ELSEVIER

Topology and its Applications 130 (2003) 87–90

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

An elementary counterexample on dense normality

A.E. McCluskey [a], T.B.M. McMaster [b,*]

[a] *Department of Mathematics, National University of Ireland, Galway, Ireland*
[b] *Department of Pure Mathematics, Queen's University Belfast, Belfast BT7 1NN, UK*

Received 4 January 2002; received in revised form 15 July 2002

Abstract

We show that the Sorgenfrei plane is not normal on any of its dense subsets, that is, is not densely normal. This addresses in the simplest possible terms Arhangel'skii's question as to whether an elementary example exists of a regular κ-normal space that fails to be densely normal.

MSC: 54D15

Keywords: Densely normal; κ-normal; Sorgenfrei plane

A.V. Arhangel'skii recently raised the question [1] of whether every κ-normal regular space is densely normal. Sophisticated counterexamples [2,3] to this conjecture have now been constructed but, as Arhangel'skii pointed out in unpublished correspondence, the possibility of the existence of a more elementary counterexample was not settled. This short paper displays a truly elementary one, as presented by the first-named author at the Fifth Galway Topology Symposium at the University of Hull in April 2001. The authors would like to express their thanks to Arhangel'skii for his advice and guidance in this pursuit.

If G and D are subsets of a topological space X, then G is said to be *concentrated* on D if their intersection $G \cap D$ is dense in G. The space X is termed *densely normal* if there exists a dense subset D of X such that each two non-empty, disjoint closed subsets of X *that are concentrated on* D possess disjoint neighbourhoods. (The term "normal on D" is also used to describe this scenario.) The space X is called κ*-normal* if each two non-

* Corresponding author.
E-mail addresses: aisling.mccluskey@nuigalway.ie, ciarandunwoody@eircom.net (A.E. McCluskey), t.b.m.mcmaster@qub.ac.uk (T.B.M. McMaster).

0166-8641/02/$ – see front matter
doi:10.1016/S0166-8641(02)00200-6

empty, disjoint canonical closed subsets of X have disjoint neighbourhoods. (Recall that a *canonical* closed set is a set which is equal to the closure of its interior.) One of the many strong properties that the Sorgenfrei plane is known to possess (such as homogeneity and subparacompactness) is that of κ-normality: because it is, in fact, κ-metrizable (see [4]). Our purpose here is to show that it is not densely normal, thus resolving in the most accessible terms possible the quest for a counterexample to the suggestion that dense normality might be weaker than κ-normality.

By the Sorgenfrei plane S^2 we intend the coordinate plane topologised by declaring that basic neighbourhoods of a typical point (a, b) take the form of 'squares'

$$\Sigma\big((a, b), h\big) = [a, a + h) \times [b, b + h)$$

where $h > 0$ may, by *abus de language*, be referred to as the 'radius' of the square. This topology is finer than the Euclidean one. The familiar failure of S^2 to be normal is most easily evidenced by noting that the diagonal line $x + y = 0$ is discrete and closed in S^2, so that its points of rational and of irrational coordinates form disjoint closed sets which, by a simple category argument based in the Euclidean plane, cannot be separated by disjoint open sets in S^2.

Intuitively and informally, the essence of the demonstration is to distort a segment of this diagonal line so that it shall pass densely through a given dense set D but retain its discrete and closed status. Then a countable selection of those points of that distorted segment which lie within D will constitute one closed set that is concentrated upon D; the remaining points of the segment constitute a second closed set, and one which cannot be separated from the first by disjoint neighbourhoods but which, unfortunately, fails utterly to be concentrated on D (in general). The remaining need is to thicken-up this second closed set into another which *is* so concentrated. The process is easier to visualise if we begin by rotating the entire problem anticlockwise by one eighth of a revolution, relocating the initial segment onto the horizontal axis, and allowing a natural interpretation of the distorted segment (and of the approximations by which we shall approach it) as graphs of real functions.

Let D be dense in the Sorgenfrei plane S^2. An elementary argument locates within D a countable subset D_0 that still is dense in S^2. We note that D_0 is dense also in the Euclidean topology, and so is the set D' created by rotating D_0 anticlockwise by $\pi/4$ radians (about the origin).

Suppose given a line-segment AB in the coordinate plane whose *slope* (that is, the modulus of whose gradient) is less than 0.9 say. We can find elements of D' arbitrarily close in the Euclidean sense to the midpoint of AB, and this allows us to choose a point M of D' so that

(i) the slopes of both AM and MB are less than 0.9,
(ii) the horizontal lengths of both AM and MB are less than $2/3$ that of AB, and
(iii) the vertical displacement of M from AB is less than any desired threshold $\varepsilon > 0$.

Let us name the procedure of replacing AB by AMB "kinking AB within ε". Now we generate a sequence of real functions each defined on the unit interval [0,1] by letting f_0 be that function whose graph is the segment from the origin (0,0) to the point (1,0) and, thereafter, creating the graph of f_{n+1} by kinking each segment of the graph of f_n within $(1/2)^{n+2}$. The iteration ensures that

(i) the graph of f_n, and of each subsequent $f_{n'}$ where $n' > n$, shall pass through $2^n - 1$ points of D' whose successive horizontal distances do not exceed $(2/3)^n$,
(ii) the (uniform norm) distance from f_n to f_{n+1} is less than $(1/2)^{n+2}$, and
(iii) the Lipschitz condition $|f_n(x) - f_n(y)| < 0.9|x - y|$ is obeyed throughout [0,1].

The completeness of $C[0,1]$ guarantees that this sequence converges to some limit function f, and we see that D' is dense in the graph of f, and that $|f(x) - f(y)|$ is strictly less than $|x - y|$ for all relevant x and y. Rotate the graph of f clockwise about the origin through $\pi/4$ to form a new set Γ; Γ is discrete and closed in the Sorgenfrei plane (for essentially the same reasons that applied for the line $x + y = 0$: appealing in particular to the continuity of f to ensure that its graph is Euclidean-compact, and Γ therefore Euclidean-compact and -closed); further, the sets Q of points common to Γ and D_0, and I of points of $\Gamma \setminus D_0$, are disjoint, non-empty and closed in S^2. Routine modification of the standard argument (see, for example, [5]) suffices to show that Q and I cannot be enclosed in disjoint open subsets of S^2; *a fortiori*, Q and any superset of I cannot be so separated.

Next, enumerating the points of Q as a sequence (q_n), we can inductively choose a basic neighbourhood $\Sigma_n = \Sigma(q_n, h_n)$ of q_n in such a way that the radius $h_n \to 0$ and that the double-sized neighbourhoods $\Sigma(q_n, 2h_n)$ are pairwise disjoint. For any $i \in I$ and any basic neighbourhood T of i notice that, with respect to i as an origin, each q_n is constrained to lie strictly within the second or fourth quadrants. It is clear that, if only *finitely* many of the sets Σ_n intersect T, then there are some interior points of T that lie outside their union. Equally, if *infinitely* many of them meet T, then amongst these there will be one (Σ_s, say) whose radius is less than that of T; and since $\Sigma_s = \Sigma(q_s, h_s)$ overlaps T, so must $\Sigma(q_s, 2h_s) \setminus \Sigma_s$ which is disjoint from all of the sets Σ_n. Hence the interior of T cannot be covered by *any* collection of the neighbourhoods Σ_n.

Putting $D_1 = D \setminus \bigcup\{\Sigma_n\colon n \geqslant 1\}$, we see that every point of I is in the closure of D_1 but that no point of Q is. Define I^+ to be this closure, and we have that Q and I^+ are disjoint, non-empty and closed, that both are concentrated upon D and that they do not possess disjoint S^2-neighbourhoods.

We conclude that the Sorgenfrei plane is not normal on any of its dense subsets. Concerning the higher products, we have been able to show that Sorgenfrei n-space (the product of *finitely* many copies of the Sorgenfrei line) cannot be densely normal, but it is presently unclear to us whether the argument or the result extends to infinite products. Hence the question appears to remain open: *can some power of the Sorgenfrei line be densely normal*?

References

[1] A.V. Arhangel'skii, Relative topological properties and relative topological spaces, Topology Appl. 70 (1996) 87–99.
[2] A.V. Arhangel'skii, Normality and dense subspaces, Proc. Amer. Math. Soc. 130 (2002) 283–291.

[3] W. Just, J. Tartir, A κ-normal, not densely normal Tychonoff space, Proc. Amer. Math. Soc. 127 (1999) 901–905.
[4] E.V. Shchepin, On κ-metrizable spaces, Izv. Akad. Nauk USSR 43 (1979) 442–479.
[5] L.A. Steen, J.A. Seebach, Counterexamples in Topology, Springer-Verlag, Berlin, 1978.