the smallest number with the following property: if $P \in \mathscr{P}_n^{\mathrm{c}}$ satisfies*

$$\left|P(\mathrm{e}^{\frac{2\pi \mathrm{i}k}{N}})\right| \geq 1, \quad k = 1, \ldots, N, \tag{6}$$

then P has at least one zero z_0 with $|z_0| \le Z(N,n)$.

For $N, n \in \mathbb{N}$ let Z be the solution of

$$8n \frac{\frac{N\,Z}{N+n}}{\frac{N\,Z}{N+n} - 1} \frac{\left(1 + \frac{N}{n}\right)^n}{\left(\frac{N\,Z}{N+n}\right)^N - 1} = 1 \tag{7}$$

satisfying $Z > 1 + \frac{n}{N}$. Then $Z(N,n) \le Z$.

In particular, we have

$$\begin{aligned} Z(n+1,n) &\le 7, \\ Z(2n+1,n) &\le 5, \\ Z(3n+1,n) &\le 3, \\ Z(6n+1,n) &\le 2. \end{aligned}$$

Note, however, that the truth of the above mentioned conjecture would rule out (6) for $N \ge 3n+1$.

As pointed out above, the minimum of $|P|$ on the Cauchy circle can be as large as $1 - \frac{1}{n}$ if $P \in \mathscr{P}_n^c$. Therefore, the theoretical descent in one step of these methods of descent can be very marginal. It may be therefore of interest that in a circle of 3 times the Cauchy radius we can expect a point where $|P|$ assumes a much smaller value:

Theorem 3. *Let $P \in \mathscr{P}_n^c$. Then there exists a point ζ with $|\zeta| \le 3$ such that $|P(\zeta)| \le \frac{2}{3}$.*

It is not clear, however, how one could possibly use this information for the numerical methods of descent.

2. Proofs

Our proofs are using an idea due to Kellenberger [4], which we briefly recall: let $P(z) = \sum_{k=0}^n a_k\, z^k \in \mathscr{P}_n^c$, and define

$$h(z) := \frac{1}{P(z)} = \sum_{j=0}^{\infty} d_j\, z^j .$$

From the recursive equations

$$d_k = \begin{cases} 1, & k = 0, \\ -\sum_{j=1}^k a_j d_{k-j}, & k = 1, \dots, n, \\ -\sum_{j=1}^n a_j d_{k-j}, & k > n, \end{cases}$$

one easily deduces

$$\sum_{j=1}^{n} |d_j| \ge \frac{1}{2}, \tag{8}$$

which implies $\sum_{j=1}^n |d_j|^2 \ge \frac{1}{4n}$, and therefore

$$\sum_{j=0}^{\infty} |d_j|^2 \geq 1 + \frac{1}{4n}. \tag{9}$$

From now on we always assume that the P we are working with has no zeros on $\partial\mathbb{D}$; the general case can be easily established through obvious limiting procedures. Since $P \neq 0$ on $\partial\mathbb{D}$ we deduce that h is analytic in the closed unit disk, and therefore the following definitions are meaningful:

$$|h(\mathrm{e}^{\mathrm{i}\phi})|^2 = \sum_{j=-\infty}^{\infty} f_j \, \mathrm{e}^{\mathrm{i}j\phi},$$

where

$$f_0 = \sum_{j=0}^{\infty} |d_j|^2, \quad f_j = \overline{f_{-j}} = \sum_{k=0}^{\infty} \overline{d_k} d_{k+j}, \quad j \in \mathbb{N}.$$

For arbitrary $N \in \mathbb{N}$ we obtain

$$\begin{aligned}
\frac{1}{N} \sum_{k=1}^{N} \frac{1}{|P(\mathrm{e}^{\frac{2\pi\mathrm{i}k}{N}})|^2} &= \frac{1}{N} \sum_{k=1}^{N} \sum_{j=-\infty}^{\infty} f_j \, \mathrm{e}^{\frac{2\pi\mathrm{i}kj}{N}} \\
&= \frac{1}{N} \sum_{j=-\infty}^{\infty} f_j \sum_{k=1}^{N} \mathrm{e}^{\frac{2\pi\mathrm{i}kj}{N}} \\
&= \sum_{j=-\infty}^{\infty} f_{Nj} \\
&\geq f_0 - 2 \sum_{j=1}^{\infty} |f_{Nj}|
\end{aligned}$$

and therefore

$$\frac{1}{N} \sum_{k=1}^{N} \frac{1}{|P(\mathrm{e}^{\frac{2\pi\mathrm{i}k}{N}})|^2} \geq f_0 - 2 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} |d_k| \, |d_{k+Nj}|, \tag{10}$$

which is the basic inequality for the subsequent proofs. We need the following lemma.

Lemma 1. *Let* $P \in \mathscr{P}_n^{\mathrm{c}}$ *and* $s := \min_{|z|=1} |P(z)|$. *Then*

$$|P(z)| \geq \left(1 + \frac{s}{2}\right) - (1 - \frac{s}{2})R^n, \quad |z| = R \geq 1. \tag{11}$$

Proof. We have $s < 1$. Thus, for $Q(z) := P(z) - s$, we get $Q \neq 0$ in $\mathbb{D}$, and $||Q|| \leq 2 - s$ where $||.||$ denotes the sup-norm in $\mathbb{D}$. Lax's generalization of the Bernstein inequality then gives $||P'|| = ||Q'|| \leq \frac{2-s}{2} n$, and another inequality of Bernstein yields

$$|P'(t)| \leq \left(1 - \frac{s}{2}\right) n |t|^{n-1}, \quad |t| \geq 1.$$

This implies for $|z| = R > 1$

$$
\begin{aligned}
|P(z)| = \left|P(\frac{z}{|z|}) + \int_{\frac{z}{|z|}}^{z} P'(t)\,dt\right| &\geq \left|P(\frac{z}{|z|})\right| - \left|\int_1^R P'(\rho e^{i\theta})e^{i\theta}\,d\rho\right| \\
&\geq s - \int_1^R |P'(\rho e^{i\theta})|\,d\rho \\
&\geq s - \left(1 - \frac{s}{2}\right) n \int_1^R \rho^{n-1}\,d\rho \\
&= \left(1 + \frac{s}{2}\right) - \left(1 - \frac{s}{2}\right) R^n,
\end{aligned}
$$

the assertion. □

Lemma 2. *Let P, s be as above. Then there exists an arc Γ on $\partial\mathbb{D}$ of length*

$$L(\Gamma) = \frac{4}{n}\sqrt{\frac{(1-\frac{1}{16n})^2 - s^2}{4 - s^2}} \tag{12}$$

such that

$$|P(z)| \leq 1 - \frac{1}{16n}, \quad z \in \Gamma, \tag{13}$$

Proof. Since $s \leq |P(e^{i\theta})| \leq 2,\ \theta \in \mathbb{R}$ we deduce

$$\left| |P(e^{i\theta})|^2 - \frac{4+s^2}{2} \right| \leq \frac{4-s^2}{2}, \quad \theta \in \mathbb{R}.$$

Two applications of Bernstein's inequality to the trigonometric polynomial $|P(e^{i\theta})| - \frac{4+s^2}{2}$ yields

$$\left| \frac{\partial^2}{\partial\theta^2} |P(e^{i\theta})|^2 \right| \leq n^2 \frac{4-s^2}{2} \tag{14}$$

for all θ. From (1) we deduce that $s^2 \leq 1 - \frac{1}{n} < (1 - \frac{1}{16n})^2$. Assume $|P(e^{i\theta_0})| = s$, and set

$$\Gamma := \left\{ e^{i\theta} : |\theta - \theta_0| \leq \frac{2}{n}\sqrt{\frac{(1-\frac{1}{16n})^2 - s^2}{4 - s^2}} \right\}.$$

Taylor's formula now guarantees a $\phi = \phi(\theta) \in \mathbb{R}$ with

$$|P(e^{i\theta})|^2 = |P(e^{i\theta_0})|^2 + \frac{1}{2}(\theta - \theta_0)^2 \frac{\partial^2}{\partial\theta^2}|P(e^{i\phi})|^2,$$

which, using (14) and the definition of Γ, proves (13). □

Proof of Theorem 1. Define s as in the previous lemma (note that $s > 0$ by our general assumption). It is now clear from Lemma 2 that

$$N_s\left(\frac{1}{16n}, n\right) \leq \frac{n\pi}{2}\sqrt{\frac{4-s^2}{(1-1/(16n))^2 - s^2}}, \tag{15}$$

where $N_s(.,.)$ is defined as $N(.,.)$, except that we consider only polynomials $P \in \mathscr{P}_n^{\mathrm{c}}$ with that corresponding s. Note that the right hand side of (15) increases with s for fixed n, and that it gives reasonable bounds for small s.

For larger values of s we make use of (10). We write $\sigma := \frac{1+s/2}{1-s/2}$, and, for n fixed, we choose R so that $\sigma > R^n$. Lemma 1 shows that $h(z)$ is analytic in $|z| \leq R$, so that a two-times application of the Cauchy-Schwarz inequality yields

$$(16) \qquad \begin{aligned} \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} |d_k|\,|d_{k+Nj}| &\leq \sum_{j=1}^{\infty} \sqrt{\sum_{k=0}^{\infty} |d_k|^2} \sqrt{\sum_{k=0}^{\infty} |d_{k+Nj}|^2} \\ &= \sqrt{f_0} \sum_{j=1}^{\infty} \sqrt{\sum_{k=Nj}^{\infty} |d_k|^2 R^{2Nj}} \sqrt{R^{-2Nj}} \\ &\leq \sqrt{f_0} \sqrt{\sum_{j=1}^{\infty} R^{-2Nj}} \sqrt{\sum_{j=1}^{\infty}\sum_{k=Nj}^{\infty} |d_k|^2 R^{2Nj}} \\ &= \sqrt{\frac{f_0}{R^{2N}-1}} \sqrt{\sum_{k=N}^{\infty} |d_k|^2 \sum_{j=1}^{[\frac{k}{N}]} R^{2Nj}} \\ &\leq \sqrt{\frac{f_0}{R^{2N}-1}} \sqrt{\frac{1}{N} \sum_{k=N}^{\infty} k |d_k|^2 R^{2k}}. \end{aligned}$$

We now use the estimate

$$(17) \qquad \begin{aligned} \sum_{k=N}^{\infty} k|d_k|^2 R^{2k} &\leq \frac{1}{2\pi} \int_0^{2\pi} R\mathrm{e}^{\mathrm{i}\phi} h'(R\mathrm{e}^{\mathrm{i}\phi}) \overline{h(R\mathrm{e}^{\mathrm{i}\phi})}\, d\phi \\ &= \frac{1}{\pi} \text{ Area } \{\, h(z) \,:\, |z| \leq R \,\} \\ &\leq \frac{n}{(1-\frac{s}{2})^2(\sigma - R^n)^2}, \end{aligned}$$

where the last line comes from the fact that

$$|h(z)| \leq \frac{1}{(1-\frac{s}{2})(\sigma - R^n)}, \quad |z| \leq R,$$

and that h, as a reciprocal of a polynomial of degree n, covers that disk at most n times. We choose

$$R := \left(\frac{\sigma}{1+\frac{n}{N}} \right)^{\frac{1}{n}},$$

(assuming that σ, n, N are such that $R > 1$) and set $\gamma := \frac{N}{n}$. Inserting this and (17) into (16) we get

$$(18) \qquad \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} |d_k|\,|d_{k+Nj}| \leq \sqrt{\frac{f_0}{\left(\frac{\gamma\sigma}{1+\gamma}\right)^{2\gamma} - 1}}\; \frac{1+\gamma}{(1+\frac{s}{2})\sqrt{\gamma}}$$

We shall now choose $\gamma \geq 3$, $\sigma \geq 2$, and hence $s \geq \frac{2}{3}$. It is then easily verified that

$$\frac{1}{\sqrt{\left(\frac{\gamma\sigma}{1+\gamma}\right)^{2\gamma} - 1}} \frac{1+\gamma}{(1+\frac{s}{2})\sqrt{\gamma}} \leq 1,$$

and is decreasing in both, s and γ. Having this in mind we deduce, using (10), (18) and (9),

$$\begin{aligned}(19)\quad \frac{1}{N}\sum_{k=1}^{N} \frac{1}{|P(\mathrm{e}^{\frac{2\pi \mathrm{i} k}{N}})|^2} &\geq f_0 - 2\sqrt{f_0}\frac{1}{\sqrt{\left(\frac{\gamma\sigma}{1+\gamma}\right)^{2\gamma} - 1}} \frac{1+\gamma}{(1+\frac{s}{2})\sqrt{\gamma}} \\ &\geq 1 + \frac{1}{4n} - 2\sqrt{1+\frac{1}{4n}}\frac{1}{\sqrt{\left(\frac{\gamma\sigma}{1+\gamma}\right)^{2\gamma} - 1}} \frac{1+\gamma}{(1+\frac{s}{2})\sqrt{\gamma}}.\end{aligned}$$

We shall make use of (19) in the calculation of Table 1. In order to prove Theorem 1 we apply further simplifications to (19). We shall use this estimate for $\sigma \geq \sigma_0 := \mathrm{e}$, i.e. $s \geq s_0 := 2(\mathrm{e}-1)/(\mathrm{e}+1) = .924\ldots$. Then

$$\begin{aligned}(20)\quad \frac{1}{N}\sum_{k=1}^{N} \frac{1}{|P(\mathrm{e}^{\frac{2\pi \mathrm{i} k}{N}})|^2} &\geq 1 + \frac{1}{4n} - \sqrt{1+\frac{1}{4n}}\,\mathrm{e}^{-\gamma}\sqrt{\gamma}\frac{(1+\frac{1}{\gamma})(1+\frac{1}{\mathrm{e}})}{\sqrt{\left(\frac{\gamma}{1+\gamma}\right)^{2\gamma} - \mathrm{e}^{-2\gamma}}} \\ &\geq 1 + \frac{1}{4n} - 4\,\mathrm{e}^{-\gamma}\sqrt{\gamma}\sqrt{1+\frac{1}{4n}}, \qquad (\gamma \geq 9).\end{aligned}$$

It follows readily that if we choose $\gamma = \gamma(n) \geq 9$ such that

$$(21)\qquad 1 + \frac{1}{4n} - 4\,\mathrm{e}^{-\gamma}\sqrt{\gamma}\sqrt{1+\frac{1}{4n}} \geq \frac{1}{(1-\frac{1}{16n})^2}$$

we must have

$$N_s\left(\frac{1}{16n}, n\right) \leq n\,\gamma(n), \quad s \geq s_0.$$

It is a matter of elementary calculus to verify that

$$\gamma(n) := \max\{9,\ \log(n) + \log(\log(n)) + 4\}$$

satisfies (21).

Choosing $s \leq s_0$ in (15) we obtain

$$N_s\left(\frac{1}{16n}, n\right) \leq 9.47n, \quad s \leq s_0,\ n \geq 2,$$

which then proves

$$N\left(\frac{1}{16n}, n\right) = \sup_{0<s<1} N_s\left(\frac{1}{16n}, n\right) \leq n\ \max\{9.47,\ \log(n) + \log(\log(n)) + 4\},$$

the assertion. □

Proof of Theorem 2. We start again from (10). We observe that $|P(z) - 1| < 1$ holds in $\mathbb{D}$, which implies $\Re h(z) > \frac{1}{2}$ in $\mathbb{D}$, and therefore

$$|d_k| \le 1, \quad k = 0, 1, \dots .$$

Now choose $N \in \mathbb{N}$ and Z according to (7), and assume that P is different from zero in $|z| \le Z^*$, with $Z^* > Z$. Then obviously $|P(z)| \ge (1 - |z|/Z^*)^n$ for those z and an application of the Cauchy estimates to h yields

$$|d_k| \le \frac{\rho^{-k}}{(1 - \frac{\rho}{Z}{}^*)^n}, \quad 1 < \rho < Z^*, \ k \in \mathbb{N}.$$

Our assumption (6), combined with (10), yields

$$\begin{aligned} 1 &\ge 1 + \frac{1}{4n} - 2 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} |d_k| \, |d_{k+Nj}| \\ &\ge 1 + \frac{1}{4n} - 2 \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} |d_{k+Nj}| \\ &\ge 1 + \frac{1}{4n} - 2 \frac{1}{(1 - \frac{\rho}{Z^*})^n} \frac{\rho}{\rho - 1} \frac{1}{\rho^N - 1}. \end{aligned} \tag{22}$$

If we choose

$$\rho = \frac{N Z^*}{N + n}$$

then we find

$$8n \frac{\frac{N Z^*}{N+n}}{\frac{N Z^*}{N+n} - 1} \frac{\left(1 + \frac{N}{n}\right)^n}{\left(\frac{N Z^*}{N+n}\right)^N - 1} \ge 1$$

which contradicts (7) because of $Z < Z^*$ and the monotonicity property of the left hand side of (7). The special cases mentioned at the end of Theorem 2 follow from elementary verifications of (7), and from (5) for a few cases of small n. □

Proof of Theorem 3. If P has a zero in $|z| \le 3$ there is nothing to prove. Hence we may assume that h is analytic in $|z| \le 3$. From (8) we obtain

$$\sum_{k=0}^{\infty} |d_k| \ge \frac{3}{2},$$

which, by a theorem of H. Bohr [1] implies

$$\max_{|z|=3} |h(z)| \ge \frac{3}{2}.$$

This proves our result. □

3. Numerical evaluation for small n

In this brief section we describe the numerical method to calculate Table 1 above, which gives the optimal values of $N(\frac{1}{16n}, n)$ presently available.

We fix n and set $t = t(n) := 1 - \frac{1}{16n}$. It is important to realize that the estimate (15) is increasing in s. For $N > 4n$ this implies that for

$$s(N) := \sqrt{\frac{(2Nt)^2 - (2n\pi)^2}{(2N)^2 - (n\pi)^2}}$$

we have

$$N_s\left(\frac{1}{16n}, n\right) \leq N, \quad s \leq s(N).$$

The right hand side of (19) decreases with s. Hence, if

$$1 + \frac{1}{4n} - 2\sqrt{1 + \frac{1}{4n}}\,\frac{1}{\sqrt{\left(\frac{\gamma\sigma(N)}{1+\gamma}\right)^{2\gamma} - 1}}\,\frac{1+\gamma}{(1 + \frac{s(N)}{2})\sqrt{\gamma}} \geq \frac{1}{t^2},$$

where

$$\sigma(N) = \frac{1 + \frac{s(N)}{2}}{1 - \frac{s(N)}{2}}. \quad \gamma = \frac{N}{n},$$

then also

$$N_s\left(\frac{1}{16n}, n\right) \leq N, \quad s \geq s(N),$$

and hence

$$N\left(\frac{1}{16n}, n\right) \leq N, \tag{23}$$

This means that if we solve the equation

$$1 + \frac{1}{4n} - 2\sqrt{1 + \frac{1}{4n}}\,\frac{1}{\sqrt{\left(\frac{\gamma\sigma(N)}{1+\gamma}\right)^{2\gamma} - 1}}\,\frac{1+\gamma}{(1 + \frac{s(N)}{2})\sqrt{\gamma}} = \frac{1}{t^2}$$

for $N, N > 4n$, then this N satisfies (23). Table 1 shows this value for N (in fact, the smallest integer greater than N) or the one obtained from (15) whichever is smaller (it turns out that (15) is better for $2 \leq n \leq 30$).

For practical purposes it may be useful to know that

$$N\left(\frac{1}{16n}, n\right) \leq 10\,n, \quad n \leq 366.$$

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