Continuous Functions that are Nowhere Differentiable

S. Kesavan
The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600 113
E-mail: kesh @imsc.res.in

Abstract. It is shown that the existence of continuous functions on the interval [0, 1] that are nowhere differentiable can be deduced from the Baire category theorem. This approach also shows that there is a preponderance of such functions.

1. Introduction

The French mathematician Hermite, in a letter written to Stieltjes, dated May 20, 1893, wrote 'I turn away with fear and horror from the lamentable plague of continuous functions which do not have derivatives . . . ' (cf. Pinkus [6]). The earliest universally acknowledged explicit example of a continuous function which is nowhere differentiable is due to Weierstrass (1872) given by

$$\sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $ab > 1 + \frac{3}{2}\pi$. It is also said that Bolzano constructed such an example (in the 1830s), which was not published. Since then a number of variants of Weierstrass' example have appeared in the literature. Here are some of them.

- $$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin(3^n x).$$
- (cf. Hardy [3]) $$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(n^2 \pi x).$$
- (cf. Rudin [7]) Define $$\varphi(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2 \end{cases}$$ and extend it to all of $\mathbb{R}$ by setting $\varphi(x+2) = \varphi(x)$. Then the function defined by the series $$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$$ is again continuous and nowhere differentiable.

In the above three examples, the series are clearly uniformly convergent by the Weierstrass M-test and so the sum defines a continuous function. One has to show that it is nowhere differentiable.

Another type of example is constructed as follows. Consider the space $\mathcal{C}[0, 1]$ (the space of continuous functions on [0, 1]) with the usual norm topology generated by the norm

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|.$$

Let

$$X = \{f \in \mathcal{C}[0, 1] \mid f(0) = 0, \quad f(1) = 1\}.$$

Then it is a closed subset of $\mathcal{C}[0, 1]$ and is hence a complete metric space in its own right. For $f \in X$, define

$$T(f)(x) = \begin{cases} \frac{3}{4} f(3x), & 0 \le x \le \frac{1}{3}, \\ \frac{1}{4} + \frac{1}{2} f(2 - 3x), & \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{1}{4} + \frac{3}{4} f(3x - 2), & \frac{2}{3} \le x \le 1. \end{cases}$$

Then it can be shown that T maps X into itself and that

$$\|T(f) - T(g)\|_\infty \le \frac{3}{4} \|f - g\|_\infty.$$

Hence, by the contraction mapping theorem, there exists $h \in X$ such that $T(h) = h$. It can be shown then that h is nowhere differentiable.

The aim of the present article is to show the existence of continuous but nowhere differentiable functions, without exhibiting one. The proof, following the ideas of Banach [1] and Mazurkiewicz [5], uses the Baire category theorem which can be stated as follows.

Theorem 1.1 (Baire). *Let X be a complete metric space. If $\{U_n\}_{n=1}^{\infty}$ is a sequence of open and dense sets in X, then*

$$\bigcap_{n=1}^{\infty} U_n$$

is also dense in X. ■

Equivalently, a complete metric space cannot be the countable union of a family of closed and nowhere dense sets. In technical parlance, a complete metric space is said to be of the 'second category' (the first category being topological spaces which are countable unions of closed and nowhere dense sets), and hence the word 'category' in the name of the theorem. For a proof, see any text on functional analysis (for instance, see Ciarlet [2] or Kesavan [4]).

Baire's theorem is the corner stone of the famous trinity of theorems in functional analysis, *viz.* the uniform boundedness principle, the open mapping theorem and the closed graph theorem. As a consequence of the uniform boundedness principle, we can show that for a large class of continuous functions, the Fourier series diverges on a large set of points (see, for instance, Kesavan [4]).

We will use Baire's theorem to prove the existence of nowhere differentiable functions in $\mathcal{C}[0,1]$. This approach also shows that the class of such functions is quite large. Our presentation is an adaptation of that found in Ciarlet [2].

2. Approximation by Smooth Functions

A celebrated theorem of Weierstrass states that any continuous function on $[0,1]$ can be uniformly approximated by polynomials. To make this presentation as self-contained as possible, we will prove a slightly weaker result which is enough for our purposes, *viz.* that any continuous function on $[0,1]$ can be uniformly approximated by smooth functions.

Consider the function

$$\rho(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

It is not difficult to see that this defines a $\mathcal{C}^\infty$ function on $\mathbb{R}$ whose support is the closed ball centered at the origin and with unit radius. For $\varepsilon > 0$, define

$$\rho_\varepsilon(x) = (k\varepsilon)^{-1}\rho\left(\frac{x}{\varepsilon}\right)$$

where

$$k = \int_{-\infty}^{\infty} \rho(x)\,dx = \int_{-1}^{1} \rho(x)\,dx.$$

Then, it is easy to see that ρ_ε is also $\mathcal{C}^\infty$ and its support is the closed ball centered at the origin with radius ε. Further

$$\int_{-\infty}^{\infty} \rho_\varepsilon(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(x)\,dx = 1.$$

Recall that if f and g are continuous real-valued functions defined on $\mathbb{R}$, with one of them having compact support, the convolution product $f * g$ defined by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} g(x-y)f(y)\,dy$$

is well defined and is a continuous function. Further, if one of them is in $\mathcal{C}^k(\mathbb{R})$, then $f * g \in \mathcal{C}^k(\mathbb{R})$ for any $1 \leq k \leq \infty$. If $\operatorname{supp}(F)$ denotes the support of a function F, then

$$\operatorname{supp}(f*g) \subset \operatorname{supp}(f) + \operatorname{supp}(g)$$

where, for subsets A and B of $\mathbb{R}$, we define

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Then $\rho_\varepsilon * f$ converges uniformly to f as $\varepsilon \to 0$.*

Proof. Let K be the support of f. Then K is a compact subset of $\mathbb{R}$. Without loss of generality, we can assume $0 < \varepsilon < 1$ so that $\rho_\varepsilon * f$ is a $\mathcal{C}^\infty$ function with support contained in the fixed compact set

$$\{x \in \mathbb{R} \mid |x| \leq 1\} + K.$$

Clearly f is uniformly continuous and so, given $\eta > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \eta$ whenever $|x-y| < \delta$. Now, since the integral of ρ_ε is unity, we can write

$$(\rho_\varepsilon * f)(x) - f(x) = \int_{-\varepsilon}^{\varepsilon} (f(x-y) - f(x))\rho_\varepsilon(y)\,dy.$$

Thus, if $\varepsilon < \delta$ then

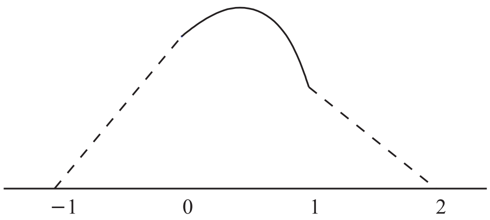
$$|(\rho_\varepsilon * f)(x) - f(x)| \leq \int_{-\varepsilon}^{\varepsilon} |f(x-y) - f(x)|\rho_\varepsilon(y)\,dy \leq \eta$$

for all x and this completes the proof. ■

Corollary 2.1. *Let $f \in \mathcal{C}[0,1]$. Then f can be uniformly approximated by smooth functions.*

Proof. Given $f \in \mathcal{C}[0,1]$, we can extend it to a continuous function with compact support in $\mathbb{R}$. For example, define

$$\tilde{f}(x) = \begin{cases} 0, & \text{if} x < -1 \text{or if} x > 2, \\ (x+1)f(0), & \text{if} x \in [-1,0], \\ f(x), & \text{if} x \in [0,1], \\ (2-x)f(1), & \text{if} x \in [1,2]. \end{cases}$$



Now $\widetilde{f}$ can be uniformly approximated by smooth functions in $\mathbb{R}$ and so their restrictions to $[0, 1]$ will approximate f uniformly on $[0, 1]$. ■

Proposition 2.2. *Let $f \in \mathcal{C}[0, 1]$. Let $\varepsilon > 0$ and n, a positive integer, be given. Then there exists a piecewise linear continuous function g, defined on $[0, 1]$ such that $\|f - g\|_\infty < \varepsilon$ and such that $|g'(t)| > n$ at all points where the derivative exists.*

Proof. In view of the corollary above, we can assume that f is a smooth function defined on $[0, 1]$.

Step 1. Since f is smooth, f' is bounded in $[0, 1]$. Let $|f'(x)| \leq M$ for all $x \in [0, 1]$. Since f is continuous on $[0, 1]$, it is uniformly continuous and so there exists $\delta > 0$ such that, whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \frac{\varepsilon}{4}$. Now, choose $h > 0$ such that

$$h < \min\left\{\delta, \frac{\varepsilon}{2(M+n)}\right\}.$$

Step 2. Now choose a partition

$$\mathcal{P} : 0 = t_0 < t_1 < \cdots < t_k = 1$$

such that

$$\max_{0 \leq i \leq k-1} (t_{i+1} - t_i) \leq h.$$

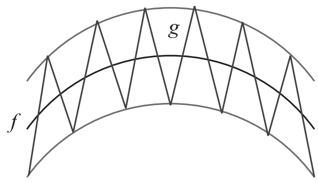
Let $g : [0, 1] \to \mathbb{R}$ be a piecewise linear and continuous function, defined on each sub-interval $[t_i, t_{i+1}]$, $0 \leq i \leq k-1$, as follows:

$$g(t_i) = f(t_i) + (-1)^i \frac{\varepsilon}{4},$$

$$g(t_{i+1}) = f(t_{i+1}) + (-1)^{i+1} \frac{\varepsilon}{4},$$

$$g(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} g(t_i) + \frac{t - t_i}{t_{i+1} - t_i} g(t_{i+1}), t_i < t < t_{i+1}.$$

The function g is differentiable except at the points $\{t_1, \cdots, t_{k-1}\}$.



Step 3. For $t \in [t_i, t_{i+1}]$, $0 \leq i \leq k - 1$, we have

$$g(t) - f(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i}(g(t_i) - f(t)) + \frac{t - t_i}{t_{i+1} - t_i}(g(t_{i+1}) - f(t))$$

so that

$$|g(t) - f(t)| \leq |f(t_i) - f(t)| + |f(t_{i+1}) - f(t)| + \frac{\varepsilon}{2} < \varepsilon$$

since $|t - t_i|$ and $|t - t_{i+1}|$ are both less than, or equal to $h < \delta$. Thus, it follows that $\|f - g\|_\infty < \varepsilon$.

Step 4. For any $t \in (t_i, t_{i+1})$, $0 \leq i \leq k - 1$, we have

$$g'(t) = \frac{f(t_{i+1}) - f(t_i) + (-1)^{i+1}\frac{\varepsilon}{2}}{t_{i+1} - t_i} = f'(\xi_i) + \frac{(-1)^{i+1}\frac{\varepsilon}{2}}{t_{i+1} - t_i}$$

where $\xi_i \in (t_i, t_{i+1})$. Thus, by our choice of h, we have

$$\begin{aligned} |g'(t)| &= \left|\frac{(-1)^{i+1}\frac{\varepsilon}{2}}{t_{i+1} - t_i} + f'(\xi_i)\right| \\ &\geq \frac{\varepsilon}{2(t_{i+1} - t_i)} - |f'(\xi_i)| \\ &\geq \frac{\varepsilon}{2h} - M \\ &> n \end{aligned}$$

which completes the proof. ■

3. The Main Result

Proposition 3.1. *Let $f \in \mathcal{C}[0, 1]$ be differentiable at some point $a \in [0, 1]$. Then, there exists a positive integer N such that*

$$\sup_{h \neq 0} \left|\frac{f(a+h) - f(a)}{h}\right| \leq N.$$

Proof. Since f is differentiable at $a \in [0, 1]$, there exists $h_0 > 0$ such that for all $0 < |h| \leq h_0$, we have

$$\left|\frac{f(a+h) - f(a)}{h} - f'(a)\right| \leq 1.$$

Thus, for all $0 < |h| \le h_0$, we have

$$\left|\frac{f(a+h)-f(a)}{h}\right| \le 1+|f'(a)|.$$

If $|h| \ge h_0$, then trivially

$$\left|\frac{f(a+h)-f(a)}{h}\right| \le \frac{2\|f\|_\infty}{h_0}.$$

Thus we only need to take

$$N \ge \max\left\{1+|f'(a)|, \frac{2\|f\|_\infty}{h_0}\right\}. \quad \blacksquare$$

Let us now define, for each positive integer n,

$$\mathcal{A}_n = \left\{ f \in \mathcal{C}[0,1] \mid \sup_{h\neq 0}\left|\frac{f(a+h)-f(a)}{h}\right| \le n \text{ for some } a \in [0,1] \right\}.$$

Proposition 3.2. *For each positive integer n, the set $\mathcal{A}_n$ is closed in $\mathcal{C}[0,1]$.*

Proof. Let $\{f_k\}$ be a sequence in $\mathcal{A}_n$ such that $f_k \to f$ in $\mathcal{C}[0,1]$. Then, there exists a sequence $\{a_k\}$ in $[0,1]$ such that, for each k,

$$\sup_{h\neq 0}\left|\frac{f_k(a_k+h)-f_k(a_k)}{h}\right| \le n.$$

Let $\{a_{k_l}\}$ be a convergent subsequence, converging to $a \in [0,1]$.

Let $h \neq 0$ be given. Choose h_{k_l} such that $a_{k_l} + h_{k_l} = a+h$. Thus the sequence $\{h_{k_l}\}$ converges to $h \neq 0$ and so we may assume, without loss of generality, that it is a sequence of non-zero real numbers. Now

$$\begin{aligned}&|f(a+h) - f_{k_l}(a_{k_l}+h_{k_l})|\\ &\quad = |(f-f_{k_l})(a+h)| \le \|f-f_{k_l}\|_\infty.\end{aligned}$$

Also

$$\begin{aligned}|f(a)-f_{k_l}(a_{k_l})| &\le |f(a)-f(a_{k_l})| + |f(a_{k_l}) - f_{k_l}(a_{k_l})|\\ &\le |f(a)-f(a_{k_l})| + \|f-f_{k_l}\|_\infty.\end{aligned}$$

By the continuity of f and the convergence of $\{f_{k_l}\}$ to f, we then deduce that

$$\left|\frac{f(a+h)-f(a)}{h}\right| = \lim_{l\to\infty}\left|\frac{f_{k_l}(a_{k_l}+h_{k_l})-f_{k_l}(a_{k_l})}{h_{k_l}}\right| \le n$$

which shows that $f \in \mathcal{A}_n$ as well, which completes the proof. $\blacksquare$

Proposition 3.3. *For each positive integer n, the set $\mathcal{A}_n$ has empty interior.*

Proof. Given $\varepsilon > 0$, a positive integer n and a function $f \in \mathcal{A}_n$, let g be constructed as in the proof of Proposition 2.2. Then it is clear that the ball centered at f and of radius ε in $\mathcal{C}[0,1]$ contains g and that $g \notin \mathcal{A}_n$. This completes the proof. $\blacksquare$

We can now prove the main theorem.

Theorem 3.1. *There exist continuous functions on the interval $[0,1]$ which are nowhere differentiable. In fact the collection of all such functions forms a dense subset of $\mathcal{C}[0,1]$.*

Proof. By Baire's theorem and the two preceding propositions, it follows that

$$\mathcal{C}[0,1] \neq \bigcup_{n=1}^{\infty} \mathcal{A}_n.$$

From the definition of the sets $\mathcal{A}_n$ and from Proposition 3.1, it follows that every function in

$$\mathcal{C}[0,1] \Big\backslash \bigcup_{n=1}^{\infty}\mathcal{A}_n = \bigcap_{n=1}^{\infty}(\mathcal{C}[0,1]\backslash\mathcal{A}_n)$$

is nowhere differentiable and also that this set is dense, since it is the countable intersection of open dense sets. $\blacksquare$

In particular, it follows that every continuous function on $[0,1]$, irrespective of its smoothness, is the uniform limit of functions that are nowhere differentiable!

References

[1] S. Banach, Über die baire'sche Kategorie gewisser Funktionen mengen, *Studia Math.*, **3** (1931) 174–179.

[2] P. G. Ciarlet, Linear and nonlinear functional analysis with applications, *SIAM* (2013).

[3] G. H. Hardy, Weierstrass' non-differentiable function, *Trans. Amer. Math. Soc.*, **17** (1916) 301–325.

[4] S. Kesavan, Functional analysis, TRIM Series No. 52, Hindustan Book Agency (2009).

[5] S. Mazurkiewicz, Sur les fonctions non dérivables, *Studia Math.*, **3** (1931) 92–94.

[6] A. Pinkus, Weierstrass and approximation theory, *J. Approx. Th.*, **107** (2000) 1–66.

[7] W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, Second Edition, International Student Edition (1964).

On the Elementary Subgroup of GL_n Over Rings

Rabeya Basu
Indian Institute of Science Education and Research (IISER), Pune
E-mail: rabeya.basu@gmail.com, rbasu@iiserpune.ac.in

1. Introduction

In linear algebra, one studies the general linear group GL_n consisting of all $n \times n$ matrices which are 'invertible'. If the entries of a matrix M are from a commutative ring R with identity, the matrix M is said to be *invertible* if there is a matrix N (again with entries from R) so that $MN = NM = \mathrm{I}_n$, the $n \times n$ identity matrix. It is easy to show that M is invertible if, and only if, its determinant is an element of R which has a multiplicative inverse. The name 'general linear' comes from the following geometric fact. A set of at least $d + 1$ points in the d-dimensional Euclidean space is said to be in *general linear position* (or just *general position*) if no hyperplane contains more than d points. That is, the points do not satisfy any more linear relations than they must. The columns of an invertible matrix are linearly independent, and hence, the vectors they define are in general linear position. The matrices in the general linear group transform points in general linear position to points in general linear position. The set of $n \times n$ matrices with entries from R is denoted by $\mathrm{M}_n(R)$, and the group of $n \times n$ invertible matrices with entries from R is denoted by $\mathrm{GL}_n(R)$.

The simplest type of invertible matrix is a so-called elementary matrix. It has 1's as diagonal entries and exactly one non-zero entry in an off-diagonal position. Observe that an elementary matrix is in the special linear group $\mathrm{SL}_n(R)$ consisting of matrices which have determinant 1. Formally, we define:

Definition 1.1. *Let $e_{ij}(\lambda)$, where $i \neq j$, and $\lambda \in R$, denote the matrix $\mathrm{I}_n + \lambda \mathrm{E}_{ij}$, where E_{ij} is the matrix with 1 in the (i, j)-th position and zeros elsewhere. The subgroup of $\mathrm{GL}_n(R)$ generated by $e_{ij}(\lambda)$, $\lambda \in R$, is denoted by $\mathrm{E}_n(R)$. It is called the elementary subgroup of $\mathrm{GL}_n(R)$, and e_{ij}'s are called its elementary generators.*

The elementary subgroup $\mathrm{E}_n(R)$ plays a crucial role for the development of classical algebraic K-theory. It turns out that it is not always equal to the special linear group $\mathrm{SL}_n(R)$. But, if R is a field, then the groups coincide. The matrices $e_{ij}(\lambda)$'s are linear operators on row and column vectors. Indeed, observe that multiplying a row vector with $e_{ij}(\lambda)$ on the right, is the elementary column operation of adding λ times the i'th column to the j'th column. Similarly, the multiplication by $e_{ij}(\lambda)$ on the left can be described. The set $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ measures the obstruction to reducing an invertible matrix to the identity matrix by applying these linear operators. Thus, the question of normality of $\mathrm{E}_n(R)$ in $\mathrm{SL}_n(R)$ or $\mathrm{GL}_n(R)$ is of interest. It was studied by many mathematicians during the 1960's and the 70's. Finally, the Russian mathematician Andrei Suslin proved that $\mathrm{E}_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$ for $n \geq 3$. Interestingly, the result is not true for the case $n = 2$. We shall discuss some counter examples. We shall sketch the proof of Suslin's theorem in this article. For $n \geq 3$, Anthony Bak proved that the group $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$ is a solvable group. The purpose of this note is to give some glimpses of aspects of the elementary subgroup including a proof of Suslin's theorem. In the next article, we will discuss the role of the elementary subgroup in a famous problem posed by J.-P. Serre on projective modules.

2. Properties of $\mathrm{E}_n(R)$

We start with some examples of elements in the elementary subgroup and discuss some useful properties of this subgroup. Later, we will use these while proving the normality of $\mathrm{E}_n(R)$.

Example 2.1.

(i) The matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_2(R)$$

Note that it has finite order.

(ii) For each $a \in R$, the matrix

$$\mathrm{E}(a) = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix} = e_{12}(1-a)e_{21}(-1)e_{12}(1) \in \mathrm{E}_2(R).$$

(iii) Upper and lower triangular $n \times n$ matrices with 1's on the diagonal, are in $\mathrm{E}_n(R)$.

(iv) For $g \in \mathrm{M}_n(R)$, the matrices $\left(\begin{smallmatrix} \mathrm{I}_n & g \\ 0 & \mathrm{I}_n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathrm{I}_n & 0 \\ g & \mathrm{I}_n \end{smallmatrix}\right)$ are in $\mathrm{E}_{2n}(R)$.

Now, we list a few properties of $\mathrm{E}_n(R)$.

(i) $\mathrm{E}_n(R) \subset \mathrm{SL}_n(R) \subset \mathrm{GL}_n(R)$, for $n \geq 2$.
For $m \geq 1$, $n \geq 1$, the embedding $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+m}(R)$, given by $\alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \mathrm{I}_m \end{smallmatrix}\right)$, induces embeddings $\mathrm{E}_n(R) \to \mathrm{E}_{n+m}(R)$ and $\mathrm{SL}_n(R) \to \mathrm{SL}_{n+m}(R)$.
This allows us to define the groups

$$\mathrm{GL}(R) = \bigcup_{n=1}^{\infty} \mathrm{GL}_n(R), \quad \mathrm{SL}(R) = \bigcup_{n=1}^{\infty} \mathrm{SL}_n(R),$$

$$\mathrm{E}(R) = \bigcup_{n=1}^{\infty} \mathrm{E}_n(R).$$

(ii) (Splitting property): $e_{ij}(x+y) = e_{ij}(x)e_{ij}(y)\ \forall\, x, y \in R$, when $i \neq j$.

(iii) (Commutator formulas):
For all $x, y \in R$,

(a) $[e_{ij}(x), e_{kl}(y)] = 1$ if $j \neq k$, $i \neq l$,
(b) $[e_{ij}(x), e_{jk}(y)] = e_{ik}(xy)$ if $i \neq j$, $j \neq k$, $i \neq k$.

For $n \geq 3$, by using commutator formula one can deduce that $\mathrm{E}_n(R)$ is generated by the set $\{e_{1j}(\lambda), e_{i1}(\mu) \mid 1 \leq i, j \leq n, \lambda, \mu \in R\}$.

(iv) The subgroup $\mathrm{E}_2(R)$ is generated by the set $\{\mathrm{E}(a) \mid a \in R\}$, where

$$\mathrm{E}(a) = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}.$$

Indeed, as mentioned above, one can check that

$$\mathrm{E}(a) = e_{12}(1-a)e_{21}(-1)e_{12}(1).$$

Moreover, $e_{12}(a) = \mathrm{E}(-a)\mathrm{E}(0)^{-1}$ and $e_{21}(a) = \mathrm{E}(0)^{-1}\mathrm{E}(a)$.

(v) (Whitehead Lemma). If $\alpha \in \mathrm{GL}_n(R)$, then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in \mathrm{E}_{2n}R.$$

Further, $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] \subset \mathrm{E}_{2n}(R)$.
In particular, $[\mathrm{GL}(R), \mathrm{GL}(R)] \subset \mathrm{E}(R)$.
This is proved as follows.

If $\alpha \in \mathrm{GL}_n(R)$, then

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0 \\ \alpha^{-1} & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n - \alpha \\ 0 & \mathrm{I}_n \end{pmatrix} \times \begin{pmatrix} \mathrm{I}_n & 0 \\ -\mathrm{I}_n & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & \mathrm{I}_n - \alpha^{-1} \\ 0 & \mathrm{I}_n \end{pmatrix} \in \mathrm{E}_{2n}(R).$$

To prove the next assertion, let $\alpha, \beta \in \mathrm{GL}_n(R)$. Then

$$\begin{pmatrix} \alpha\beta\alpha^{-1}\beta^{-1} & 0 \\ 0 & \mathrm{I}_n \end{pmatrix} = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \beta^{-1}\alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix} \in \mathrm{E}_{2n}(R)$$

Therefore, $[\mathrm{GL}_n(R), \mathrm{GL}_n(R)] \subset \mathrm{E}_{2n}(R)$ and hence, $[\mathrm{GL}(R), \mathrm{GL}(R)] \subset \mathrm{E}(R)$. $\square$

The original Whitehead's Lemma is a topological assertion; the above matrix version is due to A. Suslin.

(vi) (Perfectness): $\mathrm{E}_n(R)$ is a perfect group, if $n > 2$.
In particular, $[\mathrm{GL}(R), \mathrm{GL}(R)] = \mathrm{E}(R)$.
On the other hand, $\mathrm{E}_2(\mathbb{F}_2)$ and $\mathrm{E}_2(\mathbb{F}_3)$ are not perfect.
It is easy to deduce the perfectness of $\mathrm{E}_n(R)$, for $n \geq 3$, using the commutator formulas above. It follows then that $\mathrm{E}(R)$ is perfect, i.e., $[\mathrm{E}(R), \mathrm{E}(R)] = \mathrm{E}(R)$. The last assertion is now a consequence of (vi).

(vii) Let I be an ideal in the ring R. Then the homomorphism

$$\mathrm{E}_n(R) \to \mathrm{E}_n(R/I)$$

is surjective, for all $n \geq 2$. Indeed, the generators $e_{ij}(\overline{\lambda})$ of $\mathrm{E}_n(R/I)$ can be lifted to the generators $e_{ij}(\lambda)$ of $\mathrm{E}_n(R)$.

Remarks.

(i) The analogue of the surjectivity above is not true for $\mathrm{SL}_n(R)$ in general. Here is an example.
Let $\langle XY - ZT - 1\rangle$ denote the ideal generated by the element $XY - ZT - 1$ in the ring $\mathbb{R}[X, Y, Z, T]$. Then the homomorphism

$$\mathrm{SL}_2(\mathbb{R}[X, Y, Z, T]) \to \mathrm{SL}_2\left(\frac{\mathbb{R}[X, Y, Z, T]}{\langle XY - ZT - 1\rangle}\right)$$

is not surjective. In fact, if bar denotes the reduction modulo the above ideal, then there is no lift of the matrix

$$\begin{pmatrix} \bar{X} & \bar{Z} \\ \bar{T} & \bar{Y} \end{pmatrix} \in \mathrm{SL}_2\left(\frac{\mathbb{R}[X, Y, Z, T]}{\langle XY - ZT - 1\rangle}\right)$$

to a matrix in $\mathrm{SL}_2(\mathbb{R}[X, Y, Z, T])$. There is a proof due to C. P. Ramanujam, *cf.* ([10], pg. 11). For a simplified version, see the Appendix of [4].

But, if we consider an ideal I in a ring R such that $\dim(R/I) = 0$ (here by 'dim' we mean the Krull dimension), then one can show that R/I will be a product of fields, and hence $\mathrm{SL}_n(R/I) = \mathrm{E}_n(R/I)$. Therefore, the canonical map from $\mathrm{SL}_n(R)$ to $\mathrm{SL}_n(R/I)$ will be surjective.

(ii) Apart from the above mentioned properties one can study properties of the elementary subgroup from the analytic and topological points of view if we take $R = \mathbb{R}$ or $\mathbb{C}$. The group $\mathrm{E}_n(R)$ is a Lie group in the above cases; it coincides with the special linear group over the real and complex fields. In the complex case, it is a non-compact, simply connected Lie group with a trivial fundamental group.

More generally, let X be a topological space, and let $B = \mathbb{R}^X$ be the ring of continuous functions $X \to \mathbb{R}$. In 1986, L. N. Vaserstein proved the normality of $\mathrm{E}_n(B)$ in $\mathrm{GL}_n(B)$, for $n \geq 3$. It was observed by A. Bak that if X is the union of a finite number (say, m) compact, contractible subspaces, then $\mathrm{SL}_n(B)/\mathrm{E}_n(B)$ is a nilpotent group of nilpotency class m. In particular, the group $\mathrm{GL}_n(B)/\mathrm{E}_n(B)$ is solvable for any finite cell complex X. In 1992, Vaserstein improved his observation further by proving that the group $\mathrm{SL}_n(B)/\mathrm{E}_n(B)$ is nilpotent for any finite dimensional topological space X. The interested reader may refer to [16] and [17].

3. When $n = 2$

In this section, we show that $\mathrm{E}_2(R)$ may not be a normal subgroup of $\mathrm{SL}_2(R)$. In fact, we look at $R = k[X, Y]$, where k is afield. If we consider the matrix

$$\alpha = \begin{pmatrix} 1 + XY & X^2 \\ -Y^2 & 1 - XY \end{pmatrix}$$

then, it was shown by P. M. Cohn that $\alpha \in \mathrm{SL}_2(R)$ but not in $\mathrm{E}_2(R)$.

P. M. Cohn has also proved that $\mathrm{E}_2(\mathbb{Z}[Y]) \neq \mathrm{SL}_2(\mathbb{Z}[Y])$ by proving the matrix

$$\begin{pmatrix} 1 + 2Y & 4 \\ -Y^2 & 1 - 2Y \end{pmatrix} \notin \mathrm{E}_2(\mathbb{Z}[Y]).$$

Consider the matrix $\beta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Using some facts from linear algebra, one can deduce that $\alpha\beta\alpha^{-1} \notin \mathrm{E}_2(A)$. For a detailed proof, see [7]. More examples occur in ([8], Section 1.8).

4. Normality of $\mathbf{E}_n(R)$

The systematic study of elementary subgroups E_n started with the works of Hyman Bass (*cf.* [2]). Then from the work J. S. Wilson (*cf.* [18]), L. N. Vaserstein (*cf.* [15]), A. Suslin (*cf.* [11]) *et al.* it emerged that the elementary subgroup $\mathrm{E}_n(R)$ of $\mathrm{GL}_n(R)$ plays a crucial role in the study of K-theory of *classical groups*. In connection with the K_1-analogue of Serre's problem (which we shall discuss in a future article), A. Suslin proved the normality of the elementary subgroup $\mathrm{E}_n(R)$ in the general linear group $\mathrm{GL}_n(R)$, for $n \geq 3$ (*cf.* [14]). Later similar results for the symplectic and orthogonal groups were proven by A. Suslin and V. Kopeiko in [11] and [12] and by Fu An Li in [6]; for arbitrary Chevalley groups, these types of results were proved by Abe (*cf.* [1]) in the local case, and by G. Taddei (*cf.* [13]) in general.

In this section, we describe the proof of Suslin's theorem, using a variant of Whitehead's Lemma due to L. N. Vaserstein. For the detailed proof of this lemma, the reader is encouraged to consult the book [7]. The theorem of Suslin alluded to above is:

Theorem 4.1 (A. Suslin). *Let R be a commutative ring with identity. The elementary subgroup $\mathrm{E}_n(R)$ is normal in $\mathrm{GL}_n(R)$, for $n \geq 3$.*

Lemma 4.2 (L. N. Vaserstein). *Let $\mathrm{M}_{r,s}(R)$ denote the set of $r \times s$ matrices over R. Let $\alpha \in \mathrm{M}_{r,s}(R)$ and $\beta \in \mathrm{M}_{s,r}(R)$. If $\mathrm{I}_r + \alpha\beta \in \mathrm{GL}_r(R)$, then $\mathrm{I}_s + \beta\alpha \in \mathrm{GL}_s(R)$ and*

$$\begin{pmatrix} \mathrm{I}_r + \alpha\beta & 0 \\ 0 & \mathrm{I}_s + \beta\alpha \end{pmatrix} \in \mathrm{E}_{r+s}(R).$$

Proof. Note that

$$\mathrm{I}_s - \beta(\mathrm{I}_r + \alpha\beta)^{-1}\alpha$$

is easily verified to be the inverse of $(\mathrm{I}_s + \beta\alpha)$; a nice way to arrive at this expression is to view the sought-for inverse in analogy with a geometric series.

Hence, the invertibility of $\mathrm{I}_r + \alpha\beta$ implies that of $\mathrm{I}_s + \beta\alpha$ and vice versa. Moreover,

$$\begin{pmatrix} \mathrm{I}_r + \alpha\beta & 0 \\ 0 & (\mathrm{I}_s + \beta\alpha)^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_r & 0 \\ (\mathrm{I}_s + \beta\alpha)^{-1}\beta & \mathrm{I}_s \end{pmatrix} \begin{pmatrix} \mathrm{I}_r & -\alpha \\ 0 & \mathrm{I}_s \end{pmatrix}$$
$$\times \begin{pmatrix} \mathrm{I}_r & 0 \\ -\beta & \mathrm{I}_s \end{pmatrix} \begin{pmatrix} \mathrm{I}_r & (\mathrm{I}_r + \alpha\beta)^{-1}\alpha \\ 0 & \mathrm{I}_s \end{pmatrix}.$$

The lemma follows now from the fact (which we already noted and used) that a triangular matrix with 1 in the diagonal is a product of elementary matrices. □

Observation. As a consequence of the above lemma, if $v = (v_1, \ldots, v_n)^t$ and $w = (w_1, \ldots, w_n)^t$ are two column vectors with the property that the dot product $w^t v = 0$, then the matrix $\mathrm{I}_n + vw^t$ is invertible, and 1-stably elementary, i.e.,

$$\begin{pmatrix} \mathrm{I}_n + vw^t & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{E}_{n+1}(R).$$

In the proof of Suslin's theorem, we will use the following lemma. Before stating, we give a definition. We view elements of R^n as column vectors. A vector v is said to be *unimodular*, if the ideal generated by the v_i's is the unit ideal.

Lemma 4.3. *If v is a unimodular vector and $f : R^n \to R$ is the R-linear map given by $e_i \mapsto v_i$ (e_i being the canonical basis of R^n, for $1 \le i \le n$), then*

$$\ker(f) = \left\{ w = (w_1, \ldots, w_n)^t \in R^n \mid \sum_{i=1}^{n} w_i v_i = 0 \right\}$$

is generated by the elements

$$\{v_j e_i - v_i e_j \mid 1 \le i \le n\}.$$

For a detailed proof of this lemma, see [7].

Important Remark. *If we start with a unimodular vector v, then $\mathrm{I}_n + vw^t \in \mathrm{E}_n(R)$.*

Now we can deduce the proof of Theorem 4.1.

Proof of Suslin's Theorem 4.1. *As $\mathrm{E}_n(R)$ is generated by $e_{ij}(\lambda)$; $\lambda \in R$, it is sufficient to show that for $\alpha \in \mathrm{GL}_n(R)$ we must have $\alpha e_{ij}(\lambda)\alpha^{-1} \in \mathrm{E}_n(R)$. Let α_i and β_i ($1 \le i \le n$) be the i-th column of α and i-th row of α^{-1} respectively. Then*

$$\alpha e_{ij}(\lambda)\alpha^{-1} = \alpha(\mathrm{I}_n + \lambda \mathrm{E}_{ij})\alpha^{-1} = \mathrm{I}_n + \lambda\alpha_i\beta_j.$$

Since $\alpha, \alpha^{-1} \in \mathrm{GL}_n(R)$, we observe that the ideal generated by the entries of α_i for any i is the unit ideal; likewise, the entries of β_j generate the unit ideal for each j.

Also, $\alpha^{-1}\alpha = \mathrm{I}_n$ implies that $\beta_j\alpha_i = 0$, for $j \ne i$. Hence from the above mentioned remark, it follows that $\mathrm{I}_n + \lambda\alpha_i\beta_j \in \mathrm{E}_n(R)$. □

Remark. If we take $v = (X, -Y)^t$, and $w = (Y, X)^t$. Then $w^t v = 0$. Hence, P. M. Cohn's example

$$\begin{pmatrix} \mathrm{I}_2 + vw^t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + XY & X^2 \\ -Y^2 & 1 - XY \end{pmatrix} \in \mathrm{E}_3(A)$$

under the natural inclusion; i.e., 1-stably elementary. Hence, Cohn's matrix is stably elementary even though it is not in the elementary subgroup; there is no contradiction to the important remark made above, because v, w are not unimodular.

Acknowledgement

The author gratefully acknowledges Prof. B. Sury, for his encouragement and editorial inputs.

References

[1] E. Abe, Chevalley groups over local rings, *Tohoku Math. J.*, **21**(2) (1969) 474–494.

[2] H. Bass, Algebraic K-theory, Benjamin, New York-Amsterdam (1968).

[3] H. Bass, Introduction to some methods of algebraic K-theory, CBMS, Number 20. AMS.

[4] R. Basu, Ph.D. Thesis. Available on the site – `https://sites.google.com/site/rabeyabasu/`

[5] R. Basu, Raja Sridharan; Forster's conjecture and related results. Available on the site – `https://sites.google.com/site/rabeyabasu/`

[6] Fu An Li, The structure of orthogonal groups over arbitrary commutative rings, *Chinese Ann. Math. Ser. B*, 10 (1989) 341–350.

[7] S. K. Gupta and M. P. Murthy, Suslin's proof of Serre's conjecture, ISI lecture note.

[8] T. Y. Lam, Serre's problem on projective modules, Springer.

[9] B. R. McDonald, Linear algebra over commutative rings, Marcel Dekker, INC.

[10] C. P. Ramanujam, A TRIBUTE, Tata Institute of Fundamental Research Studies in Mathematics, Springer – Verlag (1978).

[11] A. A. Suslin, On the structure of special Linear group over polynomial rings, *Math. USSR. Izv.*, 11 (1977) 221–238.

[12] A. A. Suslin and V. I. Kopeiko, Quadratic modules and Orthogonal groups over polynomial rings, *Nauchn. Sem.*, LOMI, 71 (1978) 216–250.

[13] G. Taddei, Normalite des groups elementaries dans les groupes de Chevalley sur an anneau, Application of Algebraic K-Theory to Algebraic Geometry and Number Theory, Part II (Boulder, Colo., 1983), *Contemp. Math.*, 55, *Amer. Math. Soc.*, Providence, RI (1986) 693–710.

[14] M. S. Tulenbaev, Schur multiplier of a group of elementary matrices of finite order, Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instuta im V. A. Steklova Akad. Nauk SSSR, 86 (1979) 162–169.

[15] L. N. Vaserstein, On the normal subgroups of GL_n over a ring, Algebraic K-theory, Evanston (1980) (Proc. Conf., Northwestern Univercisy, Evanston, Ill., (1980)), pp. 456–465, Lecture Notes in Mathematics, 854, Springer, Berlin-New York (1981).

[16] L. N. Vaserstein, On K_1-theory of topological spaces, *Amer. Math. Soc. Cont. Math.*, 55 (1986) 729–740.

[17] L. N. Vaserstein, Unstable K_1-theory of topological spaces is nilpotent, *Cont. Math.*, 126 (1992).

[18] J. Wilson, The normal and subnormal structure of gneral Linear groups, *Proc. Camp. Phil. Soc.*, 71 (1972) 163–177.

Weyl-Heisenberg Frame Operators

T. C. Easwaran Nambudiri

Department of Mathematics, Government College, Kasaragod, Kerala 671 123, India

K. Parthasarathy

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, India

Elementary theory of Fourier series tells us that the exponential functions $e_m(x) = e^{2\pi \imath m x}$, $m \in \mathbb{Z}$, form an orthonormal basis for the space of square integrable functions of period one on the real line. This is equivalent to saying that $\{\chi.e_m : m \in \mathbb{Z}\}$ is an orthonormal basis for the space M_0 of square integrable functions on the real line that vanish outside the unit interval [0,1), where $\chi = \chi_{[0,1)}$ is the characteristic function of the interval. Thus, if T_n are the translation operators defined by $T_n f(x) = f(x-n)$, then $\{e_m.T_n\chi\}$ is an orthonormal basis for M_n, the subspace of $L^2(\mathbb{R})$ consisting of functions supported on $[n, n+1)$, $n \in \mathbb{Z}$. Since $L^2(\mathbb{R})$ is the orthogonal direct sum $\oplus_{n\in\mathbb{Z}}\mathsf{M}_n$, we get an orthonormal basis $\{e_m.T_n\chi : n, m \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$. In particular we have, as a consequence of the Parseval identity, $\|f\|^2 = \sum |\langle f, e_m.T_n\chi\rangle|^2$ for all $f \in L^2(\mathbb{R})$.

Consider now the space ℓ^2 of square summable sequences and let u_n be the sequence given by $u_n(m) = \frac{m}{m+1}\delta_{mn}$, $m, n \in \mathbb{N}$. Then, since $\frac{1}{2} \leq \frac{n}{n+1} < 1$ for all n, the sequence (u_n) satisfies the double inequalities

$$\frac{1}{4}\|x\|^2 \leq \sum |\langle x, u_n\rangle|^2 \leq \|x\|^2, x \in \ell^2.$$

In other words, we have two inequalities – an upper and a lower – in place of equality that we have had in the case of the orthonormal basis of the first example. Such a sequence is called a frame. The trade-off is that these objects, although mathematically weaker, are much more pliable for practical purposes. We still have series expansions: every element of the space has a series representation in terms of the frame sequence, but the coefficients are no longer unique. Rather than considering this as a setback, this could be viewed as giving flexibility to choose coefficients suitable to a given situation. This nonuniqueness also makes the mathematical theory very interesting.

Gabor, in laying the theoretical foundations of communication theory and signal processing, initiated the study of time-frequency analysis in 1946 ([6]) and, with the fundamental work of Janssen (see, e.g., [8]), it became an independent topic of mathematical investigation in the 1980's. Systematic utilisation of time shifts (translations) and frequency shifts (modulations) lie at the heart of modern time-frequency analysis. Frames were introduced as early as 1952 by Duffin and Schaeffer ([3]) in their study of nonharmonic Fourier series (or, nonuniform sampling of bandlimited functions, in the language of time-frequency analysis). It was the ground breaking work of Daubechies, Grossmann and Meyer in 1986 ([2]) that brought about the confluence of the two fields to create the theory of Gabor frames or Weyl-Heisenberg frames. It is still an extremely active area of research not only among mathematicians but also among electrical and communication engineers.

One of the most important objects in the theory, both from the theoretical and applications points of view, is the frame operator. In this expository article, we take a brief look at elementary theoretical aspects of Weyl-Heisenberg frames and their frame operators, concluding with a simple characterisation of those operators which can occur as frame operators of such frames. These frames in $L^2(\mathbb{R})$ are generated by a single function *via* translations and modulations, as in the opening example, and are among the most useful in applications. We begin with a few basics of the needed abstract Hilbert space frame theory. Only elementary Fourier analysis and basic facts about Hilbert spaces and operators on them are assumed.

Our basic references for both abstract Hilbert frame theory and the theory of Weyl-Heisenberg frames are [1] and [7]. We refer to these for unproved statements and uncited references in this article.

Definition 1. *A countable family (u_n) of vectors in a Hilbert space H is called a frame for H if there are positive constants α, β such that*

$$\alpha\|x\|^2 \le \sum |\langle x, u_n\rangle|^2 \le \beta\|x\|^2 \quad \textit{for all } \; x \in \mathsf{H}.$$

Note that the frame bounds α, β are not unique. The frame (u_n) is called a tight frame if the frame inequalities are satisfied with $\alpha = \beta$ and a normalised tight frame if $\alpha = \beta = 1$. An exact frame is a frame (u_n) with the property that $(u_n)_{n\neq k}$ is not a frame for any index k.

Here is a useful observation that is a trivial consequence of the definition.

Proposition 1. *For a frame (u_n) in H with upper frame bound β, $\|u_n\| \le \sqrt{\beta}$ for all n. In particular, $\|u_n\| \le 1$ for all n when it is a normalised tight frame.*

Proof. For any $x \in \mathsf{H}$ and any positive integer n, $|\langle x, u_n\rangle|^2 \le \beta\|x\|^2$ and the result follows on taking $x = u_n$. □

It is easy to generate many examples of frames from a given orthonormal basis or a frame. Here is a sample.

Proposition 2. *If (u_n) is a frame in H and A is an invertible bounded linear operator on H, then (Au_n) is a frame; if (u_n) is exact, so is (Au_n). If A is unitary, (Au_n) has the same frame bounds as (u_n).*

Proof. The first part follows because in this case A^* is bounded both above and below:

$\|(A^*)^{-1}\|^{-1}\|x\| \le \|A^*x\| \le \|A^*\|\|x\|$ for all $x \in \mathsf{H}$ and consequently

$$\begin{aligned}\sum |\langle x, Au_n\rangle|^2 &= \sum |\langle A^*x, u_n\rangle|^2 \le \beta\|A^*x\|^2\\ &\le \beta\|A^*\|^2\|x\|^2,\\ \sum |\langle x, Au_n\rangle|^2 &= \sum |\langle A^*x, u_n\rangle|^2 \ge \alpha\|A^*x\|^2\\ &\ge \alpha\|(A^*)^{-1}\|^{-2}\|x\|^2, x \in \mathsf{H}.\end{aligned}$$

Note that the second part is a consequence of the first since if $(Au_n)_{n\neq k}$ is a frame, then so is $(u_n)_{n\neq k} = (A^{-1}(Au_n))_{n\neq k}$. If A is unitary, $\|A^*\| = \|A\| = 1$ and so the last part is clear from proof of the first part. □

Proposition 3. *Let H be a Hilbert space. If (u_n) is a frame for H, then the u_n span a dense subspace of H.*

Proof. If $x \in \mathsf{H}$ and $\langle x, u_n\rangle = 0$ for all n, then the first of the defining inequalities for a frame implies $\|x\| = 0$, so $x = 0$. □

Corollary 1. *If a Hilbert space H has a frame, then it is separable.*

Proof. If (u_n) is a frame for H, then finite linear combinations $\sum c_j u_j$, with real and imaginary parts of c_j rational, is dense in H. □

Here is a useful result showing that the frame inequalities need be verified only on a dense subspace. The proof is not difficult.

Proposition 4. *Let (u_n) be a sequence in H. If there are constants α, β satisfying $\alpha\|x\|^2 \leq \sum_n |\langle x, u_n\rangle|^2 \leq \beta\|x\|^2$ for all x in a dense subspace D of H, then (u_n) is a frame for H.*

Recall that if (e_n) is an orthonormal basis for H, then $\sum c_n e_n$ converges for every square summable sequence (c_n). This property carries over to frames.

Proposition 5. *If (u_n) is a frame in H, then for every sequence $(c_n) \in \ell^2$, the series $\sum c_n u_n$ converges in H.*

Proof. It suffices to show that the sequence $(s_n), s_n = \sum_1^n c_k u_k$, of partial sums is a Cauchy sequence in H. If β is an upper frame bound then, for $n > m$, we have, using Cauchy-Schwarz inequality and the upper frame inequality,

$$\begin{aligned}\|s_n - s_m\| &= \sup\{|\langle s_n - s_m, y\rangle| : \|y\| \leq 1\} \\ &= \sup\left\{\left|\left\langle \sum_{m+1}^{n} c_k u_k, y\right\rangle\right| : \|y\| \leq 1\right\} \\ &\leq \sup\left\{\sum_{m+1}^{n} |c_k\langle u_k, y\rangle| : \|y\| \leq 1\right\} \\ &\leq \left(\sum_{m+1}^{n} |c_k|^2\right)^{1/2} \\ &\quad \times \sup\left\{\left(\sum_{m+1}^{n} |\langle u_k, y\rangle|^2\right)^{1/2} : \|y\| \leq 1\right\} \\ &\leq \sqrt{\beta}\left(\sum_{m+1}^{n} |c_k|^2\right)^{1/2}.\end{aligned}$$

Thus (s_n) is a Cauchy sequence and the result is proved. □

Remark 1. Note that only the upper frame inequality is used in the proof above. If (u_n) satisfies the upper frame inequality $\sum |\langle x, u_n\rangle|^2 \leq \beta\|x\|^2, x \in \mathsf{H}$, it is usually called a *Bessel sequence*. For the operator S defined below to be bounded, it suffices to assume that (u_n) is a Bessel sequence.

Proposition 6. *Let (u_n) be a frame in H. Then S defined by $Sx = \sum\langle x, u_n\rangle u_n$ for $x \in \mathsf{H}$ is a bounded linear operator on H, called the* frame operator*. It has the following properties:*

i) *S is an invertible, positive operator; in fact, $\alpha I \leq S \leq \beta I$, α, β being frame bounds.*
ii) $\beta^{-1}I \leq S^{-1} \leq \alpha^{-1}I$.

Proof. $\sum |\langle x, u_n\rangle|^2 < \infty$ from the definition of a frame and so the previous proposition ensures that the series $\sum\langle x, u_n\rangle u_n$ converges in H. This means $Sx \in \mathsf{H}$ for $x \in \mathsf{H}$ and gives the map $S : \mathsf{H} \to \mathsf{H}$ that is clearly linear. An easy calculation shows that $\langle Sx, x\rangle = \sum |\langle x, u_n\rangle|^2$ and so

$$\alpha\|x\|^2 \leq \langle Sx, x\rangle \leq \beta\|x\|^2, x \in \mathsf{H}.$$

This proves that S is bounded with $\|S\| \leq \beta$ and satisfies i). Finally, ii) is a consequence of i). □

An immediate consequence is the following useful observation.

Corollary 2. *The frame operator of a tight frame is a scalar multiple of the identity and that of a normalised tight frame is the identity operator.* □

A frame and its frame operator together give rise to new frames and we now present two such frames canonically associated to the given frame. Recall that a positive operator S has a unique positive square root $S^{1/2}$.

Proposition 7. *Let (u_n) be a frame in H with frame operator S. Then*

i) *$(S^{-1}u_n)$ is a frame and ii) $(S^{-1/2}u_n)$ is a normalised tight frame.*

Proof.

i) Since S^{-1} is an invertible operator, we already know that $(S^{-1}u_n)$ is a frame. But in this case we get neater estimates as follows. For $x \in \mathsf{H}, Sx = \sum\langle x, u_n\rangle u_n$, so $x = S(S^{-1}x) = \sum\langle S^{-1}x, u_n\rangle u_n$. Thus $\langle S^{-1}x, x\rangle = \langle S^{-1}x, \sum\langle S^{-1}x, u_n\rangle u_n\rangle = \sum |\langle S^{-1}x, u_n\rangle|^2 = \sum |\langle x, S^{-1}u_n\rangle|^2$. In view of the inequalities $\beta^{-1}I \leq S^{-1} \leq \alpha^{-1}I$, this leads to the inequalities $\beta^{-1}\|x\|^2 \leq \sum |\langle x, S^{-1}u_n\rangle|^2 \leq \alpha^{-1}\|x\|^2$ for all $x \in \mathsf{H}$.
ii) For $x \in \mathsf{H}$ we have

$$\begin{aligned}x &= S^{-1/2}(S(S^{-1/2}x)) = S^{-1/2}\left(\sum\langle S^{-1/2}x, u_n\rangle u_n\right) \\ &= \sum\langle x, S^{-1/2}u_n\rangle S^{-1/2}u_n.\end{aligned}$$

This gives $\|x\|^2 = \langle x, x\rangle = \sum |\langle x, S^{-1/2}u_n\rangle|^2$ which is precisely the desired conclusion that $(S^{-1/2}u_n)$ is a normalised tight frame. □

Definition 2. *The frame* $(\tilde{u}_n) = (S^{-1}u_n)$ *is called the (canonical) dual frame of the frame* (u_n).

The dual frame plays a vital role in frame theory. Most basically, it gives rise to *reconstruction formulas* or *frame expansions*.

Proposition 8. *Let* (u_n) *be a frame in* H *with dual frame* $(\tilde{u}_n)$. *Then*

$$x = \sum \langle x, \tilde{u}_n\rangle u_n = \sum \langle x, u_n\rangle \tilde{u}_n \quad \text{for all } x \in \mathsf{H}.$$

Proof. The first of the stated expansions is got by replacing x by $S^{-1}x$ in $Sx = \sum\langle x, u_n\rangle u_n$ (as seen in the proof of the previous proposition) and the second is obtained by applying S^{-1} on both sides of the same identity. □

When (u_n) is a normalised tight frame, S is the identity operator and such a frame is self-dual. In particular, this holds for an orthonormal basis and in this case the frame expansion reduces to the familiar 'Fourier expansion' in terms of an orthonormal basis.

The frame expansion leads to the 'duality theorem' for frames.

Corollary 3. *If S is the frame operator of the frame* (u_n), *then the frame operator of the dual frame* $(\tilde{u}_n)$ *is* S^{-1} *and the dual frame of* $(\tilde{u}_n)$*(i.e. the second dual of* (u_n)*) is* (u_n) *itself.*

Proof. If $\tilde{S}$ is the frame operator of the dual frame $(\tilde{u}_n)$, then

$$\tilde{S}x = \sum\langle x, S^{-1}u_n\rangle\tilde{u}_n = \sum\langle S^{-1}x, u_n\rangle\tilde{u}_n = S^{-1}x,$$

proving the first assertion. The second follows from the first because the dual frame of $(\tilde{u}_n)$ is $(\tilde{S}^{-1}\tilde{u}_n) = (S\tilde{u}_n) = (u_n)$. □

The frame operator S of a frame (u_n) in a Hilbert space H is positive and invertible. We conclude our discussion on abstract Hilbert frames by observing that, conversely, every positive, invertible operator can be realised as the frame operator of a suitable frame. This simple, interesting characterisation of frame operators seems to have first appeared in [4].

Theorem 1. *Let* H *be a separable Hilbert space and let* $S \in \mathscr{B}(\mathsf{H})$ *be invertible and positive. Then there is a frame (in fact, an exact frame)* (u_n) *in* H *whose frame operator is* S.

Proof. Let (x_n) be an orthonormal basis for H and let $u_n = S^{1/2}x_n$. Then (u_n) is an exact frame. If S' is the frame operator of this frame, then

$$\begin{aligned} S'x &= \sum\langle x, u_n\rangle u_n \\ &= S^{1/2}\left(\sum\langle S^{1/2}x, x_n\rangle x_n\right) = S^{1/2}(S^{1/2}x) = Sx, x \in \mathsf{H}. \end{aligned}$$

Thus $S' = S$ and the proof is complete. □

We are now ready for the special class of frames that are our main concern.

Definition 3. *For* $a \in \mathbb{R}$ *the translation operator* T_a *and the modulation operator* E_a *are defined on functions by* $T_a f(x) = f(x-a)$ *and* $E_a f(x) = e^{2\pi\imath ax}f(x)$. *These are both unitary operators on* $L^2(\mathbb{R})$. *A frame in* $L^2(\mathbb{R})$ *of the form* $(g, a, b) := \{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$, *where* $g \in L^2(\mathbb{R})$ *and* a, b *are positive real numbers, is called a Weyl-Heisenberg frame or a Gabor frame. The function* g *is called the generator or the window function of the frame and* a, b *are known as the frame parameters.*

It is worth noting that it is the value of the product ab that determines whether we can construct a frame from a given function or not, rather than the individual values of the parameters (even though, as we shall see later, every pair a, b with $ab \le 1$ gives a frame, even a normalised tight frame, (g, a, b) for some g). More precisely, we have the following simple result.

Proposition 9. *Let* $a, b > 0$. *Let* D_a *be the dilation operator defined by* $D_a f(x) = \sqrt{a}f(ax)$. *Then* D_a *is a unitary operator on* $L^2(\mathbb{R})$ *and* $D_aE_{mb}T_{na} = E_{mab}T_nD_a, m, n \in \mathbb{Z}$. *In particular, if* (g, a, b) *is a frame, then so is* $(D_ag, 1, ab)$ *with the same frame bounds.*

Proof. The first part is a direct verification and the second part follows from the first since D_a is unitary. □

The generator of a Weyl-Heisenberg frame is necessarily bounded. In fact, we have the following stronger, more precise result.

Theorem 2. *Suppose (g, a, b) is a Weyl-Heisenberg frame in $L^2(\mathbb{R})$ with bounds $0 < \alpha \le \beta$. Then*

$$\alpha b \le G_0(x) := \sum_n |g(x - na)|^2 \le \beta b \quad \text{a.e. on } \mathbb{R}.$$

In general, it is difficult to determine which $g \in L^2(\mathbb{R})$ generate frames and for which values of the parameters. For example, even as simple a function as the characteristic function of an interval $[0, c)$ poses difficulties – a detailed analysis has been carried out by Janssen. As another example, we mention the deep result that (e^{-x^2}, a, b) is a frame if and only if $ab < 1$. In view of such complex behaviour, any sufficient condition is welcome. Here is a sufficient condition, due to Casazza and Christensen, for (g, a, b) to be a frame. It is easy to see that the series $\sum_n g(x-na)\bar{g}(x-na-k/b)$ converges absolutely a.e. for all $k \in \mathbb{Z}$ when $g \in L^2(\mathbb{R})$. Write $G_k(x)$ for the sum.

Theorem 3. *Let $\alpha := \frac{1}{b}\inf_{0\le x\le a}\left(G_0(x) - \sum_{k\neq 0}|G_k(x)|\right)$, $\beta := \frac{1}{b}\sup_{0\le x\le a}\sum|G_k(x)|$. If $\alpha > 0$ and $\beta < \infty$, then (g, a, b) is a frame with bounds α, β.*

Example 1. Take $a = 1 = b$ and let g be the function defined on $\mathbb{R}$ by

$$g(x) = \begin{cases} x, & 0 < x \le 1, \\ 2x, & 1 < x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $0 < x \le 1$, $G_0(x) = x^2 + 4(x+1)^2 = 5x^2 + 8x + 4$, $G_1(x) = 2x^2 + 2x = G_{-1}(x)$. Thus $(g, 1, 1)$ is a frame with bounds 4, 25.

Corollary 4. *Suppose $g \in L^2(\mathbb{R})$ is supported in an interval of length $1/b$. Then (g, a, b) is a frame with bounds α, β if and only if $b\alpha \le G_0(x) \le b\beta$ a.e. When the condition is satisfied, the frame operator S is just the multiplication operator given by $Sf = \frac{1}{b}G_0 f$, so that $S^{-1}f = b\frac{1}{G_0}f$ for all $f \in L^2(\mathbb{R})$.*

Proof. Observe that, in the present situation, $G_k = 0$, $k \neq 0$. Thus

$$\begin{aligned}\langle Sf, f\rangle &= \sum_{m,n}|\langle f, E_{mb}T_{na}\rangle|^2 \\ &= \frac{1}{b}\int_{\mathbb{R}} G_0(x)|f(x)|^2 dx = \frac{1}{b}\langle G_0 f, f\rangle\end{aligned}$$

for all compactly supported $f \in L^2(\mathbb{R})$ and hence, by continuity, for all $f \in L^2(\mathbb{R})$. This proves the assertion about S and that about its inverse is an immediate consequence. □

Corollary 5. *Let g be a continuous function which is supported in an interval of length $1/b$ on whose interior it is nonvanishing. Then G_0 is bounded a.e. above and below and (g, a, b) is a frame whenever $a > 0$ is such that $ab < 1$.*

Proof. Suppose $\left[x_0 - \frac{1}{2b}, x_0 + \frac{1}{2b}\right]$ is the support of g. Fix $a > 0$ with $ab < 1$. Given $x \in \mathbb{R}$, $x - na \in [x_0 - a/2, x_0 + a/2] \subset \left(x_0 - \frac{1}{2b}, x_0 + \frac{1}{2b}\right)$ for some $n \in \mathbb{Z}$. Letting $\alpha := \inf\{|g(t)|^2 : x_0 - a/2 \le t \le x_0 + a/2\}$, we get that $G_0(x) \ge \alpha > 0$. To get an upper bound, observe that, by the condition on the support of g, there is an m independent of x such the series defining $G_0(x)$ has at most m nonzero terms. (The interested reader can try to find an exact value for m.) This means that $G_0(x) \le m\|g\|_\infty^2$ and the proof is complete in view of the theorem. □

Example 2. Consider the triangular function

$$g(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then g is a continuous function whose support is [-1,1] and $g(x) > 0$ for $-1 < x < 1$. Thus $(g, a, 1/2)$ is a frame whenever $0 < a < 2$. For example, suppose $a = 1$. Then, for $0 \le x \le 1$, we have $G_0(x) = (1 - x)^2 + x^2$ and so $1/2 \le G_0(x) \le 1$. Hence $(g, 1, 1/2)$ is a frame with bounds 1 and 2.

The verification of the following lemma needs nothing more than a bit of patient, simple and careful computation.

Lemma 1. *The frame operator S of a Weyl-Heisenberg frame (g, a, b) commutes with all the translations T_{na} and the modulations E_{mb}, $m, n \in \mathbb{Z}$.*

It is a nice and important fact that the dual frame of a Weyl-Heisenberg frame is again a Weyl-Heisenberg frame.

Theorem 4. *The (canonical) dual frame of a Weyl-Heisenberg frame is a Weyl-Heisenberg frame. In fact, if (g, a, b) is a Weyl-Heisenberg frame with frame operator S, then its dual frame is $(S^{-1}g, a, b)$.*

Proof. In view of the lemma above, S^{-1} commutes with T_{na} and E_{mb} and so this is immediate because the dual frame of (g, a, b) is

$$\{S^{-1}E_{mb}T_{na}g : m, n \in \mathbb{Z}\} = \{E_{mb}T_{na}S^{-1}g : m, n \in \mathbb{Z}\}. \quad \square$$

Lemma 2. *If (g, a, b) is a Weyl-Heisenberg frame with frame operator S, then $ab = \|S^{-1/2}g\|_2^2$.*

Proof. Since S commutes with the translations T_{na} and modulations E_{mb}, so does $S^{-1/2}$ and consequently

$$\{E_{mb}T_{na}S^{-1/2}g : m, n \in \mathbb{Z}\} = \{S^{-1/2}E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$$

is the canonical normalised tight frame associated with (g, a, b). Thus, by Theorem 2, $\sum |S^{-1/2}g(x - na)|^2 = b$ a.e. Integration gives

$$\begin{aligned} ab &= \sum \int_0^a |S^{-1/2}g(x - na)|^2 dx \\ &= \int_{\mathbb{R}} |S^{-1/2}g(x)|^2 dx = \|S^{-1/2}g\|_2^2. \end{aligned} \quad \square$$

The next result shows that the frame conditions put severe restrictions on the frame parameters: if $ab > 1$ there is no g for which (g, a, b) is a frame. This result has a long history and the original proof, due M.A.Rieffel, made heavy use of the theory of von Neumann algebras. Many simplifications have been found since and currently a simple, elementary proof is available.

Theorem 5. *Let a, b be positive real numbers. If there is a $g \in L^2(\mathbb{R})$ such that (g, a, b) is a Weyl-Heisenberg frame, then $ab \leq 1$.*

Proof. If (g, a, b) is a frame, then $(S^{-1/2}g, a, b)$ is a normalised tight frame and so its members have norms at most one (Proposition 1). In particular, $\|S^{-1/2}g\|_2^2 \leq 1$. Now the theorem is immediate from the lemma. $\square$

The converse of this result is true and is easy to obtain.

Proposition 10. *For each pair of translation and modulation parameters a, b satisfying the condition $0 < ab \leq 1$, there exists a $g \in L^2(\mathbb{R})$ such that (g, a, b) is a normalised tight Weyl-Heisenberg frame which, therefore, has the identity operator as its frame operator.*

Proof. This is really a simple generalised version of our opening example and could have been discussed at the very beginning. Let a, b be positive real numbers with $ab \leq 1$ and let $g = \sqrt{b}\chi_{[0,a)}$. For $f \in L^2(\mathbb{R})$, let $f_n = f\chi_{[na,(n+1)a)}$, $n \in \mathbb{Z}$. Then $\|f\|_2^2 = \sum \|f_n\|_2^2$. Each f_n is supported in an interval of length $a \leq \frac{1}{b}$ and $\{\sqrt{b}e_{mb} : m \in \mathbb{Z}\}$ is an orthonormal basis for L^2 on any interval of length $\frac{1}{b}$. Thus the Fourier coefficients of f_n are given by

$$\langle f_n, \sqrt{b}e_{mb}\rangle = \int_{na}^{(n+1)a} f(x)\sqrt{b}e^{-2\pi \imath mbx}dx = \langle f, E_{mb}T_{na}g\rangle,$$

and using Parseval identity for each f_n we get the norm of f:

$$\|f\|_2^2 = \sum \|f_n\|_2^2 = \sum_{m,n} |\langle f, E_{mb}T_{na}g\rangle|^2.$$

Thus (g, a, b) is a normalised tight frame and consequently has the identity operator as the frame operator. Observe that this normalised tight frame is actually an orthonormal basis when $ab = 1$. $\square$

We have seen that every positive, invertible bounded linear operator on a separable Hilbert space H is the frame operator of some frame in H. To conclude our discussion on Weyl-Heisenberg frames, we therefore take a look at a natural question: Which operators on $L^2(\mathbb{R})$ can arise as Weyl-Heisenberg frame operators? Since a Weyl-Heisenberg frame operator necessarily commutes with the relevant translations and modulations, not every positive, invertible operator can be expected to be the frame operator of a Weyl-Heisenberg frame. It is, in fact, easy to think of many such operators which can not be Weyl-Heisenberg frame operators. (For instance, take a positive, bounded measurable function φ which is nonperiodic and is bounded away from zero – e.g. $\varphi(x) = 1 + e^{-x^2}$. The multiplication operator M_φ on $L^2(\mathbb{R})$ is positive, invertible. It does not commute with any nontrivial translation and so can not be a Weyl-Heisenberg frame operator.) But the answer to our question turns out to be surprisingly simple, elegant and rather unexpected – this commutation property is the only extra condition needed. We now round off our discussion on Weyl-Heisenberg frames with this nice characterisation of Weyl-Heisenberg frame operators. Strangely, this simple characterisation appears to have been observed explicitly only recently, [5].

Theorem 6. *A bounded linear operator on $L^2(\mathbb{R})$ is the frame operator of some Weyl-Heisenberg frame if and only if it is positive, invertible and commutes with a translation operator T_a and a modulation operator E_b, where a, b are positive real numbers with $ab \leq 1$.*

Proof. We have observed that if $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ is a Weyl-Heisenberg frame in $L^2(\mathbb{R})$ with frame operator S, then $0 < ab \leq 1$ and S commutes with the translation operator T_a and the modulation operator E_b. Moreover, all frame operators are positive and invertible.

Conversely, suppose that S is a bounded linear operator on $L^2(\mathbb{R})$ which is positive, invertible and commutes with some translation operator T_a and modulation operator E_b satisfying $0 < ab \leq 1$. Then its positive square root $S^{1/2}$ is again a positive, invertible bounded linear operator on $L^2(\mathbb{R})$. It also commutes with E_b and T_a and hence with E_{mb} and T_{na} for all $m, n \in \mathbb{Z}$. Since $0 < ab \leq 1$, the previous proposition ensures that there is a Weyl-Heisenberg frame $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$ on $L^2(\mathbb{R})$ with the identity operator as its frame operator. But $\{S^{1/2}E_{mb}T_{na}g = E_{mb}T_{na}S^{1/2}g : m, n \in \mathbb{Z}\}$ is also a frame, by the invertibility of $S^{1/2}$. Moreover

$$\begin{aligned}
&\sum_{m,n}\langle f, E_{mb}T_{na}S^{1/2}g\rangle E_{mb}T_{na}S^{1/2}g \\
&\quad = S^{1/2}\left(\sum_{m,n}\langle S^{1/2}f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g\right) \\
&\quad = S^{1/2}I(S^{1/2}f) \\
&\quad = Sf.
\end{aligned}$$

Hence the Weyl-Heisenberg frame $\{E_{mb}T_{na}S^{1/2}g : m, n \in \mathbb{Z}\}$ has S as its frame operator. The proof of the theorem is thus complete. $\square$

Analogous to a Weyl-Heisenberg frame, we have the concept of a wavelet frame generated by $g \in L^2(\mathbb{R})$ through translations and dilations: a frame for $L^2(\mathbb{R})$ of the form $\{T_{kba^j}D_{a^{-j}}g : j, k \in \mathbb{Z})\}$ where $a > 1, b > 0$ and $D_{a^{-j}}$ is the dilation operator. However, there is no guarantee ([1], page 297) that the corresponding frame operator commutes with the associated dilation and translation operators. Thus this characterisation of Weyl-Heisenberg frame operators appears to be of special interest and significance.

References

[1] O. Christensen, Frames and Bases – An Introductory Course, Birkhauser (2008).

[2] I. Daubechies, A. Grossmann and Y. Meyer, Painless nonorthogonal expansions, *J. Math. Physics*, **27** (1986) 1271–1283.

[3] R. J. Duffin and A. C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.*, **72** (1952) 341–366.

[4] T. C. Easwaran Nambudiri and K. Parthasarathy, Generalised Weyl-Heisenberg frame operators, *Bull. Sci. Math.*, **136** (2012) 44–53.

[5] T. C. Easwaran Nambudiri and K. Parthasarathy, A characterisation of Weyl-Heisenberg frame operators, *Bull. Sci. Math.*, **137** (2013) 322–324.

[6] D. Gabor, Theory of communications, *J.IEE*, **93** (1946) 429–457.

[7] K. Gröchenig, Foundations of Time Frequency Analysis, Birkhauser (2001).

[8] A. J. E. M. Janssen, Gabor representation of generalized functions, *J. Math. Anal. Appl.*, **83** (1981) 377–394.

Srinivasa Ramanujan

Transcending Kanigel's Canvas

Parthasarathi Mukhopadhyay

Associate Professor, Department of Mathematics,

Ramakrishna Mission Residential College (Autn.), Narendrapur, Kolkata

While visiting India, in an interview given to *The Hindu* published on 26th December 2011, Robert Kanigel described his book **The Man Who Knew Infinity** on Ramanujan as a 'narrative non-fiction'. In his words, "in the case of Ramanujan, what I wrote was the first western biography . . . In some respects, I consider this almost a dual biography about

Ramanujan and Hardy . . . . For me Hardy played such an important role that their chemistry, their tension, their friendship, their relationship played a central role mathematically and personally in Ramanujan's life. And I felt it was really important for the reader to come to understand Hardy as well as Ramanujan."

Biographies of mathematicians, barring few exceptions, are hardly known to be popular, but Kanigel has trodden that seemingly impossible path a couple of decades earlier to produce the most elegantly awe-inspiring tale of the last century. His book, a psycho-analytic dual biography of Ramanujan and Hardy has, in its pursuit of capturing the quintessential Ramanujan through the odyssey of his romantic sojourn, has won many accolades since its publication and made the author a distinguished international celebrity. In USA, Robert Kanigel's book was highly praised and it was eventually translated into German in 1993, into Japanese in 1994, into Korean in 2000, into Chinese in 2002 and again in 2008, into Italian in 2003, into Thai in 2007 and into Greek in 2008. An English-language edition of the book was published in India in 1992, but until recently it has not been published in any of the Indian regional languages. Excepting Kanigel's *Magnum opus*, no full-length life-story of Srinivasa Ramanujan was available in most of the Indian languages – a biography that would do justice to the phenomenal yet tragically short life of this genius, and simultaneously laying before the reader an informed sketch of his stupendously huge Mathematical oeuvre that he left as a legacy for the generations to come and its ever-growing importance.

No wonder, Government of India declared 2012 as the National Mathematics Year to commemorate the 125th birth anniversary of Srinivasa Ramanujan. An organising committee was formed for a yearlong celebration, which along with many other mathematical activities throughout India, also took up the project of getting Kanigel's book translated into ten regional Indian languages under the auspices of Ramanujan Mathematical Society (RMS). The first one to come out was in Bengali. It was published jointly by RMS and National Book Trust, India, and was formally inaugurated on 19th July 2013 at Kolkata, by the Nobel Laureate economist, Bharatratna Prof. Amartya Sen. Within a month, the Malayalam translation was released by the Chief Minister of Kerala. The Tamil and the Kannada translations were published on 22nd December 2013, the auspicious birthday of Ramanujan, now declared as the National Mathematics Day. Six more are in the pipeline, to be published shortly.

However, water has been flowing relentlessly through the Kaveri since the book by Kanigel was first published in 1991. And with the passage of time, the information related to Ramanujan, as one may find in Kanigel's well-researched book, comprehensive and up-to-date as it was during its time of publication, appeared to require further updating here and there. This is why, while working on the Bengali translation, it became almost a moral compulsion to try one's level best and shed some light on the recent boom in the mathematical arena influenced by Ramanujan's mathematics, as well as some other socio-academic developments related to his name and life story that have taken place since then. It is with this mission at the back of mind, an addendum was proposed to this biography of epic proportions, which was duly accepted and appreciated by Robert Kanigel and finally got appended to the Bengali translation. The present article, a bouquet of various information, is an abridged modification of the same. It is divided into two parts. The first part updates some of the available information on the various academic and socio-cultural facets relating to the growing impact of Ramanujan's life in our society at large, while the second one is a brief non-technical survey on the influence of Ramanujan's works on the present day mathematical research.

Part I

At the outset, let us focus on the not-so-mathematical and rather socio-cultural aspects related to Ramanujan that took place during last two decades or so. In an article entitled "The Meaning of Ramanujan Now and for the Future" commemorating the 123rd anniversary of Ramanujan's birth on December 22, 2010, George E. Andrews, while paying homage to "this towering figure whose mathematical discoveries so affected mathematics throughout the twentieth century and into the twenty-first," pointed out to the reader that "whenever we remember Ramanujan, three things come most vividly to mind: (1) Ramanujan was a truly great mathematician; (2) Ramanujan's life story is inspiring; and (3) Ramanujan's life and work give credible support to our belief in the Universality of truth." In this article Andrews aptly points out that "almost anyone interested enough in Ramanujan to be reading these words knows the broad outline of Ramanujan's life." There have been many biographies of Ramanujan written before and after the Kanigel book, in English and also in other regional Indian languages. One such notable book is by K. Srinivasa Rao (1998). Two short biographies of Ramanujan by S. Ram (2000) in English and by Ranjan Bandyopadhyay (2009) in Bengali are worth mentioning. Another one, entitled "The Mathematical Legacy of Srinivasa Ramanujan" was published in October, 2012. Written by eminent Ramanujan scholars, M. Ram Murty and V. Kumar Murty, this book has several chapters devoted to Ramanujan's conjecture and its impact on 20th and 21st century mathematics. In any such book on Ramanujan, mention of Kanigel in the bibliography is a common feature. On the other hand, Kanigel's passionate research is evident from the similar mentions of almost all the major Ramanujan related books published beforehand. However, there is a notable exception both ways, which seems worth mentioning. Almost at the same time of Kanigel's book, Wazir Hasan Abdi, a distinguished professor of Mathematics and a Fellow of The Indian National Science Academy has written a beautiful book on Ramanujan's life and work entitled *Toils and Triumphs of Srinivasa Ramanujan the Man and the Mathematician*. Apart from containing almost all the information on Ramanujan's life that one may get in Kanigel's, it includes all the contribution that Ramanujan made to the Journal of Indian Mathematical Society before his voyage to Cambridge. Furthermore, in the last section there is a collection of six survey articles written by experts on various areas of Ramanujan's mathematical work. Subsequently, Bruce C. Berndt and Robert A. Rankin have published two wonderful books. The first one, called *Ramanujan – Letters and Commentary*, published in 1995 by AMS(Indian edition by Affiliated East West Press Private Limited, 1997) collects various letters written to, from, and about Ramanujan, and makes detailed commentaries on the letters. The second book, called *Ramanujan-Essays and Surveys* published in 2001 by AMS-LMS (Indian edition by Hindusthan Book Agency, 2003) is a collection of excellent articles by various experts on Ramanujan's life and work. The book is divided into eight parts and contains articles about certain individuals who played a major role in Ramanujan's life. A noted Ramanujan scholar from University of Florida, Prof. Krishnaswami Alladi says "both books will appeal not only to mathematicians, but to students and lay persons as well". While writing a review of the Indian edition of the second book in volume 87 of *Current Science* in 2004, Prof. M. S. Raghunathan remarked that "our curiosity about the great is by no means confined to their lofty pursuits and achievements (perhaps because it is their common place experiences that will reassure us of our kinship with them). And in the case of Srinivasa Ramanujan, that curiosity is greater as the bare outlines of his life have so much of the romantic element. The excellent biography of Ramanujan by Robert Kanigel ... is indeed a comprehensive work; yet there remain many unanswered questions. The volume under review provides some facts of human interest not to be found in Kanigel's work and at the same time caters also to those who seek an introduction to the workshop of the genius." Highlighting some such issues he further noted "A note by Berndt tells us that the original notebooks are to be found in the Librarian's office of the University of Madras .... We owe the publications of The Notebooks to the initiative of K. Chandrasekharan, one of our leading mathematicians – a fact that unfortunately finds no mention in much of the extensive writings about Ramanujan." Part three of this book has a short biography of Janaki, Ramanujan's wife, who eventually passed away on April 13, 1994 at the age of 94. An interview given by her to Pritish Nandy has been reproduced in the book, which reveals some glimpses of her personality. Part two of the book contains information regarding Ramanujan's illness and treatment. An article by Rankin reproduced here from the *Proceedings of the Indian Academy of Sciences* and another by a physician, D. A. B. Young, who did some investigations on this subject for the Royal Society, have considerable information not to be found in Kanigel's book. Raghunathan pointed out that, "among other things, it appears that the contemporary diagnosis of Ramanujan's illness was not satisfactory".

Robert Kanigel, in an interview given to *The Hindu* in December 2011, observed, "after the book (*The Man Who Knew Infinity*) came out, there was new theorising about what Ramanujan actually died from. That would have been interesting to bring to" a book on Ramanujan written after that period. In an article in the Asia Pacific Mathematics Newsletter (April 2012, vol. 2, no. 2), K. Srinivasa Rao of The Institute of Mathematical Sciences, Chennai, also pointed out the same. "Mainly due to the efforts of Prof. Robert A Rankin, a renowned mathematician and Dr. D. A. B. Young, a medical doctor, it is now common knowledge, amongst the admirers of Ramanujan, that the cause of the death of Ramanujan was not the then dreaded TB, but hepatic amoebiasis, which was the cause of his illness twice in his younger days, in India. Since TB was diagnosed by (some) doctors in England and in India after his return, as a celebrity he got the best medical attention and the full-fledged backing of the University of Madras. Since the treatment was done (not for hepatic amoebiasis but) for TB, it led to his premature death."

In his expertly written article full of medical jargon written to the Royal Society, Dr. Young pointed out that "unfortunately, no official medical records of Ramanujan's illness during his time in England have survived, so any attempt at a retrospective diagnosis must depend on information in letters and reminiscences." And on the basis of such information, he retraced the whole medical history of Ramanujan's life and arrived at his claim of mis-diagnosis of tuberculosis:

> Many may wonder that any mystery attends Ramanujan's illness, for until 1984 it was generally believed that tuberculosis was the cause of death, and that his illness had been explicitly treated as such in the various English sanatoria and nursing homes. However that diagnosis originated, not in England, but in India with Dr. P. Chandrasekhar of the Madras Medical College, who attended Ramanujan from September 1919 until his death on 26 April 1920. The ever-worsening emaciation and pulmonary symptoms that followed his relapse on arriving in India have been powerfully persuasive. The verdict of the English doctors was quite the opposite, but its publication had to wait until Rankin brought the hitherto unpublished material together in his 1984 papers. It quite clearly contradicts, for example, the statement in Ranganathan's book that 'by the end of 1918, it was definitely known that tuberculosis had set in'. It was disappointing, therefore, that Kanigel in his biography, although acknowledging the doubts, nevertheless chose to perpetuate the undue emphasis on tuberculosis.

In his ten page long analysis of the diverse possibilities that the symptoms suffered by Ramanujan might indicate, Young carefully took into account and examined all the available facts about Ramanujan's illness and the contradicting opinions of the Doctors who have attended him and finally opined that,

> although we have but little information about Ramanujan's illness, there is much that can be inferred from it. The illness began with a acute episode that was diagnosed as gastric ulcer. Later, the condition eased and the symptomatology must have changed significantly, for this diagnosis was rejected and that of tuberculosis favoured. The sign prompting this was probably the onset of 'intermittent pyrexia', which eventually became the regular nighttime fevers described by Hardy. 'Intermittent pyrexia' is found in a group of otherwise diverse diseases, by far the most important of which at that time was tuberculosis. Expert opinion was accordingly sought from Dr. H. Battty Shaw (1867–1936), a London specialist in consumption and chest diseases. His verdict, given probably in August 1917, was that it was not tuberculosis but metastatic liver cancer, derived, he believed, from a malignancy of the scrotum excised some years before. Time proved him wrong, but it must be very significant that he did not favour tuberculosis. By the summer the bouts of intermittent fever had become less frequent, and they ceased altogether during Ramanujan's stay at Fitzroy Square. Here, doubtless, advantage was taken of the wealth of medical expertise and facilities available in London. The consensus of medical opinion mentioned by Hardy in his November letter (to Dewsbury), namely that Ramanujan had been suffering from some obscure source of blood poisoning, entails that, at a minimum, blood counts were carried out. These are procedures that were, even then, a matter of routine.

Armed by this hypothesis, he then presented 'a diagnosis by exclusion', which according to him was "limited to clinical detail and diagnostic procedures available in 1918, as presented in the 1917 edition of Munro's textbook". Referring to the diseases, that might have caused the dominant symptom of 'Intermittent pyrexia', and eliminating them one-by-one through their would-be-evident manifestation in a blood count he reached the crux of his theory:

> The fourth disease is hepatic amoebiasis and the fact that it has been arrived at here by a process of elimination should not disguise the high probability that it could have been the cause of intermittent fever in someone of Ramanujan's background. For as a 'Always suspect hepatic amoebiasis in a patient with obscure pyrexia coming symptoms would be pain with tenderness in epigastrium, enlargement of the liver, and weakness. Progressive emaciation leading to cachexia is characteristic of the disease. If hepatic amoebiasis was suspected,the effect of emetine for 8-10 days on the patient's symptoms was the easiest way to confirm the diagnosis; and it remains so today, although metronidazole would now be preferred to emetine.

Elaborating on *Ramanujan's background*, in this regard, he pointed out that in the second half of 1906 while attending senior school in Madras, Ramanujan contracted a bad bout of dysentery that forced him to return home for three months. "This was probably amoebic dysentery" wrote Young, "the most common form in India at the time .... Amoebiasis, unless adequately treated, is a permanent infection, although many patients may go for long periods with no overt signs of the disease. Relapses occur when the host-parasite relationship is disturbed. Ramanujan experienced such a relapse, I believe, in 1909" when he was nursed by his friend R. Radhakrishna Aiyar. Young then brought into account the 'hydrocele' episode and argued that it was "a scrotal 'amoeboma' rather than a hydrocele, ... a lesion arising from the relapse, for amoebae can spread into adjacent tissues in the anogenital area" and "Dr. Shaw's suspicion that the 'hydrocele' operation was the excision of a malignant growth" seemed to coroborate with this fact. He then pointed out that, "in the spring of 1917 Ramanujan became acutely ill, gastric ulcer was diagnosed. This could have been a recurrence of intestinal amoebiasis, this time in the transverse colon, where it can give rise to symptoms closely resembling those of gastric ulcer, but without dysentery. This could have then led to hepatic amoebiasis, with the changed symptomatology." Young concluded the article with the following remarks:

> Two questions will inevitably arise from the suggested diagnosis of hepatic amoebiasis; namely, why the diagnosis was not made at the time and what prospect of a cure could there have been. Hepatic amoebiasis was regarded in 1918 as a tropical disease ('tropical liver abscess'), and this would have had important implications for successful diagnosis, especially in provincial medical centres. Furthermore, the specialists called in were experts in either tuberculosis or gastric medicine. Another major difficulty is that a patient with this disease would not, unless specifically asked, recall as relevant that he had two episodes of dysentery 11 and 8 years before. Finally, there is the very good reason that, because of the great variability in physical findings, the diagnosis was difficult in 1918 and remains so today: hepatic amoebiasis 'presents a severe challenge to the diagnostic skills of the clinician ... [and] should be considered in any patient with fever and an abnormal abdominal examination coming from an endemic area'. This same admonition was made by Savill in 1930 but not in earlier editions of his book (e.g. that of 1918). The treatment available in 1917 was essentially the same as that in 1962, namely administration of emetine with aspiration of any detectable abscess. Emetine alone would probably have cured Ramanujan in 1917, and perhaps as late as his departure for India. Since the abscess must have enlarged rapidly thereafter aspiration would have been essential. Pulmonary amoebiasis responds well to emetine, so that unless secondary infection or empyema had occurred, emetine and aspiration might have effected a cure as late as January 1920, when Ramanujan wrote for a last time to Hardy, full of hope, wanting to subscribe to new journals and still producing mathematics of very high quality.

However, Prof. (Dr.) D. N. Guha Mazumder, a noted Gastroenterologist, who is a member of the Task Force of Liver Disease, Indian Council of Medical Research, New Delhi and former Head of the Department of Gastroenterology, Institute of Post Graduate Medical Education and Research, Kolkata, holds contrary views. Referring to the 'diagnosis by exclusion' as proposed by Dr. Young as 'far-fetched', he points out the fact that, at the time under consideration, Madras was quite advanced in tackling tropical diseases and with a blood count data suggesting any possibility of a liver abscess, it seems quite unlikely that the eminent doctors treating Ramanujan would have failed to explore that possibility as well. He further suggests that it is highly unlikely that a case of liver abscess, if not treated properly, would take such a prolonged period of time to worsen through gradual deterioration, rather than a faster manifestation of ultimate decline. He strongly feels that the available data are not sufficient to favour 'hepatic amoebiasis' against 'tuberculosis', for which Ramanujan was treated.

Let us now turn our attention to the various socio-academic and cultural events related to Ramanujan that have taken place during the last two and a half decades or so. In 1985, a few years before Kanigel has taken up his project, Ramanujan Mathematical Society, an organisation with the aim of "promoting mathematics at all levels" was formed at Tiruchirappalli, Tamil Nadu, which had blossomed into one of the most prominent among such bodies of India in due course of time. Today the publications of Ramanujan Mathematical Society include, 'Mathematics Newsletter' – a journal catering to the needs of students, research scholars and teachers. Launched in the year 1991, The Newsletter is circulated free of cost among various academic institutions within India. 'Journal of the Ramanujan Mathematical Society', started in 1986, initially was a biannual Journal but now it has four issues per year. 'Little Mathematical Treasures', has recently been envisaged as a series of books addressed to mathematically mature readers and to bright students. Finally there is 'RMS Lecture Notes Series in Mathematics', which is a series consisting of monographs and proceedings of conferences. Recently, one of the distinguished members of the society, Prof. Manjul Bhargava of Princeton, in collaboration of the Clay Institute of Mathematics, USA, has instituted the Bhargava-Clay Fellowship of the RMS. At least one prominent mathematical journal bearing the name of Ramanujan newly came out during the last two decades. Given the name, *Ramanujan Journal*, it is devoted to the areas of mathematics influenced by Ramanujan. It is being published since 1997 by Springer.

Department of Science and Technology, Government of India, has recently instituted Ramanujan Fellowship worth Rs. 75,000 per month and a contingency grant of Rs. 5.00 lakh per annum, meant for brilliant scientists and engineers from all over the world below the age of 60 years to take up scientific research positions in India, especially those scientists who want to return to India from abroad.

The duration of the fellowship is initially for five years. The Ramanujan Fellows may work in any of the scientific institutions and universities in India and apart from the fellowship, they would be eligible for receiving regular research grants through the extramural funding schemes of various agencies of the Government of India. In 2005, an annual prize worth $10,000 for young mathematicians of age less than 45 years, from developing countries who conduct their research in a developing country, has been created in the name of Ramanujan by the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy, in cooperation with the International Mathematical Union and with support from the Niels Henrik Abel Memorial fund, Norway. The list of recipients till date includes Ramdorai Sujatha from India in 2006. The Shanmugha Arts, Science, Technology & Research Academy (SASTRA), a private university based in the state of Tamil Nadu, has instituted the SASTRA Ramanujan Prize of $10,000 to be given annually to a mathematician not exceeding the age of 32 for outstanding contributions in an area of mathematics influenced by Ramanujan. This prize is being awarded annually since 2005, at an international conference conducted by SASTRA in Kumbakonam, Ramanujan's hometown, around Ramanujan's birthday, 22 December. The joint first recipients were Manjul Bhargava of Princeton University and Kannan Soundararajan of the University of Michigan. Research chairs has also been established at the SASTRA centre at Kumbakonam in honour of Ramanujan – two by the Department of Science and Technology, Government of India and one by City Union Bank Ltd., to encourage research in the field of mathematics. SASTRA purchased the home of Ramanujan in 2003, and has since maintained it as a museum. Another single room Ramanujan Museum and Math. Education Centre was created in 1993 in the premises of the Avvai Academy in Royapuram, Chennai by Mr. P. K. Srinivasan, a dedicated school teacher in mathematics, with the help of a benevolent businessman Mr. A. T. B. Bose. In this little-known museum, there's a treasure of pictures and letters and documents which include Hardy's replies to Ramanujan's first two letters sent to him in early 1913. In 1999, Prof. Srinivasa Rao and Dr. R. Jagannathan helped establish at the Periyar Science and Technology Centre, Kotturpuram, a Ramanujan Gallery, which includes a poster-rich 'Pie Pavilion', a multimedia presentation on the life of Ramanujan and a replica of the Ramanujan Museum. A Ramanujan Photo Gallery, with 110 photographs on the life and work of the genius, was later set up by Prof. Srinivasa Rao, next to the Gallery. Today the Museum display includes Ramanujan's original passport, facsimiles of his notebooks and the bronze bust of Ramanujan by Paul Grandlund that was presented to Janaki. Another statue of Ramanujan sculpted by K. G. Ravi was installed at the entrance of the Operational Head Quarters of the Society for Electronic Transactions and Security (SETS), Taramani, Chennai, in June 2010. Prof. Srinivasa Rao's pilot multimedia project on the Life and Work of Srinivasa Ramanujan, made in 1999 for the Indian Science Congress Exhibition in Chennai, finally bloomed into two full length CD-ROM. The Institute of Mathematical Sciences (IMSc), Chennai and C-DAC's (Centre for Development of Advanced Computing) National Multimedia Resource Centre (HCDC), Pune, were sponsored by the Department of Science and Technology (DST), Government of India, to produce these CD's.

Turning to the cultural impact of the story of Ramanujan's life on the society as a whole, it is quite amazing to note that, time and again the romantic sojourn of Ramanujan against all odds, his ultimate success and tragic premature death has drawn attention of the storytellers of all varieties, be it a novel, a drama or in celluloid. Presently no less than three Hollywood films on his life are at different levels of production, of which one is based on the Kanigel book. Produced by Edward R. Pressman, this film is being adapted and directed by Matthew Brown. Popular Bollywood actor Madhavan, best known for "Three Idiots" was initially chosen to play the role of Ramanujan. However, for some reason it has been changed and now Dev Patel, the hero of the Oscar winning film "Slumdog Millionaire" is all set to play the role. Meanwhile, another Bollywood actor Siddharth of "Rang De Basanti" fame, has been signed up to play the central role in the Hollywood film on Ramanujan titled "The First Class Man" to be directed by Roger Spottiswoode, better known as the director of the Bond film "Tomorrow Never Dies". Spottiswoode has relied on multiple sources for the film, including a sizable chunk from *Ramanujan: The Man and the Mathematician* by S. R. Ranganathan. He says that the "script concentrates largely on the time Ramanujan was at Cambridge; so it deals closely with his relationship with two other Trinity mathematicians: Hardy and Littlewood. It is the story of the friendship between these three and, although they discuss their work, it is a story of relationships and not of mathematical equations.” In the year 2006, Stephen Fry and Dev Benegal together announced a biopic on Ramanujan. In early 2010, Bollywood director Rohit Jugraj announced he would make a biopic as well. Both projects are yet to see the light of day. However, in an interview given to *The Hindu* on March 22, 2011 Dev Benegal, the director of much acclaimed “English August”, who himself studied at the Cambridge University, told that he has been researching on the life of Srinivasa Ramanujan for four years. “The film will explore his life at an emotional level – the struggle of his parents, particularly his mother; Ramanujan’s relationship with his wife, which is one of the greatest love stories of our times; the sacrifices that the wife had to make which are unknown and unheard of; and the bond that Ramanujan and G. H. Hardy shared”. However, a beautiful docu-feature film on Ramanujan was made several years ago by Nandan Kudhyadi under the auspices of NFDC, India. Bollywood actors Raghubir Yadav and Tom Alter played the roles of Ramanujan and Hardy respectively. In 2012–13, Nandan has directed another documentary on Ramanujan, entitled “The Genius of Srinivasa Ramanujan”. Developed by IISER Pune, this documentary is presented mostly through mutual conversations of many notable Ramanujan scholars of present day, though anchored mainly by Prof. A. Raghuram. Another biopic on the inspiring life of Ramanujan is now being filmed at Bollywood by the national award winning director Gnana Rajasekharan, where the central casts are Abhinay Vaddi and Suhasini Mani Ratnam. The film is supposed to be released shortly. Meanwhile, in 2007, English playwright Simon McBurney conceived and directed a brilliant play inspired by the collaboration between Ramanujan and Cambridge don G. H. Hardy, entitled “A Disappearing Number” co-written and devised by the Théâtre de Complicité company. After a successful international tour, this mesmerizing performance was staged in India during 2010 ICM (International Congress of Mathematicians) at Hyderabad. Incidentally, the logo of ICM 2010, the first ever such a big mathematical event to be held in India, depicted the statement of the Ramanujan Conjecture or the so called Tau conjecture that Ramanujan proposed in a 1916 paper “On Certain Arithmetical Functions”.

Meanwhile, during a program held at the Madras University Centenary Auditorium, Chennai, on 26th December 2011, inaugurating the yearlong celebrations of the 125th birth anniversary of Ramanujan, Prime Minister of India Dr. Manmohan Singh declared December 22, Ramanujan’s birthday, as the National Mathematics Day and 2012 as the National Mathematics Year. Releasing a commemorative stamp on Srinivasa Ramanujan to mark the occasion, Dr. Singh referred to mathematics as “the mother science,” and pointed out that “the Ramanujan story illustrates the inadequacy of the university evaluation system in the early decades, while at the same time it shows that the system displayed enough flexibility to take care of mavericks like him. A genius like Ramanujan would shine bright even in the most adverse of circumstances, but we should be geared to encourage and nurture good talent which may not be of the same caliber as Ramanujan.” A hugely improved new edition of the Ramanujan Notebooks, made with the help from the archiving and digitising team at the Roja Muthiah Research Library (RMRL) in Chennai, was released by TIFR on the occasion. In this program Robert Kanigel was honoured by Tamil Nadu Governor K. Rosaiah for his excellent biography of Ramanujan. An Organising Committee with Prof. M. S. Raghunathan, the then President of the RMS as chair, and Prof. Dinesh Singh as secretary, was formed to formulate and implement programs and projects as part of the observance of the National Mathematics Year. A National Committee with the then Minister for Human Resource Development Mr. Kapil Sibal as the chair was formed to supervise the activities of the Organising Committee. Vice-chair of the national committee Prof. M. S. Raghunathan outlined the programs planned for the next one year as part of the celebrations. A plan to bring out translations of the Ramanujan’s biography written by Prof. Kanigel, in various Indian languages was chalked out. Apart from that, various mathematical activities were planned to cater to the diverse sections of society all over the country. One program meant for the public was a lecture tour by Prof. Kanigel covering Mumbai, Chennai, Bangalore, Hyderabad and Delhi, which took place eventually in due course of time. At about the same time another development took place. From the year, 2012, International Centre for Theoretical Sciences (ICTS) of TIFR has started a yearly Lecture Series named after Ramanujan. Peter Sarnak of Princeton University was the first lecturer who spoke on “The Generalized Ramanujan Conjectures and Applications” on 21 May, 2012 at TIFR Mumbai. During December 17–22, 2012 an International Conference on mathematics related to and influenced by Ramanujan’s work was organized by Delhi University in New Delhi. An International

Scientific Committee chaired by Prof. Bruce Berndt suggested a list of eminent Ramanujan Scholars to be invited as speakers. This grand conference named, '*The Legacy of Srinivasa Ramanujan*' was attended by a large number of Ramanujan enthusiasts from all over the world.

Part II

Let us now turn to the unbelievable legacy of mathematics that Ramanujan has left behind and relevant hardcore research-related developments during the period under scan. In Harvard-MIT Current Developments in Mathematics Conference 2008, Ken Ono, the Asa Griggs Candler Professor of Mathematics and Computer Science at Emory University, and a noted Ramanujan scholar pointed out that "The legend of Ramanujan has continued to grow with the ever-increasing importance of his mathematics." Shortly after Ramanujan's death Hardy wrote,

> Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on mathematics of the future . . . . He would probably have been a greater mathematician if he could have been caught and tamed a little in his youth. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain . . . .

Ono emphatically claims, "In view of the last eighty five years of progress in number theory, it is clear that the loss would have been much greater than the gain." Analyzing his claim, he points out,

> on one hand, as Hardy did, we may largely base our conclusion on the contents of Ramanujan's notebooks, which, apart from the "lost notebook" that contained the work of his last year, were known to mathematicians at the time of his death. They are a repository of thousands of cryptic entries on evaluations and identities of strangely named functions. Through the tireless efforts of B. C. Berndt, adding to the accumulated effort of earlier mathematicians such as Hardy, G. N. Watson, B. M. Wilson, and R. A. Rankin, a clear picture has emerged which reveals Ramanujan's incredible gift for formulas and combinatorial relations. Perhaps he was the greatest such mathematician of all time. On the other hand, this flattering assessment is grossly inadequate, for it does not take into account Ramanujan's impact on Hardy's "*mathematics of the future*". Indeed, number theory has undergone a tremendous evolution since Ramanujan's death, and today it bears no resemblance to the number theory of his day. The subject is now dominated by the arithmetic and analytic theory of automorphic and modular forms, the study of Diophantine questions under the rubric of arithmetical algebraic geometry, and the emergence of computational number theory and its applications. These subjects boast many of the most celebrated achievements of 20th century mathematics such as: Deligne's proof of the Weil Conjectures, the solution to Gauss' Class Number Problem by Goldfeld, Gross, and Zagier, Wiles' proof of Fermat's Last Theorem, and Borcherds's work on the infinite product expansions of automorphic forms. A proper assessment of Ramanujan's greatness must then take into account the remarkable fact that his work, the portion which was known to Hardy, makes intimate contact with all of these notable achievements.Clearly, Ramanujan was a great anticipator[1]. His work provided examples of deeper structures, and suggested important questions which are now inescapable in the panorama of modern number theory.

Then turning towards the contents of the "lost" notebook, the mathematics that Ramanujan created in his deathbed, of which Hardy was unaware at the time of Ramanujan's death, save the January 20, 1920 letter on 'mock theta' function that he had received from Ramanujan, Ono dramatically posed the question, "What are the secrets of the mathematical scrawl Ramanujan penned during his last days? What is its impact on Hardy's *mathematics of the future*?" The answer to this question shapes the core of the hundred odd pages of his masterly exposé "Unearthing the Visions of a Master: Harmonic Maass Forms and Number Theory" where he begins:

> Modular forms are central in contemporary mathematics. Indeed, modular forms play crucial roles in algebraic number theory, algebraic topology, arithmetic geometry, combinatorics, number theory, representation theory, and mathematical physics. The recent history of the subject includes (to name a few) great successes on the Birch and Swinnerton-Dyer Conjecture, Mirror Symmetry, Monstrous Moonshine, and the proof of Fermat's Last Theorem. These celebrated works are dramatic examples of the evolution of mathematics; indeed, it would have been impossible to prophesy them fifty years ago. Instead of travelling back in time to the 1950s, our story begins in 1887, in a village in India. Our mathematics, which is about harmonic Maass forms, begins with the legend of the great mathematician Srinivasa Ramanujan, and the mathematics he conjured from his death bed.

Trying to decipher what might have motivated Ramanujan to cook up 'mock theta' functions, Ono suggested, "it is not difficult to imagine Ramanujan's mindset. It seems that Ramanujan, largely motivated by his work on partitions and

[1] Ono gives the credit of such a description of Ramanujan to Prof. Manjul Bhargava, the second youngest-ever full Professor of Princeton, another major player in modern number theory.

the Rogers-Ramanujan identities, spent the last year of his life thinking deeply about the "near" modularity of Eulerian series. He understood the importance of developing a "new theory", one which overlaps in spots with the classical theory of modular forms. He discovered the mock theta functions." George Andrews' accidental discovery of the "lost notebook" in 1976 sparked off, almost immediately, a flurry of research on the mock theta functions. Andrews himself was a forerunner and there were many other notable names. By the late 1990s, works by Andrews, Y.-S. Choi, H. Cohen, F. Dyson, Garvan, B. Gordon, Hickerson, R. McIntosh, M. Wakimoto among numerous others, revealed many of the deeper properties of the mock theta functions. Due to the works done by them, and too many others to list, Ramanujan's 22 mock theta functions had been related to a surprising collection of subjects: Artin L-functions in number theory, Hypergeometric functions, Partitions, Lie theory, Mordell integrals, Modular forms, Polymer Chemistry etc. In particular, Hypergeometric functions in the context of Ramanujan's derivative formula have been explored by Balasubramanian and others. At one stage of this global development, the truth of the surprising identities, known as Mock Theta Conjecture, directly related mock theta functions to modular forms. These clues from the "lost notebook" finally placed Ramanujan's mock theta functions in the vicinity of the theory of modular forms. Unfortunately, these clues were not enough to re-construct Ramanujan's theory. In the language of Ono," despite a flurry of activity, the essence of Ramanujan's theory remained a mystery. The puzzle of his last letter to Hardy, thanks to the "lost notebook," had morphed into the enigmatic web of Ramanujan's 22 mock theta functions. The presence of this web strongly suggested the existence of a theory, and it also demanded a solution."

In 1987, The University of Illinois at Urbana-Champaign organized Ramanujan Centenary Conference. There, in his plenary address, Freeman Dyson beautifully summed up the situation related to mock theta functions:

> The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions . . . . But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further.

By the late 1990s, the vast literature on Ramanujan's mock theta functions contained many important clues for Dyson's "challenge for the future". In addition to the identities comprising the mock theta conjectures, there were further clues such as q-series identities relating mock theta functions to Lambert-type series and indefinite theta series. Some such identities served as motivation for the 2002 Ph.D. thesis of S. Zwegers, written under the supervision of Don Zagier[2]. Indeed, Zwegers researched the following two questions of Zagier:

(1) How do the mock theta-functions fit in the theory of modular forms?
(2) Is there a theory of indefinite theta functions?

The real analytic modular forms of Zwegers turned out to be examples of harmonic Maass forms which were defined about the same time by Bruinier and Funke, a coincidence which catalyzed much of the contemporary research in this area. These developments opened floodgates in a wide number of new directions. Indeed, recent works by Ono, Andrews, Eguchi, Hikami, Kac, Lawrence, Malmendier, Mellit, Okada, Wakimoto, and Zagier apply this theory to areas like Donaldson invariants, Gauge theory, Representation theory of Lie superalgebras, Knot theory, Mathematical physics, Probability Theory and Topology. Ono and his collaborators had taken up the investigation towards the answers to deep questions about many of the number theoretic topics captured by the web of Ramanujan's mock theta functions. In the article mentioned above, Ono described the implications of this theory to: Partitions and q-series, Modular forms, Traces of singular moduli, Borcherds products, Modular L-functions 'a la Kohnen-Waldspurger' and Kohnen-Zagier. In conclusion of his article Ono remarked:

> although the mock theta functions are humble in origin, they have earned a distinguished role in the legend of Ramanujan. Andrews and Berndt confirm this in their article "Your hit parade: the top ten most fascinating formulas in Ramanujan's

[2]Zagier delivered a Séminaire Bourbaki lecture on these relatively recent works on Ramanujan's mock theta functions in 2007.

lost notebook". In their amusing informal poll, Ramanujan's work on Dyson's ranks[3] and the mock theta functions rank first and second! Based on the mathematics born out of these works, as described here, it is a safe bet that ranks and mock theta functions will continue to hold these top spots into the foreseeable future . . . . Although Ramanujan's last works provided the first examples of such [harmonic Maass] forms, his untimely death and the enigmatic nature of his writings resulted in a great mystery. We will never know how he came up with the mock theta functions. We certainly cannot pretend to know what he fully intended to do with them. However, it is clear that he understood that the mock theta functions would go on to play important roles in number theory, his "visions".

How have the scholars recently tried to untangle the mythical enigma associated with the Ramanujan's creative genius? Bruce Berndt, the mathematician who has proved each of the 3,542 theorems of the Ramanujan Notebooks, a man whose 20 years of relentless research on the three notebooks of Ramanujan has been compiled into five volumes, has a pensive take on this issue. In an interview given in Chennai to *Frontline* in 1999 he said, "Many people falsely promulgate mystical powers to Ramanujan's mathematical thinking. It is not true. He has meticulously recorded every result in his three notebooks." According to him, the absence of proofs were "perhaps because for him paper was unaffordable and so he worked on a slate and recorded the results in his notebooks without the proofs, and not because he got the results in a flash." Upon being asked about any serious error made by Ramanujan in the notebooks, he said, "there are a number of misprints. I did not count the number of serious mistakes but it is an extremely small number – maybe five or ten out of over 3,000 results. Considering that Ramanujan did not have any rigorous training, it is really amazing that he made so few mistakes." Besides the five volumes, Berndt has written over 100 papers on Ramanujan's works and along with George Andrews he edited four volumes of *Ramanujan's Lost Notebooks*, published by Springer, the latest one being in 2013. When asked about the nature of Ramanujan's mathematics in the notebooks, Berndt pointed out that,

To much of the mathematical world and to the public in general, Ramanujan is known as a number theorist. Hardy was a number theorist, but he was also into analysis. When Ramanujan was at Cambridge with Hardy, he was naturally influenced by him (Hardy). And so most of the papers he published while he was in England were in number theory. His real great discoveries are in partition functions. Along with Hardy, he found a new area in mathematics called probabilistic number theory, which is still expanding. Ramanujan also wrote sequels in highly composite numbers and arithmetical functions. There are half a dozen or more of these papers that made Ramanujan very famous. They are still very important papers in number theory. However, the notebooks do not contain much of number theory. It is, broadly speaking, in analysis. I will try and break that down a little bit. I would say that the area in which Ramanujan spent most of his time, more than any other, is in elliptic functions (theta functions), which have strong connections with number theory. In particular, Chapters 16 to 21 of the second notebook and most of the unorganized portions of the notebooks are on theta functions. There is a certain type of theta functions identity which has applications in other areas of mathematics, particularly in number theory, called modular equations. Ramanujan devoted an enormous amount of effort on refining modular equations.

In recent years some remarkable work has been done in geometric function theory where modular equations have played an important role. Indeed, the so called extremal distortion functions has been defined in terms of solutions of modular equations. Examples of such functions are the distortion function of the quasiconformal analogue of the Schwarz lemma and the extremal distortion function of Schotty's theorem for analytic function. These can be found in the works of G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen and others.

In 1913, Ramanujan submitted his First Quarterly Report to the Board of Studies of the University of Madras. As per *Ramanujan's Notebooks, Part I, page 295* by Bruce Berndt, these reports have never been published. Among other formulae, it contained the beautiful theorem, as usual without formal proof, which came to be known as 'Ramanujan's Master Theorem'. However, Hardy presented this in his book on Ramanujan's work and provided a rigorous proof of it for a natural class of functions and a natural set of parameters by means of the Residue Theorem in Complex Analysis. Also in 1937, after Ramanujan's death Hardy published a paper relating Ramanujan's work to Fourier transforms. In 1997, Wolfgang Bertram of The Institut für Mathematik, Germany, proved a version of Ramanujan's Master Theorem where the multiplicative group of positive reals was replaced by certain Riemannian manifolds with a large group of symmetry. In 2012, Gestur Ólafsson and Angela Pasquale, further

[3]Although Dyson defined the notion of a partition rank in 1944, Ono thinks it is clear that Ramanujan understood the notion in 1920 because of certain identities he recorded in the "Lost Notebook".

generalized the result of Bertram for a more general class of manifolds.

There have been many works on Ramanujan's ground-breaking discoveries on the congruences of partition function $p(n)$ by G. E. Andrews, A. O. L. Atkin, F. J. Dyson and many others. Ramanujan conjectured (and in some cases proved) that there are further congruence properties in which the moduli are powers of 5, 7, or 11. Subsequent works by Atkin and Watson resolved the conjectured congruences of Ramanujan, and in an important paper Atkin somewhat experimentally discovered completely new congruences modulo some further small primes. Meanwhile, attacking the problem in a more systematic manner and using the theory of modular forms, Ono found analytically Atkin-type congruences for all prime moduli exceeding 3, and in a relatively recent work of K. Ono and S. Ahlgren in 2001, they have further extended these results to include all moduli co prime to 6. In particular, it turns out that there are such Ramanujan-type congruences for every modulus co-prime to 6. This comprehensive theory requires deep works of Deligne, Serre, and Shimura. Ramanujan's work on $p(n)$, and the research it inspired, according to Ono, "underscores the fact that the theory of partitions has historically served as a delightful 'testing ground' for some of the deepest developments in the theory of modular forms" which includes the "interplay between the Deligne-Serre theory of ℓ-adic Galois representations, the 'language' of the proof of Fermat's Last Theorem, and Shimura's theory of half-integral weight modular forms."

Ramanujan's conjecture on the size of the tau function was recast in the language of representation theory by I. Satake in the 1960's. This generalized Ramanujan conjecture occupies a central position in the theory of Automorphic Representations and has ramifications that are crucial in diverse areas of pure as well as applied Mathematics. For example, an analogue of the Ramanujan conjecture in the context of graphs (Ramanujan graph) is used in construction of high-speed communication networks. A fundamental contribution to Ramanujan Graph was made by A. Lubotzky, R. Phillips and P. Sarnak. Till date the stongest result towards the generalized Ramanujan Conjecture is due to H. Kim and P. Sarnak.

Ramanujan's congruences for the Tau function had inspired Jean-Pierre Serre, P. Deligne, H. P. F. Swinnerton-Dyer, and several other great mathematicians of the twentieth century to develop a deep theory connecting modular forms to Galois representations. Serre, the youngest ever recipient of the Fields Medal, made a famous conjecture in this theory dating back sometime in the 70's. It was solved in complete generality by Chandrasekhar Khare and Jean-Pierre Wintenberger in 2008. They were awarded the prestigious Cole prize for Number Theory in 2011 for this achievement. Prof. Khare became FRS in 2012. Currently, he is a professor at University of California, Los Angeles.

Beautiful works have been done by J. H. Conway, W. A. Schneeberger, M. Bhargava and J. Hanke that are related to Ramanujan's work on universal quaternary forms and his question of determining all universal quadratic forms. In 1993 John H. Conway and W. A. Schneeberger announced a proof of the "15 theorem" which states that if a quadratic form with integer matrix represents all positive integers up to 15, then it represents all positive integers. But the proof was complicated, and was never published. Manjul Bhargava (born August 8, 1974), a Canadian-American mathematician of Indian origin, who is presently R. Brandon Fradd Professor of Mathematics at Princeton University, found a much simpler proof which was published in 2000. In 2005 Manjul Bhargava and Jonathan P. Hanke announced a proof of Conway's conjecture that a similar theorem holds for quadratic forms with integer coefficients, with the constant 15 replaced by 290.

The classical Rogers-Ramanujan identities are continued to be studied by researchers in various fields that include Number Theory, Modular Forms, Representation Theory of Lie Algebras and Combinatorics. Among many important works in this area, recently D. S. Lubinsky used these identities to disprove a conjecture due to G. A. Baker, J. L. Gammel, and J. G. Wills from 1961 in the theory of Padé approximation which deals with fast approximation of analytic functions by rational functions, generalizing the notion of Taylor expansion of smooth functions which deals with polynomial approximations.

An odyssey of recent developments on Ramanujan's mathematics remains incomplete unless one points out the tremendous influence of the 'circle method' introduced by Hardy and Ramanujan in 1918. It has been applied in hundreds of papers and has been refined and modified in many directions and it is still very much in use today. In particular, Hardy and Littlewood were the first to develop the method, followed by Vinogradov, Lou Keng Hua and Kloosterman, who are among the major contributors. Traditionally its main use was limited to additive problems in number theory and Diophantine questions such as the Waring problem or the Goldbach problem. In recent times one may find them in the works of Robert Vaughan, Trevor Wooley and Kevin Ford among many others. However, this powerful method has found applications in diverse areas of Mathematics as well. Some of the major applications are being found in Combinatorics in the works of J. Bourgain, B. Green among others; Analytic Theory of Modular Forms and L-functions in the works of H. Iwaniec, M. Jutila and R. Munshi among others, Arithmetic Algebraic Geometry in the works of R. Heath-Brown, T. D. Browning and others; Arithmetic in function fields in the works by Yu-Ru. Liu and T. D. Wooley and also in Harmonic Analysis in the works by E. M. Stein, S. Wainger, A. Magyar and others. Very recently a result has been announced by J. Bourgain and A. Kontorovich, that makes a great progress towards proving a 1971 conjecture of Zaremba predicting that every integer appears as the denominator of a finite continued fraction whose partial quotients are bounded by an absolute constant. The circle method is crucial in their work.

So much so for an utterly incomplete highlight of the mathematical developments that took place through the last twenty odd years, since the publication of the book by Kanigel in 1991.

Ramanujan once told Janaki that his 'name will live for one hundred years'. How amazingly right he was! On the eve of his 126th Birth Anniversary, his name appears in the titles of about 1700 research articles in Mathematics, as a quick search in MathSciNet reveals. This numeral arguably may not be proportional to his greatness, but this may be regarded as some measure to judge the relevance of Ramanujan's beautiful mathematics even in this 21st century.

Ramanujan left us about 93 years ago. But he still lives and will continue to live through his immortal work. Indeed, the impact of the story of his life as a fountainhead of inspiration on the society as a whole, seems to be growing day by day.

Ramanujans never die.

Acknowledgement

I am grateful to Prof. M. S. Raghunathan, Prof. C. S. Aravinda, Prof. M. K. Sen, Prof. Mahan Maharaj for their kind encouragement. Cordial thanks to Prof. Swagato K. Ray and Prof. Satadal Ganguly of ISI, Kolkata. Heartfelt thanks to medical persons Dr. Arunaloke Bhattacharya and Dr. D. N. Guha Majumdar for their help. I also acknowledge the overall help rendered by Mr. Bijoyaditya Chakraborty of my parent institution in shaping up this article. I thank wholeheartedly the learned referees for their meticulous suggestions, particularly with the mathematics related part.

References

[1] W. H. Abdi, Toils and Triumphs of Srinivasa Ramanujan the Man and the mathematician, *National* (1992).

[2] P. V. Seshu Aiyar and R. Ramachandra Rao, Srinivasa Ramanujan (1887–1920), Collected papers of Ramanujan (Ed. G. H. Hardy, *et al.*), Cambridge Univ. Press, Cambridge (1927) xi–xx.

[3] G. D. Anderson, S.-L. Qiu, M. K. Vamanamurthy and M. Vuorinen, Generalized elliptic integrals and modular equations, *Pacific J. Math.*, **192** (2000) 1–37.

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Conformal Invariants, inequalities and quasiconformal mappings, *J. Wiley*, (1997) 500.

[5] G. D. Anderson and M. Vuorinen, Reflections on Ramanujan's Mathematical Gems, Mathematics Newsletter, 19 (Sp. no. 1), August (2010) 87–108, arXiv:1006.5092 [math.CV], `http://www.ramanujanmathsociety.org/mnl/August2010/mnl-spl-fullfiles.pdf`

[6] G. D. Anderson, M. Vuorinen and X. Zhang, Topics in Special Functions III., Analytic Number Theory, Approximation Theory and Special Functions, ed. by K. Alladi, G. Milovanovic and M. Th. Rassias, Springer-Verlag (2014) 307–356, DOI 10.1007/978-1-4939-0258-3$_{11}$arxiv.org/abs/1209.1696[math.CA]

[7] G. E. Andrews, An introduction to Ramanujan's lost notebook, *Amer. Math. Monthly*, **86** (1979) 89–108.

[8] G. E. Andrews, Combinatorics and Ramanujan's "lost" notebook, Surveys in combinatorics 1985 (Glasgow, 1985), London Math. Soc. Lecture Note Ser., 103, Cambridge Univ. Press, Cambridge (1985) 1–23.

[9] G. E. Andrews, The fifth and seventh order mock theta functions, *Trans. Amer. Math. Soc.*, **293** no. 1 (1986) 113–134.

[10] G. E. Andrews, Ramanujan's fifth order mock theta functions as constant terms, Ramanujan revisited

(Urbana-Champaign, Ill., 1987), Academic Press, Boston, MA (1988) 47–56.

[11] G. E. Andrews, Mock theta functions, Theta functions – Bowdoin 1987, Part 2 (Brunswick, ME, 1987), 283–298, *Proc. Sympos. Pure Math.*, **49**, Part 2, *Amer. Math. Soc., Providence*, RI (1989).

[12] G. E. Andrews,Partitions with short sequences and mock theta functions, *Proc. Natl. Acad. Sci. USA*, **102** no. 13 (2005) 4666–4671.

[13] G. E. Andrews and B. C. Berndt, Ramanujan's lost notebook. Part I, Springer-Verlag, Berlin, New York (2005).

[14] G. E. Andrews and B. C. Berndt, Ramanujan's lost notebook. Part II, Springer-Verlag, Berlin, New York (2009).

[15] G. E. Andrews and B. C. Berndt, Ramanujan's lost notebook. Part III, Springer-Verlag, Berlin, New York (2012).

[16] G. E. Andrews and B. C. Berndt, Ramanujan's lost notebook. Part IV, Springer-Verlag, Berlin, New York (2013).

[17] G. E. Andrews and F. Garvan, Ramanujan's "lost" notebook. VI. The mock theta conjectures, *Adv. in Math.*, 73 no.2 (1989) 242–255.

[18] G. E. Andrews and D. Hickerson, Ramanujan's "lost" notebook. VII. The sixth order mock theta functions, *Adv. Math.*, **89** no. 1 (1991) 60–105.

[19] R. Balabubramanian, S. Naik, S. Ponnusamy and M. Vuorinen, Elliott's identity and hypergeometric functions, *J. Math. Anal. and Appl.*, **271(1)** (2002) 232–256.

[20] R. Bandyopadhayay, Ramanujan, Ganit O Nioti (in Bengali), Gangchil (2009).

[21] B. C. Berndt, The Continued Fractions Found in the Unorganized Portions of Ramanujan's Notebooks (Memoirs of the American Mathematical Society), by Bruce C. Berndt, L. Jacobsen, R. L. Lamphere, George E. Andrews (Editor), Srinivasa Ramanujan Aiyangar (Editor), American Mathematical Society (1993).

[22] B. C. Berndt, Number Theory in the Spirit of Ramanujan (American Mathematical Society) (2006).

[23] B. C. Berndt, Ramanujan's Notebooks, Part I, Springer (1985).

[24] B. C. Berndt, Ramanujan's Notebooks, Part II, Springer (1999).

[25] B. C. Berndt, Ramanujan's Notebooks, Part III, Springer (2004).

[26] B. C. Berndt, Ramanujan's Notebooks, Part IV, Springer (1993).

[27] B. C. Berndt, Ramanujan's Notebooks, Part V, Springer (2005).

[28] B. C. Berndt, Rediscovering Ramanujan, *Interview in Frontline*, **16(17)** August, (1999) 14–27.

[29] B. C. Berndt, S. Bhargava and F. G. Garvan, Ramanujan's theories of elliptic functions to alternative bases, *Trans. Amer. Math. Soc.*, **347** no. 114 (1995) 163–4244.

[30] B. C. Berndt and George E. Andrews, Ramanujan's Lost Notebook, Part II, Springer (2008).

[31] B. C. Berndt and R. A. Rankin, Ramanujan: Letters and commentary, History of Mathematics, *Amer. Math. Soc.*, **9**, Providence, RI (1995).

[32] B. C. Berndt and R. A. Rankin, Ramanujan: essays and surveys, History of Mathematics, *Amer. Math. Soc.*, **22**, Providence, RI (2001).

[33] B. C. Berndt, Y.-S. Choi and S.-Y. Kang, The Problems Submitted by Ramanujan to the Journal of the Indian Mathematical Society, Continued Fractions: From Analytic Number Theory to Constructive Approximations (B. C. Berndt and F. Gesztesy, eds.), *Contemp. Math.*, Vol. 236, American Mathematical Society, Providence, RI, 15–56.

[34] W. Bertram, Ramanujan's master theorem and duality of symmetric spaces, *Journal of Functional Analysis*, **148** (1997) 117–151.

[35] M. Bhargava, On the Conway-Schneeberger Fifteen Theorem, Quadratic forms and their applications (Dublin, 1999), 27–37, *Contemp. Math.*, **272**, *Amer. Math. Soc.*, Providence, RI (2000).

[36] J. H. Bruinier and J. Funke, On Two Geometric Theta Lifts, *Duke Math. Journal*, **125** (2004) 45–90.

[37] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook, *Invent. Math.*, **136** no. 3 (1999) 497–569.

[38] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. II, *Adv. Math.*, **156** no. 21 (2000) 80–285.

[39] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. III, *Proc. Lond. Math. Soc.*, **94(3)** no. 12, (2007) 6–52.

[40] Y.-S. Choi, Tenth order mock theta functions in Ramanujan's lost notebook. IV, *Trans. Amer. Math. Soc.*, **354** no. 2 (2002) 705–733

[41] Y.-S. Choi, Identities for Ramanujan's sixth-order mock theta functions, *Q. J. Math.*, **53** no. 2 (2002) 147–159.

[42] H. Cohen, q-identities for Maass waveforms, *Invent. Math.*, **91** no. 3 (1988) 409–422.

[43] F. Dyson, A walk through Ramanujan's garden, Ramanujan revisited (Urbana-Champaign, Ill. 1987), Academic Press, Boston, **198**, 87–28.

[44] T. Eguchi and K. Hikami, Superconformal algebras and mock theta functions, preprint.

[45] B. Gordon and R. McIntosh, Some eighth order mock theta functions, *J. London Math. Soc.*, **62** no. 2 (2000) 321–335.

[46] B. Gordon and R. McIntosh, Modular transformations of Ramanujan's fifth and seventh order mock theta functions, *Ramanujan J.*, **7** (2003) 193–222.

[47] G. H. Hardy, Ramanujan and the theory of Fourier transforms, *Quart. J. Math.*, **8** (1937) 245–254; Hardy, "Collected Papers", Vol. VII, p. 280, Clarendon Press, Oxford (1979).

[48] G. H. Hardy, Srinivasa Ramanujan (1887–1920), Collected papers of Ramanujan (Edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson), Third printing of the 1927 original of Cambridge Univ. Press, With a new preface and commentary by Bruce C. Berndt. AMS Chelsea Publishing, Providence, RI (2000) xxxviii+426.

[49] G. H. Hardy, Ramanujan: Twelve lectures on subjects suggested by his life and work, Cambridge Univ. Press, Cambridge (1940).

[50] G. H. Hardy and S. Ramanujan, Asymptotic Formulae in Combinatory Analysis, *Proc. London Math. Soc.*, **17** (1918).

[51] D. Hickerson, A proof of the mock theta conjectures, *Invent. Math.*, **94** (1988) 639–660.

[52] D. Hickerson, On the seventh order mock theta functions, *Invent. Math.*, **94** no. 3 (1988) 661–677.

[53] K. Hikami, Mock (false) theta functions as quantum invariants, *Regul. Chaotic Dyn.*, **10** (2005) 163–174.

[54] K. Hikami, Transformation formula of the "second" order mock theta function, *Lett. Math. Phys.*, 75 (2006) 93–98.

[55] K. Hikami, Hecke type formula for the unified Witten-Reshetikhin invariants as higher order mock theta functions, *Int. Math. Res. Not.*, **7** (2007) 32.

[56] V. G. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and number theory, Lie theory and geometry, *Progr. Math.*, **123** Birkhaüser, Boston (1994) 415–456.

[57] V. G. Kac and M. Wakimoto,Integrable highest weight modules over affine superalgebras and Appell's function, *Comm. Math. Phys.*, 215 (2001) 631–682.

[58] R. Kanigel, The Man Who Knew Infinity: A Life of the Genius Ramanujan, Scribners, 1991, Rupa 1992 (Indian edition).

[59] C. Khare, Serre's conjecture and its consequences, *Japanese Journal of Mathematics*, **5** no. 1 (2010) 103–125.

[60] R. Lawrence and D. Zagier, Modular forms and quantum invariants of 3-manifolds, *Asian J. Math.*, **3** (1999) 93–107.

[61] D. S. Lubinsky, Rogers-Ramanujan and the Baker-Gammel-Wills Padé)conjecture, *Ann. of Math.*, **2** no. 3 (2003) 157.

[62] A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, *Combinatorica*, **8(3)** (1988) 261–277.

[63] A. Malmendier and K. Ono, SO(3)-Donaldson invariants of CP2 and mock theta functions, submitted for publication (as in 2012).

[64] A. Mellit and S. Okada, Joyce invariants for K3 surfaces and mock theta functions, preprint (as in 2012).

[65] G. Ólafsson and A. Pasquale, Ramanujan's Master Theorem for Riemannian Symmetric Spaces, Journal of functional analysis (Accepted for Publication [as in 2012]); arXiv:1103.5126v3[math.RT] 12 March (2012).

[66] K. Ono, Honoring a gift from Kumbakonam, *Notices Amer. Math. Soc.*, 53 no. 6 (2006) 640–651.

[67] K. Ono, 2008, Harvard-MIT Developments in Mathematics Conference, Int. Press , Somerville, MA (2009) 347–454;

[68] K. Ono and S. Ahlgren, Congruence properties for the partition function, *Proc. Natl. Acad. Sci. USA*, **98** no. 23 (2001).

[69] S. Ponnusamy and M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, *Mathematika*, **44** (1997) 278–301.

[70] S. Ram, Srinivasa Ramanajuan, National Book Trust, New Delhi, 2nd ed., 2000.

[71] S. Ramaseshan, Srinivasa Ramanujan, *Current Science*, **59** (1990) 1309–1316.

[72] M. Ram Murty, V. Kumar Murty The Mathematical Legacy of Srinivasa Ramanujan, Springer India (2012).

[73] K. Srinivasa Rao, Srinivasa Ramanujan, East West Books, Madras (1998).

[74] K. Srinivasa Rao, Srinibasa Ramanujan—from Kumbakonam to Cambridge, Asia Pacific Mathematics Newsletter, Vol. 2, No. 2, World Scientific Publishing.

[75] P. Sarnak, Notes on the generalized Ramanujan conjectures, Harmonic analysis, the trace formula, and Shimura varieties, *Clay Math. Proc.*, **4**, *Amer. Math. Soc.*, Providence, RI (2005).

[76] E. Shils, Reflections on tradition, centre and periphery, and the universal validity of science: The significance of the life of S. Ramanujan, Minerva **29** (1991) 393–419.

[77] K. Venkatachaliengar, Development of elliptic functions according to Ramanujan, Edited, revised and with a preface by Shaun Cooper, Monographs in Number Theory, 6. World Scientific Publishing Co. Pvt. Ltd., Hackensack, NJ (2012) xvi+168.

[78] M. Wakimoto, Representation theory of affine superalgebras at the critical level, *Proc. Int. Congress of Math.*, Vol. II, (Berlin, 1998), *Doc. Math.* (1998) Extra, Vol. II, 605–614.

[79] D. A. B. Young, Ramanujan's Illness, *Notes Rec. Royal Soc. London*, **48** (1994) 107–119.

[80] D. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, *Topology*, **40** (2001) 945–960.

[81] D. Zagier, Ramanujan's mock theta functions and their applications [after Zwegers and Bringmann-Ono], Séminaire Bourbaki, 2007–2008, Astérisque, no. 326 (2009).

[82] S. P. Zwegers, Mock θ-functions and real analytic modular forms, q-series with applications to combinatorics, number theory, and physics (Ed. B. C. Berndt and K. Ono), *Contemp. Math.*, **291**, *Amer. Math. Soc.* (2001) 269–277.

[83] S. P. Zwegers, Mock theta functions, Ph.D. Thesis (Advisor: D. Zagier), Universiteit Utrecht (2002).

Problems and Solutions

Edited by Amritanshu Prasad

E-mail: `problems@imsc.res.in`

This section of the Newsletter contains problems contributed by the mathematical community. These problems range from mathematical brain-teasers through instructive exercises to challenging mathematical problems. Contributions are welcome from everyone, students, teachers, scientists and other maths enthusiasts. We are open to problems of all types. We only ask that they be reasonably original, and not more advanced than the MSc level. A cash prize of Rs. 500 will be given to those whose contributions are selected for publication (for administrative reasons, payments will be made within India only). Please send solutions along with the problems. The solutions will be published in the next issue. Please send your contribution to `problems@imsc.res.in`, with the word " **problems**" somewhere in the subject line. Please also send solutions to these problems (with the word "**solutions**" somewhere in the subject line). Selected solutions will be featured in the next issue of this Newsletter.

1. **Parameswaran Sankaran, IMSc Chennai.** If a_1, $a_2, a_3, \ldots$ is a sequence of non-zero numbers such that each member of this sequence (starting from the second one) is one less than the product of its neighbors, show that

$a_{n+5} = a_n$ for all $n \geq 1$. Find all such sequences which are constant[4].

2. **Amritanshu Prasad, IMSc Chennai.** If a stick is broken into three pieces randomly, what is the probability that these three pieces can be used to form the sides of a triangle?

3. **Amritanshu Prasad, IMSc Chennai.** Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. For each positive integer i and each permutation w, let $x_i(w)$ denote the number of cycles of length i in w. For example, if w is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 4 & 6 & 7 & 8 & 9 & 10 \\ 9 & 6 & 4 & 3 & 1 & 5 & 7 & 10 & 2 & 8 \end{pmatrix}$$

then w has cycle decomposition $(1, 9, 2, 6, 5)(3, 4)(8, 10)(7)$, so $x_1(w) = 1$, $x_2(w) = 2$, $x_5(w) = 1$ and $x_i(w) = 0$ for all other i. Show that every class function on S_n can be represented by a polynomial in $x_1, x_2, \ldots, x_n$.

4. **B. Sury, ISI Bangalore.** Write $\{1, 2, \cdots, 2^{2014}\} = A \cup B$ into subsets A, B of equal size such that

$$\sum_{a\in A} a = \sum_{b\in B} b$$

$$\sum_{a\in A} a^2 = \sum_{b\in B} b^2$$

$$\sum_{a\in A} a^3 = \sum_{b\in B} b^3$$

$$\vdots \; \vdots \; \vdots$$

$$\sum_{a\in A} a^{2013} = \sum_{b\in B} b^{2013}.$$

5. **B. Sury, ISI Bangalore.** Recall:
A Fermat prime is a prime number of the form $2^{2^n} + 1$.
A Mersenne prime is a prime number of the form $2^n - 1$.
A Wieferich prime is a prime number p satisfying $2^{p-1} \equiv 1 \pmod{p^2}$.
Prove that a Wieferich prime cannot be a Fermat prime or a Mersenne prime.

[4] In the book *Mathematicians: an outer view of the inner world* (Princeton University Press, 2009) the mathematician Don Zagier says "I like explicit, hands-on formulas. To me they have a beauty of their own. They can be deep or not. As an example, imagine you have a series of numbers such that if you add 1 to any number you will get the product of its left and right neighbors. Then this series will repeat itself at every fifth step! For instance, if you start with 3, 4 then the sequence continues: 3, 4, 5/3, 2/3, 1, 3, 4, 5/3, etc. The difference between a mathematician and a non-mathematician is not just being able to discover something like this, but to care about it and to be curious about why it's true, what it means, and what other things in mathematics it might be connected with. In this particular case, the statement itself turns out to be connected with a myriad of deep topics in advanced mathematics: hyperbolic geometry, algebraic K-theory, the Schrodinger equation of quantum mechanics, and certain models of quantum field theory. I find this kind of connection between very elementary and very deep mathematics overwhelmingly beautiful." Some insight into this statement can be found at `http://math.stackexchange.com/q/11650/10126`.

Solutions to Problems from the September Issue

1. **D. Surya Ramanna, Harish-Chandra Research Institute, Allahabad.** Find all solutions to the equation $x^y = y^x$ in the positive integers.

Solution. Raising both sides of the given equation to the power $\frac{1}{xy}$ shows that it is equivalent to the equation

$$x^{1/x} = y^{1/y}.$$

The derivative of the function $f(x) = x^{1/x}$ is given by

$$f'(x) = x^{1/x-2}(1 - \log x).$$

Therefore $f(x)$ has at most one extremum, which occurs at at $x = e$. Therefore, if there exist integers $x < y$ such that $x^{1/x} = y^{1/y}$, then $x < e < y$. Thus the only possibilities for x are $x = 1$ and $x = 2$. The case $x = 1$ is ruled out, because $y^{1/y} > 1$ for all $y > e$. For $x = 2$, there is a solution: take $y = 4$. Thus, besides $x = y$, the only solutions are $x = 2$ and $y = 4$ and, by symmetry, $x = 4$ and $y = 2$.

2. **Thomas Moore, Bridgewater State University.** Squares $s_n = n^2$ and triangular numbers $t_n = \frac{n(n+1)}{2}$ are well known. The Jacobsthal numbers $j_n = \frac{2^n-(-1)^n}{3}$ for $n \geq 1$ are somewhat less well-known. Find a second degree polynomial with integer coefficients $f(x)$, such that, whenever the input s_n, t_n, or j_n, $n \geq 1$, the output is a triangular number.

Solution. Let $f(x) = 2x^2 - x$. We have

$$f(x) = 2x^2 - x = \frac{(2x)(2x-1)}{2}.$$

So for every integer n, $f(n)$ is a triangular number t_{2n-1} (the sequences s_n and j_n are red herrings).

3. **Kamalakshya Mahatab, IMSc and Kannappan Sampath, ISI Bangalore** Let $f(x)$ be a monic polynomial in $\mathbf{Z}[x]$ with a factorization

$$f(x) = \prod_{i=1}^{m} f_i(x),$$

with $f_1, \ldots, f_m$ are monic polynomials having no common factors. Let $g(x) \in \mathbb{Z}[x]$ be such that the remainder of $g(x)$ upon division by $f_i(x^n)$ is a polynomial in x^n for each i. Then the residue of $g(x)$ upon division by $f(x)$ is a polynomial in x^n.

Solution. Using induction, it is easy to reduce to the case where $m = 2$. So assume that $f(x) = f_1(x)f_2(x)$ where f_1 and f_2 have no common factors.

The remainder of $g(x)$ upon division by $f_1(x^n)$ is a polynomial in x^n, i.e.,

$$g(x) = q_1(x)f_1(x^n) + r_1(x^n)$$

for some $r_1(x)$ with $\deg r_1 < degf_1$. On the other hand,

$$q_1(x) = q_{12}(x)f_2(x^n) + r_{12}(x)$$

with $\deg r_{12} < n \deg f_2$. Combining these, we get

$$g(x) = q_{12}(x)f(x^n) + r_{12}(x)f_1(x^n) + r_1(x^n).$$

Now $\deg(r_{12}(x)f_1(x^n) + r_1(x^n)) < \deg f$, so $r_{12}(x)$ $f_1(x^n) + r_1(x^n)$ is the remainder when $g(x)$ is divided by $f(x)$.

Similarly, we can write

$$\begin{aligned} g(x) &= q_2(x)f_2(x^n) + r_2(x^n) \\ &= q_{21}(x)f(x^n) + r_{21}(x)f_2(x^n) + r_2(x^n), \end{aligned}$$

with $\deg r_2 < \deg f_2$, $\deg r_{21} < n \deg f_1$, and hence $\deg(r_{21}(x)f_2(x^n) + r_2(x^n)) < \deg f$. By the uniqueness of remainder in Euclid's algorithm,

$$r_{12}(x)f_1(x^n) + r_1(x^n) = r_{21}(x)f_2(x^n) + r_2(x^n).$$

Taking $r'_{ij}(x)$ to be the sum of monomials in $r_{ij}(x)$ of the form $c_r x^r$ where r does not divide n gives

$$r'_{12}(x)f_1(x^n) = r'_{21}(x)f_2(x^n)$$

Since f_1 and f_2 have no common factors, nor do $f_1(x^n)$ and $f_2(x^n)$. Therefore, $f_2(x^n)|r'_{21}(x)$ and $f_1(x^n)|r'_{12}(x)$. But $\deg r'_{12} < n \deg f_1$ and $\deg r'_{21} < n \deg f_2$, whence $r'_{12} = 0$ and $r'_{21} = 0$. It follows that $r_{12}(x)$ is a polynomial in x^n, and therefore, so is $g(x)$.

4. **Rahul Dattatraya Kitture, Bhaskaracharya Pratishthana, Pune.**

(1) Prove that a group G can be written as a (set-theoretic) union of proper subgroups if and only if G is not cyclic.
(2) Use (1) to prove that if G is a non-abelian group, then $G/Z(G)$ is not cyclic (here $Z(G)$ denotes the centre of G).

Solution.

(1) If G is cyclic with generator x, then x can not lie in any proper subgroup, and so G can not be written as a union of proper subgroups. If G is not cyclic, then $G = \bigcup_{x \in G} \langle x \rangle$ expresses G as a union of proper subgroups.
(2) For $x \in G$, let $Z_G(x) = \{g \in G \mid xg = gx\}$, the centralizer of x in G. Suppose G is non-abelian. Then $Z(G) \neq G$, and for any $x \in G - Z(G)$, $Z_G(x)$ is a proper subgroup of G which also contains $Z(G)$. Therefore

$$G = \bigcup_{x \in G - Z(G)} Z_G(x), \quad \text{hence}$$

$$G/Z(G) = \bigcup Z_G(x)/Z(G).$$

But for any $x \in G - Z(G)$, $Z_G(x)/Z(G)$ is a proper subgroup of $G/Z(G)$, and so by (2), $G/Z(G)$ is not cyclic.

Proposer's Comment. In almost all books on Algebra or Group Theory, the statement (2) usually appears in the form

$$\text{If } G/Z(G) \text{ is cyclic then } G \text{ is abelian.} \qquad (*)$$

But after proving that G is abelian, we have $Z(G) = G$ and $G/Z(G) = 1$. So the hypothesis holds only vacuously.

Some books, such as *Groups and Representations* by Alperin and Bell state $(*)$ as in (2), but the proof is given by contradiction. The purpose of this problem is to give a proof without contradiction and I could not see such a proof anywhere.

5. **K. N. Raghavan, IMSc.** Let n be a a positive integer. Define d_n to be $(n+1)(n-1)$ if n is odd, and to be $(n+2)(n-2)$ if n is even; define e_n to be 4 or 0 accordingly as n is divisible by 3 or not. Show that the number of ways in which n can be written as a sum of three positive integers is $(d_n+e_n)/12$. (Can you write down an analogous expression for the number of ways of writing n as a sum of four positive integers?)

Solution. Let $P = P(x,t) := (1-xt)(1-xt^2)(1-xt^3)\cdots$. Then the number we are seeking, viz., the number of ways to write n as a sum of three natural numbers is the coefficient of x^3t^n in $1/P$. We may therefore find this number as $1/6$ times the coefficient of t^n in $d^3(1/P)/dx^3|_{x=0}$.

We observe that $d(1/P)/dx = Q/P$ where $Q = t/(1-xt)+t^2/(1-xt^2)+t^3/(1-xt^3)+\cdots$. By the product rule of differentiation, we get

$$\frac{d^2(1/P)}{dx^2} = Q^2/P + Q'/P$$

$$\frac{d^3(1/P)}{dx^3} = \frac{Q^3}{P} + 3\frac{QQ'}{P} + \frac{Q''}{P}$$

where as usual $Q' = dQ/dx$ and $Q'' = d^2Q/dx^2$. An easy calculation shows:

$$\frac{Q'}{1!} = \frac{t^2}{(1-xt^2)} + \frac{t^4}{(1-xt^4)} + \frac{t^6}{(1-xt^6)} + \cdots$$

and

$$\frac{Q''}{2!} = \frac{t^3}{(1-xt^3)} + \frac{t^6}{(1-xt^6)} + \frac{t^9}{(1-xt^9)} + \cdots$$

On setting $x = 0$, we get

$$P|_{x=0} = 1; \quad Q|_{x=0} = (t+t^2+t^3+\cdots);$$
$$Q'|_{x=0} = (t^2+t^4+t^6+\cdots);$$
$$Q''|_{x=0} = 2(t^3+t^6+t^9+\cdots)$$

The coefficient of t^n in $(t+t^2+t^3+\cdots)^3$ is $\binom{n-1}{2}$, in $(t+t^2+t^3+\cdots)(t^2+t^4+t^6+\cdots)$ is $\lfloor\frac{n-1}{2}\rfloor$, and in $2(t^3+t^6+t^9+\cdots)$ is of course $e_n/2$.

The number of ways of writing n as a sum of three natural numbers is therefore:

$$\frac{1}{6}\left(\binom{n-1}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor + \frac{e_n}{2}\right)$$

which is easily seen to simplify to $(d_n+e_n)/12$.

6. **K. N. Raghavan, IMSc.** Let n be a positive integer, $n \geq 2$. On the real two dimensional plane, consider the following two linear transformations: s is the reflection in the x-axis; and r rotation counter-clockwise by the angle $2\pi/n$.

$$sx = x \quad rx = x\cos 2\pi/n + y\sin 2\pi/n$$
$$sy = -y \quad ry = -x\sin 2\pi/n + y\cos 2\pi/n$$

Define a polynomial $f(x,y)$ to be *invariant* if $f(rx,ry) = f(sx,sy) = f(x,y)$. For example, the polynomial $f_0(x,y) = x^2+y^2$ is invariant. (Geometrically, the two transformations s and r being respectively a reflection and a rotation, they preserve the inner product, which explains the invariance of $f_0(x,y)$.) Show that there exists a homogeneous invariant polynomial $f_1(x,y)$ of degree n such that, together with $f_0(x,y)$, it generates the ring of polynomial invariants (that is, any invariant polynomial is a polynomial with real coefficients in f_0 and f_1). For example, for the values 2 and 3 of n, we could take f_1 respectively to be x^2-y^2 and x^3-3xy^2. Find an explicit closed formula for such an f_1 in terms of n. Show moreover that the expression as a polynomial in f_0 and f_1 of any invariant polynomial is unique.

Solution. The trick is to complexify. Put $\zeta = e^{2\pi i/n}$, $\bar{\zeta} = e^{-2\pi i/n}$, $z = x+iy$, and $w = \bar{z} = x-iy$. Then z and w are eigenvectors for r with eingenvalues $\bar{\zeta}$ and ζ respectively. And s just switches z and w.

So, in order that a polynomial f (with complex coefficients) be invariant under s, that is, $f(sz,sw) = f(z,w)$, it is necessary and sufficient that it be symmetric in z and w. And, in order that f be invariant under r, it is necessary and sufficient that it is a polynomial in z^n, w^n, and zw (for, every monomial in z and w is an eigenvector for r, and those monomials with eigenvalue 1 are precisely those of the form $z^{na}(zw)^bw^{nc}$, where we may of course assume that at most one of a and c is non-zero).

Let f be an invariant polynomial. Since f is invariant under r, we may write it uniquely as $f = \sum_{b\geq 0} c_{0,b,0}(zw)^b + \sum_{a>0,b\geq 0} c_{a,b,0}z^{na}(zw)^b + \sum_{b\geq 0,c>0} c_{0,b,c}(zw)^bw^{nc}$. Since f is also invariant under s, it is symmetric with respect to z and w, which means $c_{a,b,0} = c_{0,b,a}$, so that $f = \sum_{b\geq 0} c_{0,b,0}(zw)^b + \sum_{a>0,b\geq 0} c_{a,b,0}(z^{na}+w^{na})(zw)^b$. A routine argument (e.g., using induction on the highest value of a such that $c_{a,b,0}$

is non-zero (for some b)) then shows that f can be written uniquely as a polynomial in $z^n + w^n$ and zw.

Observe that $zw = (x+iy)(x-iy) = x^2+y^2$. We could thus take $f_1 = z^n + w^n$.

7. **K. N. Ragahavan, IMSc** A prime number p is called a *Fermat prime* if it is of the form $2^{2^k} + 1$ for some integer k. Prove or disprove the following: for any Fermat prime p with $k \geq 1$, the multiplicative group of units modulo p is generated by 3.

   Solution. The claim is true: 3 is a generator of the group G of units modulo p. To prove the claim, first observe that the cyclic group G, being of order 2^{2^k}, has a unique maximal proper subgroup H, namely the subgroup of order 2^{2^k-1}. It is enough to show that 3 does not belong to H.

   Observe that H consists of the squares in G, in other words of the (non-zero) quadratic residues modulo p. It is thus enough to show that 3 is not a quadratic residue modulo p. We now appeal to the law of quadratic reciprocity. Observe that $p \equiv 1 \bmod 4$ (since $k \geq 1$) and that $p \equiv 2 \bmod 3$. So p is not a quadratic residue modulo 3 and so by the reciprocity law we conclude that nor is 3 a quadratic residue modulo p.

22nd Mathematics Training and Talent Search Programme

(Funded by National Board for Higher Mathematics)

Aims: The aim of the programme is to expose bright young students to the excitement of doing mathematics, to promote independent mathematical thinking and to prepare them for higher aspects of mathematics.

Academic Programmes: The programme will be at three levels: Level O, Level I and Level II. In Level O, there will be courses in Linear Algebra, Analysis and Number Theory/Combinatorics. In Levels I and II, there will be courses on Algebra, Analysis and Topology and Linear Algebra. There will be seminars by students in all Levels.

The faculty for this programme will be active mathematicians with a commitment to teaching and from various leading institutions of the country. The aim of the instructions is not to give routine lectures and presentation of theorem-proofs but to stimulate the participants to think and discover mathematical results.

Eligibility:

- **Level O:** Second year undergraduate students with Mathematics as one of their subjects.
- **Level I:** Final year undergraduate students with Mathematics as one of their subjects.
- **Level II:** First year postgraduate students with Mathematics as one of their subjects. Participants of the previous year Level I Programme.

Venues & Duration: MTTS 2014 will be held at three different places:

- Regional Institute of Education (RIE), Mysore (Main Camp), Duration: May 19 – June 14, 2014.
- Goa University, Panjim.
  Duration: May 12 – June 07, 2014.
- Indian Institute of Technology (IIT), Guwahati
  Duration: June 23 – July 19, 2014.
- There is a plan to have a fourth camp at the Central University of Bihar. If it materializes, the duration will be announced later on the MTTS website.

How to Apply?: Details and application forms can be had from the Head, Department of Mathematics of your Institution. The application forms can also be downloaded from the MTTS website: `http://www.mtts.org.in/`

Students can also apply online by logging in to this site from **December 26, 2014**.

The completed application form should reach the programme director latest by **February 22, 2014**. Please go through the MTTS-FAQ at the website before sending any queries.

Programme Director:

Prof. S. Kumaresan
Director, MTTS Programme
School of Mathematics and Statistics
University of Hyderabad
Hyderabad 500 046

E-mail: `kumaresa@gmail.com`
`mttsprogramme@gmail.com`
Tel: (040) 6679 4059 (O)

Selection: The selection will be purely on merit, based on consistently good academic record and the recommendation letter from a mathematics professor closely acquainted with the candidate.

Only selected candidates will be informed of their selection by the 2nd week of March 2014. The list of selected candidates will be posted on the MTTS website in the 2nd week of March 2014.

Candidates selected for the programme will be paid sleeper class return train fare by the shortest route and will be provided free board and lodging for the duration of the course.

Those who wish to organize mini-MTTS programme or PTMT programme can visit the mtts or ptmt website and apply online.

`http://www.mtts.org.in/`; `http://ptmt.mtts.org.in/`

Mathematics Training and Talent Search Programme

Funded by: NBHM

Websites: http://www.mtts.org.in/; http://www.ptmt.mtts.org.in/

Background: In 1992, Prof. S. Kumaresan, founding Director of MTTS programme, identified a need for a national level training programme in mathematics for bright young students at the B.Sc. level. The first programme was held in the summer of 1993 aimed at exposing bright young students to the excitement of doing mathematics, to promoting independent mathematical thinking and to preparing them for higher aspects of mathematics. Since then, this programme has been conducted with the financial support from the National Board for Higher Mathematics (NBHM). So far, we have trained more than 3000 young bright minds and a large number of them are eminent professionals in various disciplines across the globe. The success of this programme depends mainly on a very small group of mathematicians committed to the improvement of the mathematical scene in the country.

The programme has three levels, Level-O is for 2nd year B.Sc. students of mathematics, Level-I is for 3rd year B.Sc. students of mathematics and Level-II is for 1st year M.Sc. students of mathematics.

With over 20 years of experience, MTTS team has come up with two new initiatives to serve the mathematics community better, namely, **Mini-MTTS and Pedagogical Training for Mathematics Teachers (PTMT)**.

Mini-MTTS Programme: The MTTS programme is one of the most popular and significant summer training programmes in the country. We receive a large number of applications from all parts of the country and the selection is extremely difficult. Due to limitation of resources and seats, a large number of deserving students do not get a chance to participate in MTTS Programme.

In order to provide opportunities to a large number of students at regional levels, the Director, in a meeting held during the International Congress of Mathematicians (ICM) at Hyderabad in August 2010, made an appeal to the mathematics community to organize mini camps in line with MTTS programme. Since then, several Mini-MTTS programmes have been arranged at different parts of the country. The list of such programmes is available on the MTTS website. We have been receiving a large number of requests to organize mini-MTTS programmes. As a result, we have started requesting online application for the same.

PTMT Rogramme: Ever since MTTS started, the participants always came up with the suggestion that such camps may be organized for teachers, so that they can adapt the methodology of MTTS. With the success of MTTS year after year and also on seeing the impact of MTTS on their students, teachers themselves started asking for a workshop which will introduce them to the methodology of MTTS as well as train them in pedagogical aspects of teaching mathematics.

We organized two such camps at Thiagarajar College of Engineering, Madurai. On seeing how well they were received and the appreciation from teacher participants, we approached the NBHM with a proposal to organize such workshops on a regular basis for teachers. In the conference on "Mathematics Education – Trends and Challenges", held in August 2011 at the University of Hyderabad, an official announcement of PTMT programme was made.

The first official camp of PTMT was held at Bhaskaracharya Pratishthana, Pune during 5–10 April 2012. Based on the feedback from the participants, and brainstorming sessions within the MTTS fraternity, we have come up with a detailed plan of such future camps. The academic structure of the programme and other details can be found on the PTMT-website.

Each such camp will be of 11–12 days duration and will concentrate on a single topic such as Real Analysis, Linear Algebra etc. During the first 6 days, the session will be highly interactive and conducted by experienced MTTS faculty on the theme of the workshop. In the next 6 days, the teachers will give seminars on topics of their choice closely related to the theme and moderated by MTTS faculty members.

Due to diversity of academic schedule across the nation, it has been decided to encourage organizing of PTMT at the regional level.

Organizing Mini-MTTS and PTMT Programmes: Those who wish to organize MTTS, mini-MTTS or PTMT programme can visit the MTTS website and apply online. The proposals which assure some partial financial support may be given preference. The proposal can be submitted on

http://www.mtts.org.in/proposal-submission/index.php

Jammu Kashmir Institute of Mathematical Sciences

This Institute inaugurated in June 2013, by the state Chief Minister Omar Abdullah, will be an autonomous Institute of deemed university status. Currently the Institute is functioning in the Amar Singh College Campus, Srinagar. Prof. Wali Mohammad Shah is the Director of the institute and Dr. Raj Shree is the Nodal Officer to its Sub office at Jammu. The main purpose of the institute is to develop a mathematics culture in the state and facilitate the visits of eminent mathematicians and theoretical scientists from within and outside the country. This will help the students, teachers and scholars of the state to interact with them. The institute also proposes to start an integrated masters degree programme in mathematics after the +2 level.

Contact Information:

Jammu And Kashmir Institute of Mathematical Sciences
Amar Singh College Campus, Srinagar 190008.
E-mail: `jkims.director@gmail.com`
Fax no. is: 01942311472

Announcement

A special volume of the Journal of the Indian Mathematical Society to commemorate the 125$^{\text{th}}$ Birth Anniversary of Srinivasa Ramanujan and the National Mathematics Year-2012 was released on 28$^{\text{th}}$ December, 2013 during the inaugural function of the 79th Annual Conference of the Indian Mathematical Society held at Rajagiri School of Engineering & Technology, Cochin, 28–31 December, 2013.

Editor: A. K. Agarwal

Publisher: The Indian Mathematical Society

Contributors:

S. D. Adhikari, S. Bhargava, M. D. Hirschhorn, J. Lovejoy & R. Osburn, A. M. Mathai, M. Ram Murty, N. Robbins, J. A. Sellers.

Preface

Srinivasa Ramanujan at One Hundred Twenty Five

On December 26, 2011, the Hon'ble Prime Minister of India, Dr. Manmohan Singh declared the Year 2012 as the 'National Mathematics Year' to commemorate the 125th birth anniversary of the legendary Indian mathematician Srinivasa Ramanujan and the date December 22, being his birthday has been declared as the 'National Mathematics Day'. Consequently, several academic programs in his memory were organized during the Year 2012. The Indian Mathematical Society has always been in forefront for the celebration of such events, specifically, if they are related to Ramanujan – rightly so, after all, Ramanujan was the discovery of its founder Ramaswamy Aiyar and that Ramanujan published his first paper entitled "Some properties of Bernoulli's numbers" in its journal [J. Indian Math. Soc. 3 (1911), 219–234]. This time also the Society rose to the occasion and decided to bring out a special issue of its journal dedicated to Ramanujan. I was given the responsibility of its editorship. I contacted several mathematicians working in areas influenced by Ramanujan such as hypergeometric functions, partition theory, modular equations, continued fractions and mock theta functions through E-mails and invited them for their contributions. I also met many of them personally in two conferences: (1) International Conference on the Works of Srinivasa Ramanujan and Related Topics, University of Mysore, December 12–14, 2012, and (2) International Conference on the Legacy of Srinivasa Ramanujan, University of Delhi, December 17–22, 2012 and reminded them of my invitation. In all eight mathematicians responded positively to our invitation. The present volume is the collection of their articles. The Indian Mathematical Society is presenting this special volume as its tribute to the great mathematician who was described by G.H. Hardy as "a natural genius". Hardy also compared him with the eminent mathematicians Jacobi, Gauss and Euler. There are two sides of Ramanujan's personality. We all know that he was a great mathematician, as my personal tribute to him on this occasion, I recall the other side of his personality, which is: he was a great human being, most humble and completely free from egoism. He lived and worked in the true spirit of the Bhagavad Geeta wherein Lord Krishna says (Chapter 3, verse 30):

mayi sarvani karmani
amnyasa dhyatmacetasa
nirasir nirmano bhutva
yudhyasva vigatajvarah

(Renouncing all actions in Me, with the mind centred on Me free from lethargy and egoism, discharge your duties without claims of proprietorship).

Ramanujan did not take the credit for his research but renounced it in the Goddess Namagiri of Namakhal.

I thank all the invited mathematicians for their valuable contributions and those experts who have helped us in refereeing the articles but preferred to remain anonymous.

A.K. Agarwal, Editor
Centre for Advanced Study in Mathematics
Panjab University Chandigarh
Chandigarh 160 014, India

IITM Summer Fellowship 2014

Applications are invited for IITM Summer Fellowship Programme from the students pursuing 3rd year of B.E./B.Tech/B.Sc. (Engg.)/Integrated M.E./M.Tech. programme, 1st year of M.Sc./M.A./MBA with outstanding academic background in terms of high ranks (within three ranks) in university examinations. IIT students are not eligible. The fellowship is for a maximum period of two months with stipend of Rs.5,000/- per month. Further details are at the following URL and in the document linked along.

http://www.iitm.ac.in/internship/

Details of Workshop/Conferences in India

For details regarding Advanced Training in Mathematics Schools
Visit: http://www.atmschools.org/

Name: Nonlinear Dynamics and complex systems – 3rd International Symposium on Complex Dynamical Systems and Applications
Date: March 10–March 12, 2014
Location: Indian Statistical Institute, Kolkata
Visit: http://www.isical.ac.in/CDSA2014.pdf

Name: ACODS 2014 – Third International Conference on Advances in Control and Optimization of Dynamical Systems
Date: March 13–March 15, 2014
Location: Indian Institute of Technology Kanpur
Visit: http://www.iitk.ac.in/acods2014/Home_ACODS-2014.html

Name: 11th Annual Conference on Theory and Applications of Models of Computation
Date: April 11–13, 2014
Location: Anna University, Chennai
Visit: http://www.annauniv.edu/tamc2014/

Name: Recent Advances in Modeling Rare Events
Date: May 29–June 1, 2014
Location: Kumarakom, Kerala
Visit: http://www.annauniv.edu/tamc2014/

Name: ICMS-2014 – International Conference on Mathematical Sciences
Date: July 17–July 19, 2014
Location: Sathyabama University, Chennai
Visit: http://icms2014.com/

Details of Workshop/Conferences in Abroad

For details regarding ICTP (International Centre for Theoretical Physics)
Visit: http://www.ictp.it/

Name: WSCG 2014 – 22nd International Conference on Computer Graphics, Visualization and Computer Vision 2014
Date: June 2–5, 2014
Location: Primavera Hotel and Congress Centrum, Plzen (close to Prague), Czech Republic
Visit: http://www.wscg.eu

Name: Computational Nonlinear Algebra
Date: June 2–6, 2014
Location: Institute for Computational and Experimental Research in Mathematics, (ICERM), Brown University, Providence, Rhode Island
Visit: http://icerm.brown.edu/tw-14-3-cna

Name: Conference on Ulam's type stability
Date: June 2–6, 2014
Location: Rytro, Poland
Visit: http://cuts.up.krakow.pl

Name: Discrete Groups and Geometric Structures, with Applications V
Date: June 2–6, 2014
Location: KU Leuven, Arenberg Castle, Heverlee (nearby Leuven), Belgium
Visit: http://www.kuleuven-kulak.be/workshop

Name: Hamiltonian Systems and Celestial Mechanics (HAMSYS 2014)
Date: June 2–6, 2014
Location: Centre de RecercaMatematica, Bellaterra, Barcelona, Spain.
Visit: http://www.crm.cat/2014/CHamsys

Name: Joint ICTP - TWAS School on Coherent State Transforms, Time-Frequency and Time-Scale Analysis, Applications
Date: June 2–21, 2014
Location: The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy
Visit: http://agenda.ictp.it/smr.php?2585

Name: Moduli – Operads – Dynamics II
Date: June 3–6, 2014
Location: Tallinn University of Technology, Tallinn, Estonia
Visit: http://www.astralgo.com/cweb/mod14/

Name: School in Dynamical Systems and Ergodic Theory
Date: June 4–14, 2014
Location: Cheikh Anta Diop University, Dakar, Senegal and African Institute for Mathematical Sciences, Mbour, Senegal
Visit: http://agenda.ictp.it/smr.php?2622

Name: Number Theory at Illinois: A Conference in Honor of the Batemans
Date: June 5–7, 2014
Location: University of Illinois, Urbana, Illinois
Visit: http://www.math.illinois.edu/nt2014

Name: 7th Chaotic Modeling and Simulation International Conference (CHAOS2014)
Date: June 7–10, 2014
Location: Lisbon, Portugal
Visit: http://www.cmsim.org

Name: AIM Workshop: The Cauchy-Riemann equations in several variables
Date: June 9–13, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: http://www.aimath.org/ARCC/workshops/crscv.html

Name: Categorification and Geometric Representation Theory
Date: June 9–13, 2014
Location: Centre de recherchesmathématiques, Université de Montréal, Pavillon André-Aisenstadt, Montréal (Québec), Canada
Visit: http://www.crm.umontreal.ca/2014/Categorification14/index_e.php

Name: String Math 2014
Date: June 9–13, 2014
Location: University of Alberta, Edmonton, Alberta, Canada
Visit: http://sites.google.com/a/ualberta.ca/stringmath2014/

Name: Tenth edition of the Advanced Course in Operator Theory and Complex Analysis
Date: June 9–13, 2014
Location: Sevilla, Spain
Visit: http://congreso.us.es/ceacyto/2013

Name: Representations, Dynamics, Combinatorics: In the Limit and Beyond. A conference in honor of Anatoly Vershik's 80th birthday
Date: June 9–14, 2014
Location: Saint-Petersburg, Russia
Visit: http://www.pdmi.ras.ru/EIMI/2014/RDC/

Name: School on Nonlinear Analysis, Function Spaces and Applications 10
Date: June 9–15, 2014
Location: Trest, Czech Republic
Visit: http://nafsa.cuni.cz/2014/

Name: Interactions between Dynamics of Group Actions and Number Theory
Date: June 9–July 4, 2014
Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom
Visit: http://www.newton.ac.uk/programmes/GAN/index.html

Name: Geometry of Banach Spaces - A conference in honor of StanimirTroyanski
Date: June 10–13, 2014
Location: Albacete, Spain
Visit: http://sites.google.com/site/geometryofbanachspaces/

Name: Riemann, topology and physics
Date: June 12–14, 2014
Location: Institut de RechercheMathematiqueAvancee, University of Strasbourg, Strasbourg, France
Visit: `http://www-irma.u-strasbg.fr/article1375.html`

Name: Matm2014-Methodological Aspects of Teaching Mathematics
Date: June 14–15, 2014
Location: Faculty of Education in Jagodina, University of Kragujevac, Serbia
Visit: `http://matm2014.blogspot.com/`

Name: 4th European Seminar on Computing (ESCO 2014)
Date: June 15–20, 2014
Location: Pilsen, Czech Republic
Visit: `http://esco2014.femhub.com`

Name: 8th Annual International Conference on Mathematics & Computer Science
Date: June 16–19, 2014
Location: Athens, Greece
Visit: `http://www.atiner.gr/mathematics.htm`

Name: Conference on stochastic processes and high dimensional probability distributions
Date: June 16–20, 2014
Location: Euler International Mathematical Institute of the Russian Academy of Sciences, Saint Petersburg, Russia
Visit: `http://www.pdmi.ras.ru/EIMI/2014/Sppd/index.html`

Name: Strathmore University International School on Spatial Modelling (ISSM-2014)
Date: June 16–20, 2014
Location: Strathmore University, Nairobi, Kenya
Visit: `http://www.strathmore.edu/carms`

Name: Fifth International Conference and School Geometry, Dynamics, IntegrableSystems. GDIS 2014: Bicentennial of The Great Poncelet Theorem and Billiard Dynamics
Date: June 16–27, 2014
Location: The Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy
Visit: `http://agenda.ictp.it/smr.php?2586`

Name: Summer Graduate School: Dispersive Partial Differential Equations
Date: June 16–27, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/summer_schools/713`

Name: First International Congress on Actuarial Science and Quantitative Finance
Date: June 17–20, 2014
Location: Universidad Nacional de Colombia, Bogota, Colombia
Visit: `http://www.matematicas.unal.edu.co/icasqf/`

Name: XIV Encuentro de Algebra Computacional y Aplicaciones EACA 2014
Date: June 18–20, 2014
Location: Barcelona, Spain
Visit: `http://www.ub.edu/eaca2014/`

Name: Summer school on the Gan-Gross-Prasad conjectures
Date: June 18–27, 2014
Location: Institut de Mathématiques de Jussieu – Paris, Rive Gauche 4, place Jussieu, Paris, France
Visit: http://ggp-2014.sciencesconf.org/?lang=en

Name: MSRI-UP 2014: Arithmetic Aspects of Elementary Functions
Date: June 21–August 3, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: http://www.msri.org/msri_ups/735

Name: The 52nd International Symposium on Functional Equations
Date: June 22–29, 2014
Location: Innsbruck, Austria
Participation at these annual meetings is by invitation only. Those wishing to be invited should send details of their interest and, preferably, publications with their postal and E-mail address to: Roman Ger, Institute of Mathematics, Silesian University, Bankowa 14, PL-40-007 Katowice, Poland; E-mail: romanger@us.edu.pl before February 15, 2014.

Name: 17th Conference on Integer Programming and Combinatorial Optimization (IPCO 2014)
Date: June 23–25, 2014
Location: University of Bonn, Bonn, Germany
Visit: http://www.or.uni-bonn.de/ipco/

Name: Boltzmann, Vlasov and related equations: Last results and open problems
Date: June 23–27, 2014
Location: University of Cartagena, Cartagena, Colombia
Visit: http://matematicas.unicartagena.edu.co/

Name: Microlocal analysis and applications
Date: June 23–27, 2014
Location: Université de Nice Sophia Antipolis, Nice, France
Visit: http://math.unice.fr/MAA2014/

Name: What Next? The mathematical legacy of Bill Thurston
Date: June 23–27, 2014
Location: Cornell University, Ithaca, New York
Visit: http://www.math.cornell.edu/~festival

Name: 6th International Conference on Advanced Computational Methods in Engineering
Date: June 23–28, 2014
Location: NH Gent Belfort, Gent, Belgium
Visit: http://www.acomen.ugent.be

Name: Mathematics Meets Physics
Date: June 24–27, 2014
Location: University of Helsinki, Finland
Visit: http://wiki.helsinki.fi/display/mathphys/mathphys2014
http://wiki.helsinki.fi/download/attachments/105154551/mmp_poster.pdf

Name: Sixth International Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'14)
Date: June 26–July 1, 2014

Location: Black-Sea resort, Albena, Bulgaria
Visit: `http://2014.eac4amitans.eu`

Name: 26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC)
Date: June 29–July 3, 2014
Location: DePaul University, Chicago, Illinois
Visit: `http://sites.google.com/site/fpsac2014/`

Name: 8th Annual International Conference on Statistics
Date: June 30–July 3, 2014
Location: The Mathematics & Statistics Research Unit (ATINER), Athens Institute for Education and Research, Athens, Greece
Visit: `http://www.atiner.gr/statistics.htm`

Name: 25th International Conference in Operator Theory
Date: June 30–July 5, 2014
Location: West University of Timisoara, Timisoara, Romania
Visit: `http://operatortheory25.wordpress.com`

Name: Clay Mathematics Institute Summer School 2014 Periods and Motives: Feynman amplitudes in the 21st century
Date: June 30–July 25, 2014
Location: ICMAT, Instituto de Cienciasmatemáticas, Madrid, Spain
Visit: `http://www.icmat.es/summerschool2014/`

Name: The 2014 International Conference of Applied and Engineering Mathematics
Date: July 2–4, 2014
Location: Imperial College London, London, United Kingdom
Visit: `http://www.iaeng.org/WCE2014/ICAEM2014.html`

Name: 2014 International Conference on Topology and its Applications
Date: July 3–7, 2014
Location: University of Patras and Technological Educational Institute of Western Greece, Nafpaktos, Greece
Visit: `http://www.lepantotopology.gr`

Name: International Conference "Mathematics Days in Sofia"
Date: July 7–10, 2014
Location: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria
Visit: `http://www.math.bas.bg/mds2014`

Name: 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications
Date: July 7–11, 2014
Location: Universidad Autónoma de Madrid, Madrid, Spain
Visit: `http://www.aimsciences.org/conferences/2014/index.html`

Name: Conferences on Intelligent Computer Mathematics, CICM 2014
Date: July 7–11, 2014
Location: University of Coimbra, Coimbra, Portugal
Visit: `http://cicm-conference.org/2014/cicm.php`

Name: Summer Graduate School: Stochastic Partial Differential Equations
Date: July 7–18, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/summer_schools/714`

Name: The Geometry, Topology and Physics of Moduli Spaces of Higgs Bundles
Date: July 7–August 29, 2014
Location: Institute for Mathematical Sciences, National University of Singapore, Singapore
Visit: `http://www2.ims.nus.edu.sg/Programs/014geometry/index.php`

Name: 2014 World Conference on Natural Resource Modeling
Date: July 8–11, 2014
Location: Vilnius University, Vilnius, Lithuania
Visit: `http://www.resourcemodellingconference2014.com`

Name: 10th International Workshop on Automated Deduction in Geometry, ADG 2014
Date: July 9–11, 2014
Location: University of Coimbra, Coimbra, Portugal
Visit: `http://www.uc.pt/en/congressos/adg/adg2014/`

Name: Applications of Computer Algebra, ACA 2014
Date: July 9–12, 2014
Location: Fordham University, New York, New York
Visit: `http://faculty.fordham.edu/rlewis/aca2014/`

Name: 8th International Conference on Modelling in Industrial Maintenance and Reliability (MIMAR)
Date: July 13–15, 2014
Location: St. Catherine's, Oxford, United Kingdom
Visit: `http://www.ima.org.uk/conferences/conferences_calendar/mimar8.cfm`

Name: AIM Workshop: Mori program for Brauer log pairs in dimension three
Date: July 14–18, 2014
Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://www.aimath.org/ARCC/workshops/moribrauerlog.html`

Name: The 30th International Colloquium on Group Theoretical Methods in Physics
Date: July 14–18, 2014
Location: Ghent University, Ghent, Belgium
Visit: `http://www.group30.ugent.be`

Name: Theory of Water Waves
Date: July 14–August 8, 2014
Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom
Visit: `http://www.newton.ac.uk/programmes/TWW/`

Name: Sixteenth International Conference on Fibonacci Numbers and Their Applications
Date: July 20–25, 2014
Location: Rochester Institute of Technology, Rochester, New York
Visit: `http://www.mathstat.dal.ca/fibonacci/`

Name: Mixed Integer Programming (MIP) workshop 2014
Date: July 21–24, 2014
Location: The Ohio State University, Columbus, Ohio
Visit: `http://mip2014.engineering.osu.edu/home`

Name: Geometric and Asymptotic Group Theory with Applications (GAGTA)
Date: July 21–25, 2014

Location: The University of Newcastle, Australia
Visit: `http://sites.google.com/site/gagta8/`

Name: Mathematics and Engineering in Marine and Earth Problems
Date: July 21–25, 2014
Location: University of Aveiro, Portugal
Visit: `http://meme.glocos.org`

Name: Perspectives of Modern Complex Analysis
Date: July 21–25, 2014
Location: Banach Conference Center, Bedlewo, Poland
Visit: `http://bcc.impan.pl/14Perspectives/`

Name: Quantum Control Engineering: Mathematical Principles and Applications
Date: July 21–August 15, 2014
Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom
Visit: `http://www.newton.ac.uk/programmes/QCE/`

Name: International Symposium on Symbolic and Algebraic Computation (ISSAC 2014)
Date: July 23–25, 2014
Location: Kobe University, Japan
Visit: `http://www.issac-conference.org/2014`

Name: 99 years of General Relativity: ESI-EMS-IAMP Summer school on Mathematical Relativity
Date: July 28–August 1, 2014
Location: Erwin Schrödinger Institute, Vienna, Austria
Visit: `http://homepage.univie.ac.at/piotr.chrusciel/SummerSchool2014/`

Name: Summer Graduate School: Geometry and Analysis
Date: July 28–August 8, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/summer_schools/712`

Name: XIX EBT – 19th Brazilian Topology Meeting
Date: August 3–9, 2014
Location: State University of São Paulo (UNESP), São José do Rio Preto, Brazil
Visit: `http://www.mat.ibilce.unesp.br/EBT2014/`

Name: 10th International Conference on Clifford Algebras and their Applications in Mathematical Physics (ICCA10)
Date: August 4–9, 2014
Location: University of Tartu, Tartu, Estonia
Visit: `http://icca10.ut.ee`

Name: SIAM Conference on Nonlinear Waves and Coherent Structures (NW14)
Date: August 11–14, 2014
Location: Churchill College, University of Cambridge, Cambridge, United Kingdom
Visit: `http://www.siam.org/meetings/nw14/`

Name: AIM Workshop: Neglected infectious diseases
Date: August 11–15, 2014

Location: American Institute of Mathematics, Palo Alto, California
Visit: `http://www.aimath.org/ARCC/workshops/neglectedinfect.html`

Name: New geometric methods in number theory and automorphic forms
Date: August 11–December 12, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/web/msri/scientific/programs/show/-/event/Pm8996`

Name: Recent Developments in Adaptive Methods for PDEs, Collaborative Workshop and Short Course
Date: August 17–22, 2014
Location: Memorial University of Newfoundland, St. John's, Newfoundland, Canada
Visit: `http://www.math.mun.ca/anasc`

Name: Geometric Representation Theory
Date: August 18–December 19, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/web/msri/scientific/programs/show/-/event/Pm8951`

Name: Understanding Microbial Communities; Function, Structure and Dynamics
Date: August 11–December 19, 2014
Location: Isaac Newton Institute, Cambridge, United Kingdom
Visit: `http://www.newton.ac.uk/programmes/UMC/`

Name: Connections for Women: New Geometric Methods in Number Theory and Automorphic Forms
Date: August 14–15, 2014
Location: Mathematical Sciences Research Institute, Berkeley, California
Visit: `http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9806`

Name: Seventh International Conference on Differential and Functional Differential Equations
Date: August 22–29, 2014
Location: Peoples' Friendship University of Russia, Moscow, Russia.
Visit: `http://dfde2014.mi.ras.ru`

Name: First Brazilian Workshop in Geometry of Banach Spaces BWB 2014
Date: August 25–29, 2014
Location: Maresias Beach Hotel, Maresias (Sao Sebastio), Brazil
Visit: `http://www.ime.usp.br/~banach/bwb2014/index.html`

Name: Integrability and Cluster Algebras: Geometry and Combinatorics
Date: August 25–29, 2014
Location: Brown University (ICERM), Providence, Rhode Island
Visit: `http://icerm.brown.edu/tw14-4-ica`

The Mathematics Newsletter may be download from the RMS website at
`www.ramanujanmathsociety.org`