

Remembering Somesh

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Somesh Bagchi (11.09.1946 – 05.10.2012) was a mathematician of extraordinary breadth and depth. His area of research was harmonic analysis on Euclidean spaces and on Lie groups but his mastery extended to many other branches of mathematics.

Somesh Chandra Bagchi was born in Bangladesh (in what was East Pakistan) in a town called Gouripur, in the Netrokona Sub Division of Mymensingh District on 11 September 1946 to Minati and Dinesh Chandra Bagchi. He passed away in Kolkata on 5 October 2012 at the age of 66, after half a year of brave and tough battle against lung cancer and its consequent complications.

Somesh's family continued to live in East Pakistan, even after independence and the partition of Bengal into East and West Bengal, and moved to India (Kolkata, then spelt Calcutta) only after Somesh completed his matriculation in Mymensingh Zilla School. In Kolkata, Somesh completed his Pre-University at Jaipuria College, B.Sc. in Physics from St. Xavier's College, M.Stat., and Ph.D. from the Indian Statistical Institute (ISI) for his dissertation on Vector-Valued Stationary Stochastic Processes under the supervision of Prof. M. G. Nadkarni.

Somesh married Ratna (daughter of Sabita and Gopal Majumdar of Raiganj, North Dinajpur) and has a son Ramanuj. He is survived by his widow and son.

Soon after his Ph.D., he spent the years 1973–75 at the Tata Institute of Fundamental Research in Mumbai, before joining the faculty of the Indian Statistical Institute, Kolkata, where he remained until he retired as a Professor of Mathematics in 2011 at the age of 65. In all these years at the Institute, he took sabbatical leave only for a year which he spent at the University of Puget Sound in Tacoma, Washington State, USA.

Although his Bachelor's degree was in Physics and his Master's degree was in Statistics, his passion was Mathematics, which he did and taught with a great deal of commitment. Even after retiring, he taught at the Vivekananda University in Belur across the Hooghly river from Kolkata. Besides teaching regular courses at the ISI, Somesh taught regularly in various summer schools for researchers or refresher courses and nurture programs for undergraduates, all over the country, whether it was organized by the ISI or other organizations.

Somesh was a fabulous teacher at all levels. He was inspirational. His own passion for Mathematics was conveyed to his students. His lectures were lucid and extraordinarily clear. Scores of students from other institutions, including doctoral students would drop in to get his help, which he unhesitatingly gave. Despite his busy teaching schedule in the ISI and outside, he actively collaborated in research in harmonic analysis with his colleagues and guided three very good doctoral dissertations. He was a reluctant publisher of research papers, even when he had good results. He contributed to the national efforts for improvement of research and training in Mathematics as a member of the National Board for Higher Mathematics, in which capacity he participated in organizing the International Congress of Mathematicians in Hyderabad in August 2010. He was a member of the Executive Committee of the Ramanujan Mathematical Society (2010–2012). He never shirked administrative work and in fact carried out much more than his fair share of it, both academic and general. He was the Dean of Studies, Professor-in-Charge of the Theoretical Statistics and Mathematics Division, and he was even the Acting Director of the ISI for a brief period. Besides a great deal of teaching, he participated whole-heartedly in the admission work of setting up test papers, conducting selection tests, evaluating tests, and interviewing candidates. He was also a member of the Works Advisory Committee, Library Committee, etc. from time to time and carried out these tasks with promptness, efficiency, and good cheer and zeal.

Somesh was a great and popular story teller. He specialized in amusing his friends with witty and humorous real stories conveying the idiosyncracies of our colleagues in the academic world in general and mathematicians in particular. Many a story used to get repeated on different occasions to different audiences but he had this uncanny ability to render them in precisely the same way with exactly the same words every time he repeated them! He was great company and he enlivened the great Bengali institution of adda.

Generations of grateful and admiring students, friends and colleagues spread all over the country and abroad mourn the loss of Somesh Chandra Bagchi and share the grief with his family.

Comparing Cardinalities of Sets

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Abstract. Students often have motivational difficulty in accepting the definitions of comparing cardinalities of sets. The key to the application of the concepts lies in the so called Schröder-Bernstein theorem, the proof of which is often avoided due to its difficulty level. In this short article, we discuss the remarkable results around this theorem and finally give a very simple proof. On the way, we also construct a simple proof of uncountability of the set of real numbers, which follows directly from the completeness property.

1. Introduction

To determine the number of elements in a finite set, we count the elements. If the result is a certain natural number n , then by such a counting, we have put the set in one-one correspondence with the set $\{1, 2, 3, \dots, n\}$. Thus even if we do not know the names of different numbers, we still can determine whether two given finite sets have the same number of elements or not. Well, we simply try to put them in one-one correspondence; if we succeed, then they have the same number of elements, else, they have different number of elements. For arbitrary sets, the *number of elements* is called *cardinality*.

In the sequel, we denote the cardinality of a set A by $|A|$. For sets A and B , we write $A - B$ instead of $A \setminus B$, which is the set of elements of A that are not in B . If $f : A \rightarrow B$ is any map, $x \in A$, and $X \subseteq A$, then we write $f(x)$ for that element in B to which f associates x , and $f(X)$ denotes the set of all elements $f(x)$ where $x \in X$. Thus $f(A)$ is the range of f . For any subset C of B , $f^{-1}(C)$ denotes the set $\{x \in A : f(x) \in C\}$. If $C \cap f(A) = \emptyset$, then $f^{-1}(C) = \emptyset$. When f is one-one and onto, $f^{-1} : B \rightarrow A$ is a map, and $f^{-1}(b) = a$ if and only if $f(a) = b$ for $a \in A$ and $b \in B$.

We agree that if A and B are sets, then $|A| = |B|$ if and only if there exists a one-one map from one onto the other. But then how to define the notion of $|A| \leq |B|$? Looking at the definition of $|A| = |B|$, it is tempting to break it into two parts, that is,

There exists a one-one map from A to B .

There exists a map from A onto B .

But there is a gap between the existence of a one-one map from A onto B and the conjunction of the two conditions above. It is quite possible that even if the above two conditions are

satisfied, there may not exist a map which is both one-one and onto. It suggests to formulate another alternative of breaking $|A| = |B|$ into the following two parts:

There exists a one-one map from A to B .

There exists a one-one map from B to A .

Comparing these two formulations, we ask whether the second parts of them are equivalent or not. That is, whether the following are equivalent:

There exists a map from A onto B .

There exists a one-one map from B to A .

We see that if there exists a one-one map from B to A , then we have a bijection from B to $f(B) \subseteq A$. Then $f^{-1} : f(B) \rightarrow B$ is onto B . Now, if $A - f(B)$ is nonempty, then associate all elements in $A - f(B)$ to a particular element of B . This extension of the map f^{-1} is clearly a map from A onto B .

What about the converse? Suppose there exists a map g from B onto A . Then look at the reverse arrows of this g . For any $a \in A$, consider the set $g^{-1}(\{a\})$. Using Axiom of Choice, get an element $x_a \in g^{-1}(\{a\})$. The map that takes a to x_a is thus one-one from A to B . In the absence of the axiom of choice, it does not seem possible to prove this part.

A natural alternative is to take $|A| \leq |B|$ when there exists a one-one map from A onto a subset of B . However, this is equivalent to having a one-one map from A to B .

2. Uncountability

With these considerations, we compare cardinalities of sets as follows.

Definition 1. Let A and B be sets.

$|A| = |B|$ if there exists a one-one map from A onto B .

$|A| \leq |B|$ if there exists a one-one map from A to B . $|A| \leq |B|$ is also written as $|B| \geq |A|$.

$|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$. $|A| < |B|$ is also written as $|B| > |A|$.

If $A \subseteq B$, then the identity map $I : A \rightarrow B$ is one-one. Hence $|A| \leq |B|$. The set E of all even positive numbers has cardinality less than or equal to that of \mathbb{N} . However, $|E| = |\mathbb{N}|$ since the map $f : \mathbb{N} \rightarrow E$ defined by $f(n) = 2n$ is a one-one and onto map. It is easy to see that for sets A, B, C , if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$ using composition of maps.

Let A be a set. A is said to be *finite* if either $A = \emptyset$ or $|A| = n = |\{1, 2, \dots, n\}|$; A is called *denumerable* if $|A| = |\mathbb{N}|$; and A is *countable* if $|A| \leq |\mathbb{N}|$. All subsets of a countable set are countable; thus all supersets of uncountable sets are uncountable. It can be shown by induction that if A is any infinite set, then $|\mathbb{N}| \leq |A|$. Every finite set is countable and all infinite subsets of \mathbb{N} are denumerable and thus countable. All elements of a denumerable set can be enumerated in a non-ending sequence of distinct elements:

$$x_1, x_2, \dots, x_n, \dots$$

The set \mathbb{Z} of all integers is denumerable since the map $f : \mathbb{Z} \rightarrow \mathbb{N}$ defined by

$$f(m) = \begin{cases} -1 - 2m & \text{if } m < 0 \\ 2(1 + m) & \text{if } m \geq 0 \end{cases}$$

is a one-one and onto map. The set \mathbb{Q} of all rational numbers is denumerable since it contains \mathbb{Z} and the map $g : \mathbb{Q} \rightarrow \mathbb{Z}$ defined by

$$g\left(\frac{p}{q}\right) = \begin{cases} 2^p 3^q & \text{if } p \in \mathbb{N} \cup \{0\}, q \in \mathbb{N} \\ -2^p 3^q & \text{if } -p \in \mathbb{N}, q \in \mathbb{N} \end{cases}$$

is one-one; assuming that p/q is in reduced form. Wait! Have we exhibited a map from \mathbb{Q} to \mathbb{Z} which is one-one and onto? No! Then how do we conclude that \mathbb{Q} is denumerable? By now, we have only proved that

$$|\mathbb{Q}| \leq |\mathbb{Z}| \quad \text{and} \quad |\mathbb{Z}| \leq |\mathbb{Q}|.$$

Does it follow that $|\mathbb{Q}| = |\mathbb{Z}|$? Yes, by Schröder-Bernstein theorem, which says the following:

If there exist a one-map from A to B and a one-one map from B to A , then there exists a one-one map from A onto B .

We will prove this result in Section 3. What about the set \mathbb{R} of real numbers? Recall that \mathbb{R} is a (the) complete ordered field containing \mathbb{Q} . Along with the usual properties of addition, multiplication and of the order relation \leq , it satisfies the following, called the *completeness principle*:

Every subset of \mathbb{R} which is bounded above, has a least upper bound (lub), and

every subset of \mathbb{R} which is bounded below, has a greatest lower bound (glb).

In fact, existence of lub guarantees the existence of glb and vice versa. Using this property, we give a proof of uncountability of \mathbb{R} .

Theorem 1. \mathbb{R} is uncountable.

Proof. On the contrary, suppose \mathbb{R} is countable. Then $[0, 1]$ is countable since $[0, 1] \subseteq \mathbb{R}$. But $[0, 1]$ is not a finite set since $f : \mathbb{N} \rightarrow [0, 1]$, defined by $f(n) = 1/n$, is one-one. Hence $[0, 1]$ is denumerable. Then, let

$$x_1, x_2, \dots, x_n, \dots$$

be an enumeration of $[0, 1]$. For each $n \in \mathbb{N}$, construct a sub-interval $[a_n, b_n]$ of $[0, 1]$ that does not contain x_n , inductively, as in the following:

Initially, set $a_0 := 0, b_0 := 1$.

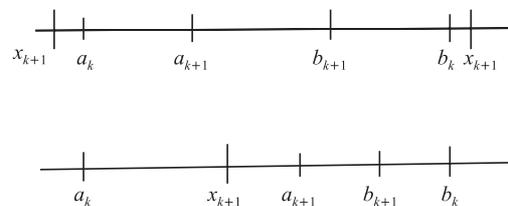
Suppose, for $k \geq 0, a_k, b_k$ have already been chosen.

Choose a_{k+1}, b_{k+1} as follows:

If $a_k < x_{k+1} < b_k$, then $y_k := x_{k+1}$, else, $y_k := a_k$.

$a_{k+1} := y_k + (b_k - y_k)/3, b_{k+1} := y_k + 2(b_k - y_k)/3$.

The construction says that if x_{k+1} lies in the open interval (a_k, b_k) , then choose the interval $[a_{k+1}, b_{k+1}]$ as the middle third of $[x_{k+1}, b_k]$; and if $x_{k+1} \leq a_k$ or $x_{k+1} \geq b_k$, then choose $[a_{k+1}, b_{k+1}]$ as the middle third of $[a_k, b_k]$.



We observe that for each $n \in \mathbb{N}, [a_n, b_n] \neq \emptyset$ and $x_n \notin [a_n, b_n]$. Moreover,

$$0 < a_1 < a_2 \dots < a_n < \dots < \dots < b_n < \dots < b_2 < b_1 < 1.$$

Write $a = \text{lub} \{a_n : n \in \mathbb{N}\}$ and $b = \text{glb} \{b_n : n \in \mathbb{N}\}$. Then, $0 < a \leq b < 1$. Thus $[a, b]$ is a nonempty sub-interval of $[0, 1]$. Moreover,

$$x_n \notin [a_n, b_n] \supseteq [a, b] \quad \text{for each } n \in \mathbb{N}.$$

That is, no number in $[a, b]$ is enumerated in the assumed enumeration, a contradiction. \square

It can be shown that each real number has a decimal expansion; see [1]. This representation is not unique since any number having a finite number of digits after the decimal point does have exactly another representation having an infinite number of digits after the decimal point. For example, $0.12 = 0.11999 \dots$. We agree to always use the infinite one whenever a choice exists. The usual proof of uncountability of the semi-open interval $(0, 1]$ uses the *diagonalization method* of Cantor, which we now explain. Suppose, on the contrary, that $(0, 1]$ is countable. Then we have an enumeration of all numbers in $(0, 1]$ as

$$\begin{array}{l} 0.a_{11} a_{12} a_{13} \dots a_{1n} \dots \\ 0.a_{21} a_{22} a_{23} \dots a_{2n} \dots \\ \vdots \\ 0.a_{n1} a_{n2} a_{n3} \dots a_{nn} \dots \\ \vdots \end{array}$$

where each a_{ij} is a digit ranging from 0 to 9, and in which a finite decimal ends with 9s rather than 0s. It is of course immaterial whether we choose ending with 0s or 9s, but we must resort to one and not the other so that repetitions are avoided. We then construct a decimal number

$$0.b_1 b_2 b_3 \dots b_n \dots$$

where $b_i = 0$ if $a_{ii} = 9$ else $b_i = a_{ii} + 1$. Then this new decimal number is in $(0, 1]$ but it differs from each in the list above. Therefore, no enumeration of numbers in $(0, 1]$ can have all the numbers in $(0, 1]$.

Another proof uses the representation of any real number as a binary decimal. Once again, we choose to use a binary decimal number with an infinite number of digits and discard one with finite number of digits after the decimal point, if such a choice exists. The uncountability of the closed interval $[0, 1]$ is accomplished by showing that $\mathcal{P}(\mathbb{N})$, the power set of \mathbb{N} and $[0, 1]$ are in one-one correspondence; and then resorting to Cantor's theorem that for any set A , $|\mathcal{P}(A)| > |A|$.

To see that $|\mathcal{P}(\mathbb{N})| \leq |[0, 1]|$, define $f : \mathcal{P}(\mathbb{N}) \rightarrow [0, 1]$ as follows.

Let $S \subseteq \mathbb{N}$. Let $n \in \mathbb{N}$. Then $f(S)$ is the decimal number $0.a_1 a_2 \dots$ in base 10 such that its n th digit $a_n = 3$ if $n \in S$, and $a_n = 4$ if $n \notin S$.

For example, $f(\{1, 2, 3\}) = 0.3334444 \dots$. Notice that f is not an onto map since for no subset A of \mathbb{N} , $f(A) = 0.1$. Clearly f is one-one. Hence $|\mathcal{P}(\mathbb{N})| \leq |[0, 1]|$.

For the other inequality, define $g : [0, 1] \rightarrow \mathcal{P}(\mathbb{N})$ by

Let $x = 0.b_1 b_2 \dots \in [0, 1]$, where $b_i \in \{0, 1\}$. Then $g(x) = \{i \in \mathbb{N} : b_i = 1\}$.

For example, $g(0.10111 \dots) = \{1, 3, 4, 5, \dots\}$. Again, notice that g is not an onto map, since there is no binary decimal $a \in [0, 1]$ such that $g(a) = \{1, 2\}$. For, according to the definition of g , the only suitable number a would have been 0.11 , which has been discarded in favour of $0.10111 \dots$. Obviously, g is one-one. Hence $|[0, 1]| \leq |\mathcal{P}(\mathbb{N})|$.

By Schröder-Bernstein theorem, $|\mathcal{P}(\mathbb{N})| = |[0, 1]|$. It remains to prove Cantor's theorem.

Theorem 2 (Cantor). For any set A , $|A| < |\mathcal{P}(A)|$.

Proof. Let A be any set. The function $f : A \rightarrow \mathcal{P}(A)$ defined by $f(x) = \{x\}$ is a one-one map. Therefore, $|A| \leq |\mathcal{P}(A)|$. We next show that no function from A to $\mathcal{P}(A)$ can be onto. On the contrary, suppose that $g : A \rightarrow \mathcal{P}(A)$ is an onto map. Notice that for any $x \in A$, $g(x) \subseteq A$. Let $B = \{x \in A : x \notin g(x)\}$. Since g is an onto map, there exists $y \in A$ such that $B = g(y)$. Then $y \in B$ iff $y \notin g(y)$ iff $y \notin B$, a contradiction. \square

\mathbb{R} and \mathbb{N} are not in one-one correspondence. But \mathbb{R} and the open interval $(0, 1)$ are in one-one correspondence, since the map $f : (0, 1) \rightarrow \mathbb{R}$ defined by

$$f(x) = \tan(-(\pi/2) + \pi x)$$

is one-one and onto. If you accept Schröder-Bernstein theorem, then it is trivial to guarantee the existence of a one-one map from $[0, 1]$ onto \mathbb{R} . Can you construct a one-one map from $[0, 1]$ onto \mathbb{R} ?

Notice that Cantor's theorem establishes a hierarchy of infinities:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \dots$$

It raises the question whether there exists a set whose cardinality is in between those of \mathbb{N} and $\mathcal{P}(\mathbb{N})$? Cantor conjectured that there exists no such set, however he could not prove it. Later it was shown by Gödel that the truth of this conjecture is consistent with ZFC, a widely accepted formalization of set theory by Zermelo and Fraenkel. Cohen proved that the falsity of this conjecture is also consistent with ZFC. These two proofs established the independence of this conjecture from ZFC. Thus, the conjecture is considered a hypothesis, the *continuum hypothesis*. The generalized continuum hypothesis states that if A is any infinite set, then there does not exist a set B such that $|A| < |B| < |\mathcal{P}(A)|$. It is not yet known whether there exists a formalization of set theory in which (generalized) continuum hypothesis becomes a theorem; see [5].

Another consequence of Cantor's theorem is that the collection of all sets is not a set. For, suppose that S is the set of all sets. Then $\mathcal{P}(S) \subseteq S$. In that case, $|\mathcal{P}(S)| \leq |S|$, contradicting Cantor's theorem. This also shows that we cannot build a set corresponding to every property. For otherwise, the property "x is a set" would define the set of all sets. In fact, as the history goes, this concerns gave rise to various formalizations of set theory, one of which is ZFC.

Central to this discussion is comparing cardinalities, which rests on Schröder-Bernstein Theorem.

3. Cantor-Schröder-Bernstein Theorem

As to the name of this section, Cantor first formulated the theorem and gave a proof relying on the well ordering principle; this principle, as we know, is equivalent to the axiom of choice. Schröder gave a proof without using the axiom of choice, in which there were gaps. The first published correct proof without using the axiom of choice was by Bernstein, though it is said that Dedekind proved it a bit earlier but did not publish. Zermelo and König have also proved the theorem; see [2,4]. All these proofs reveal partitions of the sets into a fixed finite number of parts each, where the individual parts of both the sets are in one-one correspondence. This is the content of Banach-Mapping theorem, where the sets are partitioned into two subsets. We prove this first.

Theorem 3 (Banach Mapping). *Let A, B be nonempty sets. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions. Then there*

exist subsets A_1, A_2 of A and subsets B_1, B_2 of B such that

$$A_2 = A - A_1, B_2 = B - B_1, f(A_1) = B_1, g(B_2) = A_2.$$

Proof. Consider the collection

$$\mathcal{C} = \{D \subseteq A : g(B - f(D)) \subseteq A - D\}.$$

$f(\emptyset) = \emptyset, B - f(\emptyset) = B, g(B - f(\emptyset)) = g(B) \subseteq A = A - \emptyset$. Hence, $\emptyset \in \mathcal{C}$. That is, \mathcal{C} is a nonempty collection. Define

$$E = \cup_{D \in \mathcal{C}} D.$$

Now,

$$\begin{aligned} g(B - f(E)) &= g(B - f(\cup_{D \in \mathcal{C}} D)) = g(B - \cup_{D \in \mathcal{C}} f(D)) \\ &= g(\cap_{D \in \mathcal{C}} (B - f(D))) \\ &\subseteq \cap_{D \in \mathcal{C}} (g(B - f(D))) \subseteq \cap_{D \in \mathcal{C}} (A - D) \\ &= A - \cup_{D \in \mathcal{C}} D = A - E. \end{aligned}$$

That is,

$$g(B - f(E)) \subseteq A - E.$$

We want to show that $g(B - f(E)) = A - E$. On the contrary, suppose $g(B - f(E)) \neq A - E$. Then there exists an $x \in A - E$ such that $x \notin g(B - f(E))$. The conditions $x \notin g(B - f(E))$ and $g(B - f(E)) \subseteq A - E$ imply that

$$g(B - f(E)) \subseteq A - (E \cup \{x\}).$$

Since $f(E) \subseteq f(E \cup \{x\}), B - f(E \cup \{x\}) \subseteq B - f(E)$. Then

$$g(B - f(E \cup \{x\})) \subseteq g(B - f(E)) \subseteq A - (E \cup \{x\}).$$

That is, $E \cup \{x\} \in \mathcal{C}$. As $E = \cup_{D \in \mathcal{C}} D, E \supseteq E \cup \{x\}$. That is, $x \in E$, This contradicts $x \in A - E$.

We conclude that $g(B - f(E)) = A - E$.

Finally, take $E = A_1, A - E = A_2, B_1 = f(E)$ and $B_2 = B - f(E)$. \square

Cantor-Schröder-Bernstein theorem can be derived from Theorem 2 by defining the map $h : A \rightarrow B$ with $h(x) = f(x)$ for $x \in A_1$ and $h(x) = g^{-1}(x)$ for $x \in A_2$. Notice that $f : A_1 \rightarrow B_1$ is one-one and onto; so is the map $g : B_2 \rightarrow A_2$. Therefore, $h : A \rightarrow B$ is one-one and onto.

The proof of Banach mapping theorem constructs the set $E \subseteq A$ satisfying

$$g(B - f(E)) = A - E.$$

That is, $E = A - g(B - f(E))$. It means that if $\phi(X) = A - g(B - f(X))$ for subsets X of A , then E is a fixed point of this map ϕ . But under what condition(s) a map ϕ taking subsets of A to subsets of A will have a fixed point? The proof of Theorem 2 suggests this condition.

Theorem 4 (Knaster Fixed Point). *Let $\mathcal{P}(A)$ denote the power set of a nonempty set A . Let a map $\psi : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ satisfy*

$$X \subseteq Y \text{ implies } \psi(X) \subseteq \psi(Y) \text{ for } X, Y \subseteq A.$$

Then there exists $G \subseteq A$ such that $\psi(G) = G$.

Proof. The collection $\mathcal{K} = \{X \subseteq A : X \subseteq \psi(X)\}$ of subsets of A is nonempty since $\emptyset \subseteq \psi(\emptyset)$. Define

$$G = \bigcup_{Y \in \mathcal{K}} Y.$$

It says that each set in \mathcal{K} is a subset of G . Now,

$$G = \bigcup_{Y \in \mathcal{K}} Y \subseteq \bigcup_{Y \in \mathcal{K}} \psi(Y) = \psi(\bigcup_{Y \in \mathcal{K}} Y) = \psi(G).$$

If $G \neq \psi(G)$, then there exists $x \in \psi(G)$ such that $x \notin G$. Since $G \subseteq \psi(G)$ and $x \in \psi(G)$, we see that

$$G \cup \{x\} \subseteq \psi(G) \subseteq \psi(G \cup \{x\}).$$

That is, $G \cup \{x\} \in \mathcal{K}$. Hence $G \cup \{x\} \subseteq G$. That is, $x \in G$, a contradiction.

Therefore, $\psi(G) = G$. □

Generalization has made the things simpler. The proof of Theorem 3 looks simpler than that of Theorem 2. To derive Theorem 2 from Theorem 3, all that we do is take a particular map ψ which satisfies the required condition. For this purpose, define the map $\psi : P(A) \rightarrow P(A)$ by

$$\psi(X) = A - g(B - f(X)) \text{ for } X \subseteq A.$$

Now,

$$\begin{aligned} X \subseteq Y &\Rightarrow f(X) \subseteq f(Y) \Rightarrow B - f(X) \supseteq B - f(Y) \\ &\Rightarrow g(B - f(X)) \supseteq g(B - f(Y)) \\ &\Rightarrow A - g(B - f(X)) \subseteq A - g(B - f(Y)). \end{aligned}$$

That is, $X \subseteq Y$ implies $\psi(X) \subseteq \psi(Y)$. An application of Theorem 3 completes the proof of Theorem 2.

In fact, Knaster fixed point theorem holds true in a much more general setting. Suppose A is a partially ordered set with

a partial order \leq . It is called a *complete lattice* if every subset of A has an infimum and a supremum with respect to the partial order. We say that a map $\phi : A \rightarrow A$ is *order preserving* if $\phi(x) \leq \phi(y)$ whenever $x \leq y$. Then Knaster-Tarski fixed point theorem can be stated as follows:

Every order preserving map on a complete lattice has a fixed point.

Further, the set of all such fixed points of the map is a complete lattice.

The proof of this result is similar to that of Theorem 3, but is to be formulated in terms of the generalized notions appropriate to a complete lattice. Moreover, the converse of Knaster-Tarski fixed point theorem for lattices holds. It states that a lattice, i.e., a partially ordered set in which every finite subset has a minimum and a maximum, is complete if each order preserving map has a fixed point.

The generalizations have helped in constructing an independent proof of Cantor-Schröder-Bernstein theorem, which we give below. Find out in the proof, how the ideas of fixed point and partition of the sets are in action.

Theorem 5 (Cantor-Schröder-Bernstein). *Let A and B be nonempty sets. Let $f : A \rightarrow B$ and $g : B \rightarrow A$ be one-one functions. Then there exists a function $h : A \rightarrow B$, which is both one-one and onto.*

Proof. Define $S = (A - g(B)) \cup \bigcup_{n \in \mathbb{N}} (g \circ f)^n (A - g(B))$. See the figure below. Then $S = (A - g(B)) \cup (g \circ f)(S) \subseteq A$. Since $g(B) \subseteq A$ and $(g \circ f)(S) \subseteq A$, we obtain

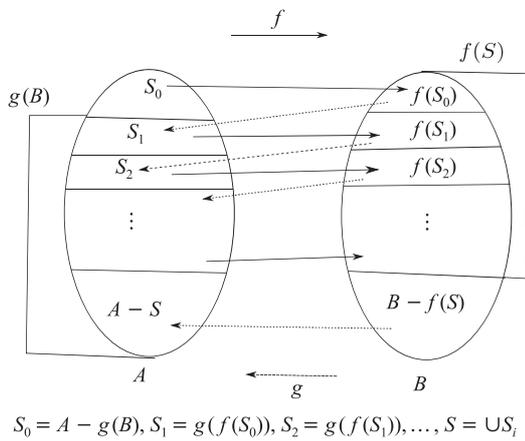
$$\begin{aligned} A - S &= A - ((A - g(B)) \cup (g \circ f)(S)) \\ &= (A - (A - g(B))) - (g \circ f)(S) \\ &= g(B) - g(f(S)) = g(B - f(S)). \end{aligned}$$

The last equality follows since g is one-one. Hence $g : B - f(S) \rightarrow A - S$ is one-one and onto. Also, $f : S \rightarrow f(S)$ is one-one and onto. Hence the map $h : A \rightarrow B$ defined by

$$h(x) = \begin{cases} f(x) & \text{for } x \in S \\ g^{-1}(x) & \text{for } x \in A - S \end{cases}$$

is both one-one and onto. □

Compare this proof with the classical one in text books on Set Theory, for example, in [1]. For completeness, prove the facts contained in the following exercise.



Exercise. Let $f : A \rightarrow B$ be a function where A and B are nonempty sets. Suppose that $A_1 \subseteq A$ and $A_2 \subseteq A$.

1. If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.
2. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
3. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. Equality holds if f is one-one.
4. $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$. Equality holds if f is one-one.
5. Formulate and prove (2)–(3) when the operations of union and intersection are taken over a collection of subsets of A .

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Ramanujan's Awesome Sums

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Abstract. In 1918, Ramanujan published a paper where he proved several nice properties of certain finite sums which are now known as Ramanujan sums. Ramanujan sums have numerous other applications in combinatorics and graph theory. Further, in physics, they have applications in the processing of low-frequency noise and in the study of quantum phase locking. In this exposition, we describe some beautiful properties of these sums and discuss some of their applications.

Keywords. Ramanujan sum, cyclotomic polynomial, Ramanujan expansion, Cayley graph, roots of unity

2001 Mathematics Subject Classification: 11A25; 05A15

Introduction

In his famous paper ([SR]), Ramanujan discussed the properties of certain finite sums – the so-called Ramanujan sums. Even though Dirichlet and Dedekind had already

considered these sums in the 1860's, according to G. H. Hardy, Ramanujan was the first to appreciate the importance of the sum and to use it systematically. Ramanujan sums play a key role in the proof of a famous result due to Vinogradov asserting that every large odd number is the sum of three primes.

These sums have numerous other applications in diverse branches of mathematics as well as in some parts of physics. So, what are these sums?

For integers $n \geq 1, k \geq 0$, the sum

$$c_n(k) = \sum_{(r,n)=1; r \leq n} e^{2ikr\pi/n}$$

is called a Ramanujan sum. In other words, it is simply the sum of the k -th powers of the primitive n -th roots of unity – ‘primitive’ here means that the number is not an m -th root of unity for any $m < n$. Note that the primitive n -th roots of unity are the numbers $e^{2ikr\pi/n}$ for all those $r \leq n$ which are relatively prime to n . The first remarkable property they have is that they are integers. Ramanujan showed that several arithmetic functions (that is, functions defined from the set of positive integers to the set of complex numbers) have ‘Fourier-like’ of expansions in terms of the sums; hence, nowadays these expansions are known as Ramanujan expansions. They often yield very pretty elementary number-theoretic identities. Recently, the theory of group representations of the permutation groups (specifically, the so-called super-character theory as in [FGK]) has been used to re-prove old identities in a quick way and also, to discover new identities. Thus, this subject is very much alive.

1. Properties of Ramanujan Sums

It is convenient to write

$$\Delta_n = \{e^{2ir\pi/n} : (r, n) = 1, 1 \leq r \leq n\}$$

Then, the set of all n -th roots of unity $\{e^{2ik\pi/n} : 0 \leq k < n\}$ is a union of the disjoint sets Δ_d as d varies over the divisors of n . This is because an n -th root of unity is a primitive d -th root of unity for a unique divisor d of n . It is also convenient to introduce the ‘characteristic’ function $\delta_{k|n}$ which has the value 1 when k divides n and the value 0 otherwise. Before stating some properties of the $c_k(n)$ ’s, let us recall two arithmetic functions which are ubiquitous in situations where elementary number-theoretic counting is involved. The first one is Euler’s totient function

$$\phi(n) = |\{r : 1 \leq r \leq n, (r, n) = 1\}|.$$

The other arithmetic function is the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = \prod_{i=1}^k p_i \\ 0, & \text{otherwise.} \end{cases}$$

The Möbius function keeps tab when we use the principle of inclusion-exclusion to do counting. The basic result which can be easily proved by induction on the number of prime factors, is the Möbius inversion formula:

If g is an arithmetic function and

$$f(n) = \sum_{d|n} g(d),$$

then

$$g(n) = \sum_{d|n} f(d)\mu(n/d).$$

With these notations, here are some elementary properties of the Ramanujan sums.

Properties of $c_k(n)$

- (i) $c_n(k) = c_n(-k) = c_n(n - k)$.
- (ii) $c_n(0) = \phi(n)$ and $c_n(1) = \mu(n)$.
- (iii) $c_n(ks) = c_n(k)$ if $(s, n) = 1$. In particular, $c_n(s) = \mu(n)$ if $(s, n) = 1$.
- (iv) $c_n(k) = c_n(k')$ if $(k, n) = (k', n)$. In particular, $c_n(k) \equiv c_n(k') \pmod n$ if $k \equiv k' \pmod n$.
- (v) $\sum_{k=0}^{n-1} c_n(k) = 0$.
- (vi) $\sum_{d|n} c_d(k) = \delta_{n|k}n$ and $c_n(k) = \sum_{d|n} d\mu(n/d)\delta_{d|k} = \sum_{d|(n,k)} d\mu(n/d)$; in particular, for prime powers p^r , we have

$$c_{p^r}(k) = \begin{cases} p^r - p^{r-1}, & \text{if } p^r | k; \\ -p^{r-1}, & \text{if } p^{r-1} || k; \\ 0, & \text{otherwise.} \end{cases}$$

- (vii) $c_{mn}(k) = c_m(k)c_n(k)$ if $(m, n) = 1$.
- (viii) $\sum_{k=1}^n c_m(k)c_n(k) = \delta_{mn}n\phi(n)$.

The property (vi) shows that these sums actually have integer values.

The proof of (i) follows already from the definition and, so do the first parts of (ii) and (iii). The second parts of (ii), (iii) as well as the assertions (iv) and (vii) will follow from (vi). We shall prove (v) and (vi).

For (v), we have

$$\sum_{k=0}^{n-1} c_n(k) = \sum_{k=0}^{n-1} \sum_{\zeta \in \Delta_n} \zeta^k = \sum_{\zeta \in \Delta_n} \sum_{k=0}^{n-1} \zeta^k = 0$$

where the last equality is because

$$\sum_{k=0}^{n-1} \zeta^k = \frac{1 - \zeta^n}{1 - \zeta} = 0$$

for each $\zeta \in \Delta_n$.

For proving (vi), we note that the second statement follows from the first by the Möbius inversion formula. Let us prove the first one now. We have

$$\sum_{d|n} c_d(k) = \sum_{d|n} \sum_{\zeta \in \Delta_d} \zeta^k = \sum_{m=0}^{n-1} e^{2imk\pi/n}$$

because, as we observed, the disjoint union of Δ_d as d varies over the divisors of n is the set of all n -th roots of unity. Now, if the above sum $\sum_{m=0}^{n-1} e^{2imk\pi/n}$ is multiplied by $e^{2ik\pi/n}$, we get the same sum which means that it is equal to 0 unless $n|k$. When $k|n$, the sum is clearly equal to n . This proves (vi).

The other parts easily follow from (vi).

Von Sterneck's Function

The equality $c_n(k) = \sum_{d|n} d\mu(n/d)\delta_{d|k}$ is very useful.¹ For instance, if n is a prime power p^r , as we noted above in (vi), we have

$$c_{p^r}(k) = p^r \delta_{p^r|k} - p^{r-1} \delta_{p^{r-1}|k}.$$

Using this expression in (vii) above, we get

$$c_n(k) = \frac{\mu\left(\frac{k}{(k,n)}\right)\phi(n)}{\phi\left(\frac{k}{(k,n)}\right)}.$$

The right hand side was studied by R. D. Von Sterneck in 1902 and is known by his name. The equality above itself was known before Ramanujan and is due to J. C. Kluyver in 1906.

2. Connection with Cyclotomic Polynomials

The cyclotomic polynomials $\Phi_n(x) = \prod_{\zeta \in \Delta_n} (x - \zeta)$ have some fascinating properties and have surprising consequences

¹Note that even computationally the defining sum for $c_n(k)$ requires approximately n operations whereas the other sum requires roughly $\log(n)$ operations.

(see [BS], where applications such as the infinitude of primes in arithmetic progressions of the form $\{1 + an\}$ are proved).

We have:

$$x^n - 1 = \prod_{d|n} \prod_{\zeta \in \Delta_d} (x - \zeta) = \prod_{d|n} \Phi_d(x)$$

and – by Möbius inversion, we deduce

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

Taking the logarithmic derivative, we obtain

$$\frac{\Phi'_n(x)}{\Phi_n(x)} = \sum_{d|n} \frac{dx^{d-1}\mu(n/d)}{x^d - 1}.$$

Multiplying by $x(x^n - 1)$, we get a polynomial in x , viz.,

$$x(x^n - 1) \frac{\Phi'_n(x)}{\Phi_n(x)} = \sum_{d|n} d\mu(n/d)(x^d + x^{2d} + \dots + x^n).$$

Thus, the coefficient of x^k in the polynomial on the right is $\sum_{d|(n,k)} d\mu(n/d)$, which is simply the Ramanujan sum $c_n(k)$. Hence, we have:

Proposition. For each $k < n$, the Ramanujan sum $c_n(k)$ is the coefficient of x^{k-1} in the polynomial $(x^n - 1) \frac{\Phi'_n(x)}{\Phi_n(x)}$.

3. Ramanujan Expansions of Arithmetic Functions

An arithmetic function $f(n)$ is often stored in terms of the generating function $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ which is a function of a variable s . Instead of a power series, one uses these type of series – called Dirichlet series – in this set-up. One reason is that the product of two such series produces the “convolution” of the corresponding arithmetic functions $f(n), g(n)$ which is something that appears naturally in number theory. That is, if f, g are two arithmetic functions and $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ and $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$, then

$$G(s)F(s) = \sum_{n=1}^{\infty} \frac{\sum_{d|n} f(d)g(n/d)}{n^s}.$$

One writes this new multiplication (the ‘convolution’) as

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Under this multiplication, the set of all arithmetic functions forms what is called a “commutative algebra with unit”. In other words, with the obviously defined addition of

arithmetic functions, the set of arithmetic functions form a complex vector space and this multiplication is compatible with that vector space structure. The unit element is the arithmetic function $\delta(n) = \delta_{n|1}$ in the notation we have used above.

Ramanujan found ‘natural’ expansions of standard arithmetic functions in terms of the $c_n(k)$ ’s as infinite series. “Natural” here is not a precisely defined notion. For instance, for an arithmetic function f , one may look for a series $\sum_{n=1}^{\infty} \widehat{f(n)} c_n(r)$ converging point-wise to f . This kind of convergence has been established so far only for a restricted class of arithmetic functions. Later researchers have tried to produce “exotic” Ramanujan expansions for several arithmetic functions which include those obtained classically by Ramanujan. It should be borne in mind that in general such an expansion may not be unique because one has equalities like $\sum_{n=1}^{\infty} \frac{c_n(k)}{n} = 0$.

We stick to giving some examples and later discuss certain classes of functions which have unique Ramanujan expansions.

Examples.

- Let $\sigma_r(k) := \sum_{d|k} d^r$ for all $r \geq 1$. Then

$$\begin{aligned} \frac{\sigma_r(k)}{k^r} &= \sum_{d|k} \frac{1}{d^r} = \sum_{d=1}^{\infty} \frac{1}{d^{r+1}} \sum_{n|d} c_n(d) \\ &= \sum_{n=1}^{\infty} \frac{c_n(k)}{n^{r+1}} \sum_{m=1}^{\infty} \frac{1}{m^{r+1}} = \zeta(r+1) \sum_{n=1}^{\infty} \frac{c_n(k)}{n^{r+1}} \end{aligned}$$

where $\zeta(s)$ denotes the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for any real $s > 1$.

- The divisor function $d(k) = \sum_{d|k} 1$ has an expansion

$$d(k) = \sum_{n=1}^{\infty} -c_n(k) \frac{\log(n)}{n}$$

- For any $m \geq 1$, a generalization of the Euler totient function is Jordan’s function

$$\phi_m(k) = k^m \prod_{p|k} (1 - p^{-m}),$$

where the product on the right is over all the prime divisors of k . Note that ϕ_1 is the Euler totient function. Ramanujan showed for any $m \geq 1$ that

$$\phi_m(k) = \frac{k^m}{\zeta(m+1)} \sum_{n=1}^{\infty} \frac{\mu(n) c_n(k)}{\phi_{m+1}(n)}.$$

- Let $r_m(k) = |\{(a, b) : a, b \in \mathbf{Z}, a^m + b^m = k\}|$, the number of ways to write k as a sum of two m -th powers. Then, Ramanujan obtained expressions for r_2, r_4, r_6, r_8 and a few other related arithmetic functions. For $r_2(k)$, this is:

$$r_2(k) = \pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} c_{2n-1}(k)$$

Note the curiosity that a form of the famous prime number theorem is the assertion that $\sum_n \frac{\mu(n)}{n} = 0$ and this is also equivalent to the assertion that $\sum_{n \geq 1} \frac{c_n(k)}{n} = 0$ for all k !

Unique Ramanujan Expansions for ‘even’ Functions

Recall that $c_n((k, n)) = c_n(k)$; thus, for each fixed n , one may say that the function $k \mapsto c_n(k)$ is “even modulo n ”. This is in analogy with even functions which are ‘even modulo 2’. The beautiful general theorem which holds good is the following one.

Theorem. *Let n be a fixed positive integer and let f be any arithmetic function which is even modulo n . Then, there exist unique numbers a_d for each $d|n$ which satisfy*

$$f(k) = \sum_{d|n} a_d c_d(k).$$

In fact, for each $d|n$, we have

$$a_d = \frac{1}{n} \sum_{e|n} f(n/e) c_e(n/d).$$

This pretty theorem has a tedious though elementary proof but the theory of super-characters mentioned in the introduction gives a very quick proof. We do not discuss the proof here but state some identities which follow quickly in that set-up.

Orthogonality Relations

- $\sum_{r|n} \phi(r) c_d(n/r) c_e(n/r) = n\phi(d)$ or 0 according as to whether $d = e$ or not.
- $\sum_{r|n} \frac{1}{\phi(r)} c_r(n/d) c_r(n/e) = \frac{n}{\phi(d)}$ or 0 according as to whether $d = e$ or not.
- If $(mu, nv) = 1$, then $c_{mn}(uv) = c_m(u) c_n(v)$.
- $\sum_{d|n} c_d(n/d) = \sqrt{n}$ or 0 according as to whether n is a perfect square or not.
- $c_d(n/e) \phi(e) = c_e(n/d) \phi(d)$ if d, e are divisors of n .
- $\sum_{d, e|n} c_d(n/e) c_e(n/d) = nd(n)$ for divisors d, e of n .

Mixed Orthogonality Relations

- For divisors d, e of n , we have $\sum_{r|n} c_d(n/r)c_r(n/e) = n$ or 0 according as to whether $d = e$ or not.
- For a divisor d of n , we have $\sum_{r|n} c_d(n/r)\mu(r) = n$ or 0 according as to whether $d = n$ or not.

A Matrix of Ramanujan Sums

Theorem. For any $n > 1$, consider the $d(n)$ divisors of n and fix them in some order. Look at the $d(n) \times d(n)$ matrix C whose (d, e) -th entry is $c_d(e)$ for divisors d, e . Then, we have:

The determinant of the above matrix C is $n^{d(n)/2}$ and the inverse matrix has (d, e) -th entry $\frac{1}{n}c_{n/d}(n/e)$.

Proof. Consider the ‘divisibility’ matrix X of size $d(n) \times d(n)$ whose (d, e) -th entry is 1 if $d|e$ and 0 otherwise. Note that X^{-1} has (d, e) -th entry is $\mu(e/d)$ if $d|e$ and 0 otherwise. This simply follows from the equality

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases}$$

If D is the diagonal matrix whose (d, d) -th entry is d for each $d|n$, then we have

$$XD(X^t)^{-1} = C$$

because (d, e) -th entry of the left hand side is $\sum_{r|(d,e)} r\mu(e/r)$ which we know to be equal to $c_d(e)$. This also shows that the determinant is as asserted. We may compute $C^{-1} = X^tDX^{-1}$ and get the result as asserted. \square

Note that the equality $CC^{-1} = I$ expresses the identity $\sum_{r|n} c_{n/r}(d)c_{n/d}(r) = n$ or 0 according as to whether $d = n$ or not.

Finally, we end this section with the following beautiful statement which is a consequence of the theory of super-characters:

Theorem. If $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, and the divisors of n are arranged in any fixed order, then consider, as above, the $d(n) \times d(n)$ matrix A with (d, e) -th entry $c_d(n/e)$. Then, $\det A = n^{d(n)/2}(-1)^{\sum_{i=1}^r [(\alpha_i+1)/2]d(n)/(\alpha_i+1)}$.

4. A Graph-Theoretic Interpretation

Both in graph theory and in group theory, the Cayley graphs figure prominently ([KS]). If G is a group generated by a subset

S which is symmetric – that is, $S = \{s^{-1} : s \in S\}$ – then, one forms the Cayley graph $\text{Cay}(G, S)$ whose vertices are elements of G and, elements g, h are connected by an (undirected) edge \overline{gh} if and only if $gh^{-1} \in S$.

Consider the Cayley graph $\text{Cay}(\mathbf{Z}_n, \mathbf{Z}_n^*)$ where \mathbf{Z}_n^* is the group of all units in \mathbf{Z}_n . In other words, this is a graph whose vertices can be identified with $\{0, 1, \dots, n-1\}$ and one connects i and j if and only if $i - j$ is relatively prime to n . The adjacency matrix of this graph has a very nice form – it is a circulant matrix. This matrix is

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$

where $a_i = 1$ or 0 according as whether i is relatively prime to n or not. Evidently, the vectors $v_k = (1, e^{2ik\pi/n}, e^{4ik\pi/n}, \dots, e^{2(n-1)ik\pi/n})$ are linearly independent for $k = 0, 1, \dots, n-1$ and satisfy

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{2ik\pi/n} \\ \vdots \\ e^{2(n-1)ik\pi/n} \end{pmatrix} = \lambda_k \begin{pmatrix} 1 \\ e^{2ik\pi/n} \\ \vdots \\ e^{2(n-1)ik\pi/n} \end{pmatrix}$$

where $\lambda_k = \sum_{m=0}^{n-1} a_m e^{2imk\pi/n} = \sum_{0 \leq m < n; (m,n)=1} e^{2imk\pi/n}$ which is the Ramanujan sum $c_n(k)$.

Therefore, we have shown:

Proposition. The eigenvalues of the Cayley graph $\text{Cay}(\mathbf{Z}_n, \mathbf{Z}_n^*)$ are the integers $c_n(k)$ for $0 \leq k \leq n-1$.

5. Counting Generators in Cyclic Groups

In discussing heuristics on the Artin primitive root conjecture, one naturally looks at a positive integer a and needs to keep track of the index of the subgroup generated by a in \mathbf{Z}_p^* for

various primes p (see [PM]). In this context, the following characteristic function plays a role and, once again, Ramanujan sums help in evaluating these functions. More precisely, we have:

Lemma. *Let G be a cyclic group of order n and $g \in G$. Then, for each divisor d of n ,*

$$\sum_{h|d} c_h(O(G)/O(g)) = \begin{cases} d, & \text{if } d \text{ divides } O(G)/O(g) \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We have already seen that

$$\sum_{h|d} c_h(k) = \sum_{h|d} c_h((k, d)).$$

We have also noted earlier that when $k|d$, we have

$$\sum_{h|d} c_h(k) = \begin{cases} d, & \text{if } k = d \\ 0, & \text{otherwise.} \end{cases}$$

Putting $k = O(G)/O(g)$, we have the assertion. \square

6. Counting Cyclic Orbifolds

In this section, we discuss an interesting way to use Ramanujan sums to count the number of actions of a finite cyclic group as orientation-preserving automorphisms on a Riemann surface (see [VAL]). Such actions of a finite group G on a Riemann surface S of genus g correspond to epimorphisms of the fundamental group of the corresponding orbifold onto G . The fundamental group $\pi_1\Omega$ of an orbifold Ω can be described by $2g + r$ generators $x_1, y_1, \dots, x_g, y_g, z_1, z_2, \dots, z_r$ and relations

$$z_1 z_2 \dots z_r \prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} = 1, \quad z_i^{m_i} = 1 \quad \forall i \leq r,$$

for certain positive integers $1 < m_i \leq |G|$.

To count epimorphisms from this group to a cyclic group \mathbf{Z}_k , one needs to simply look at the corresponding presentation of the finitely generated abelian group $\pi_1\Omega/[\pi_1\Omega, \pi_1\Omega]$. It is easy to prove that the number of order-preserving epimorphisms (that is, those epimorphisms which have torsion-free kernels) from $\pi_1\Omega(g, m_1, \dots, m_r)$ to \mathbf{Z}_k is:

$$m^{2g} \phi_{2g}(k/m) \frac{1}{M} \sum_{s=1}^M c_{m_1}(s) c_{m_2}(s) \dots c_{m_r}(s),$$

where $m = \text{LCM of } m_1, \dots, m_r$ divides M and, the above expression is independent of the choice of M (!).

Interestingly, the expression $\frac{1}{M} \sum_{s=1}^M c_{m_1}(s) c_{m_2}(s) \dots c_{m_r}(s)$ on the right hand side above also counts the number of solutions to the congruence

$$x_1 + x_2 + \dots + x_r \equiv 0 \pmod{M},$$

with $\text{GCD}(x_i, M) = M/m_i$, where M is a multiple of the m_i 's.

7. A Curious Application

Let $n > 1$ be odd. Fix $\zeta \in \Delta_n$ a primitive n -th root of unity. Writing the product $\prod_{i=1}^{(n-1)/2} (\zeta^i + \zeta^{-i}) = \sum_{k=1}^n M_k \zeta^k$, the number M_k counts the number of ways k can be written modulo n as $\pm 1 \pm 2 \dots \pm (n-1)/2$. Also, for any divisor d of n ,

$$\begin{aligned} \prod_{i=1}^{(n-1)/2} (\zeta^{di} + \zeta^{-di}) &= \sum_{k=1}^n M_k \zeta^{dk} = \sum_{e|n} \sum_{(k,n)=n/e} M_k \zeta^{dk} \\ &= \sum_{e|n} c_e(d) M_{n/e} \end{aligned} \quad (\heartsuit)$$

The left hand side can be evaluated independently to be $\pm 2^{(d-1)/2}$ where the sign is described as follows.

For any positive integer m , define $\binom{2}{m}$ to be 1 or -1 according as to whether the product $\prod a_p^{\alpha_p}$ is 1, and where α_p is 1 or -1 according as to whether 2 is a square modulo p or not and $\prod_p p^{\alpha_p} = m$.

Then, in (\heartsuit) , the left hand side turns out to be $2^{(d-1)/2} \binom{2}{n/d}$.

The equations (\heartsuit) give a system of $d(n)$ linear equations, one for each divisor d of n . As we know how to invert the matrix of Ramanujan sums, we have the following beautiful result:

Theorem. *Let $n > 1$ be odd. Consider, for each $k \leq n$, the number*

$$M_k = \left| \left\{ (\epsilon_1, \dots, \epsilon_{d(n)}) : \epsilon_i = \pm 1, \sum_{i=1}^{(n-1)/2} i \epsilon_i \equiv k \pmod{n} \right\} \right|.$$

Then, we have $M_k = \frac{1}{n} \sum_{d|n} c_d(k) 2^{\binom{2}{d-1}/2} \binom{2}{d}$.

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Problems and Solutions

Edited by Amritanshu Prasad

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This section of the Newsletter contains problems contributed by the mathematical community. These problems range from mathematical brain-teasers through instructive exercises to challenging mathematical problems. Contributions are welcome from everyone, students, teachers, scientists and other maths enthusiasts. We are open to problems of all types. We only ask that they be reasonably original, and not more advanced than the MSc level. A cash prize of Rs. 500 will be given to those whose contributions are selected for publication. Please send solutions along with the problems. The solutions will be published in the next issue. Please send your contribution to problems@imsc.res.in with the word “**problems**” somewhere in the subject line. Please also send solutions to these problems (with the word “**solutions**” somewhere in the subject line). Selected solutions will be featured in the next issue of this Newsletter.

1. **D. Surya Ramanna, Harish-Chandra Research Institute, Allahabad.** Find all solutions to the equation $x^y = y^x$ in the positive integers.
2. **Thomas Moore, Bridgewater State University.** Squares $s_n = n^2$ and triangular numbers $t_n = \frac{n(n+1)}{2}$ are well known. The Jacobsthal numbers $j_n = \frac{2^n - (-1)^n}{3}$ for $n \geq 1$ are somewhat less well-known. Find a second degree polynomial with integer coefficients $f(x)$, such that, whenever the input $s_n, t_n, \text{ or } j_n, n \geq 1$, the output is a triangular number.

3. **Kamalaskhya Mahatab, IMSc and Kannappan Sampath, ISI Bangalore.** Let $f(x)$ be a monic polynomial in $\mathbf{Z}[x]$ with a factorization

$$f(x) = \prod_{i=1}^m f_i(x),$$

with f_1, \dots, f_m are monic polynomials having no common factors. Let $g(x) \in \mathbb{Z}[x]$ be such that the remainder of $g(x)$ upon division by $f_i(x^n)$ is a polynomial in x^n for each i . Show that the residue of $g(x)$ upon division by $f(x)$ is a polynomial in x^n .

4. **Rahul Dattatraya Kitture, Bhaskaracharya Pratishthana, Pune.**
 - (1) Prove that a group G can be written as a (set-theoretic) union of proper subgroups if and only if G is not cyclic.
 - (2) Use (1) to prove that if G is a non-abelian group, then $G/Z(G)$ is not cyclic (here $Z(G)$ denotes the centre of G).
5. **K. N. Raghavan, IMSc.** Let n be a positive integer. Define d_n to be $(n+1)(n-1)$ if n is odd, and to be $(n+2)(n-2)$ if n is even; define e_n to be 4 or 0 accordingly as n is divisible by 3 or not. Show that the number of ways in which n can be written as a sum of three positive integers is $(d_n + e_n)/12$. (Can you write down an analogous expression for the number of ways of writing n as a sum of four positive integers?)

6. **K. N. Raghavan, IMSc.** Let n be a positive integer, $n \geq 2$. On the real two dimensional plane, consider the following two linear transformations: s is the reflection in the x -axis; and r rotation counter-clockwise by the angle $2\pi/n$.

$$\begin{aligned} sx &= x & rx &= x \cos 2\pi/n + y \sin 2\pi/n \\ sy &= -y & ry &= -x \sin 2\pi/n + y \cos 2\pi/n \end{aligned}$$

Define a polynomial $f(x, y)$ to be *invariant* if $f(rx, ry) = f(sx, sy) = f(x, y)$. For example, the polynomial $f_0(x, y) = x^2 + y^2$ is invariant. (Geometrically, the two transformations s and r being respectively a reflection and a rotation, they preserve the inner product, which explains the invariance of $f_0(x, y)$.) Show that there exists a homogeneous invariant polynomial $f_1(x, y)$ of degree n such that, together with $f_0(x, y)$, it generates the ring of polynomial invariants (that is, any invariant polynomial is a polynomial with real coefficients in f_0 and f_1). For example, for the values 2 and 3 of n , we could take f_1 respectively to be $x^2 - y^2$ and $x^3 - 3xy^2$. Find an explicit closed formula for such an f_1 in terms of n . Show moreover that the expression as a polynomial in f_0 and f_1 of any invariant polynomial is unique.

7. **K. N. Raghavan, IMSc.** A prime number p is called a *Fermat prime* if it is of the form $2^{2^k} + 1$ for some integer k . Prove or disprove the following: for any Fermat prime p with $k \geq 1$, the multiplicative group of units modulo p is generated by 3.

Solutions to Problems from the June Issue

1. **Abhishek Khetan, IIT Kharagpur.** Let $I_{2s} = \{1, 2, \dots, 2s\}$. Let \mathcal{F} be a collection of non-empty subsets of I_{2n} such that any two members of \mathcal{F} have at least one element in common. Show that $|\mathcal{F}| \leq \binom{2s-1}{s-1}$.

First Solution. Define a function ϕ from \mathcal{F} to the power set of I_{2s} by $\phi(A) = I_{2s} \setminus A$. Since any two members of \mathcal{F} have at least one element in common, $\phi(A) \cap \phi(B) = \emptyset$. Thus $|\mathcal{F} \cup \phi(\mathcal{F})| = 2|\mathcal{F}| \leq \binom{2s}{s}$. Therefore $|\mathcal{F}| \leq \frac{1}{2} \binom{2s}{s} = \binom{2s-1}{s-1}$.

Second Solution. Partition \mathcal{F} into two subsets:

$$\mathcal{F}_1 = \{A \in \mathcal{F} \mid 2s \in A\}, \quad \mathcal{F}_2 = \{A \in \mathcal{F} \mid 2s \notin A\},$$

and finally let $\mathcal{F}_2^* = \{B \cup \{2s\} \mid B \in \mathcal{F}_2\}$. If $A \in \mathcal{F}_2^* \cap \mathcal{F}_1$, then $I_{2s} - A \in \mathcal{F}_2$, while $A \in \mathcal{F}_1$, so the disjoint sets A and $I_{2s} - A$ would both be in \mathcal{F} , contradicting the hypothesis. It follows that \mathcal{F}_2^* is disjoint from \mathcal{F}_1 . Now each element of $\mathcal{F}_2^* \cup \mathcal{F}_1$ contains $2s$. Therefore, $|\mathcal{F}_2^*| + |\mathcal{F}_1| \leq \binom{2s-1}{s-1}$. The result now follows from the fact that $|\mathcal{F}_2| = |\mathcal{F}_2^*|$.

2. **Tom Moore, Bridgewater State University.** Prove that the Diophantine equations

$$x^3 + y^3 = z^2, \quad x^4 + y^4 = z^3, \quad x^5 + y^5 = z^4$$

each have infinitely many positive integer solutions (x, y, z) .

Solution. We found the following solutions for respective n values:

$$\begin{aligned} n = 3 : x &= y = 2^{k+1} & \text{and} & \quad z = 2^{3k+2} \\ n = 4 : x &= y = 2^{3k+2} & \text{and} & \quad z = 2^{4k+3} \\ n = 5 : x &= y = 2^{4k+3} & \text{and} & \quad z = 2^{5k+4}, \end{aligned}$$

and these suggested a general solution, for any positive integer $n \geq 3$, namely

$$x = y = 2^{(n-1)k+(n-2)} \quad \text{and} \quad z = 2^{nk+(n-1)}.$$

Checking the equation, we find that

$$\begin{aligned} x^n + y^n &= 2 \times 2^{n((n-1)k+(n-2))} \\ &= 2^{n(n-1)k+n(n-2)+1} \\ &= 2^{(n-1)(nk+(n-1))} \\ &= z^{n-1}. \end{aligned}$$

Solution received. Hari Kishan of D. N. College, Meerut has suggested $x = y = 2^{n-2}u^{n-1}$ and $z = 2^{n-1}u^n$ as solutions for $x^n + y^n = z^{n-1}$. For $x^3 + y^3 = z^2$, he has also suggested $x = u^4 + 8uv^3$, $y = -4u^3v + 4v^4$, and $z = u^6 - 20u^3v^3 - 8v^6$ as solutions.

3. **S. Kesavan, IMSc.** Let $I \subset \mathbf{R}$ be an interval and let $f : I \rightarrow \mathbf{R}$ be a continuous function which is not monotonic. Then, given any $\epsilon > 0$, show that there exist points $x(\epsilon)$ and $y(\epsilon)$ in I such that

$$\begin{aligned} x(\epsilon) &\neq y(\epsilon), & |x(\epsilon) - y(\epsilon)| &< \epsilon; & \text{and} \\ f(x(\epsilon)) &= f(y(\epsilon)). \end{aligned}$$

Solution. Since f is not monotonic, there exist distinct points $x_0 < y_0$ in I such that $f(x_0) = f(y_0)$. Let $T = y_0 - x_0$. Now consider

$$g(x) = f(x + T/2) - f(x)$$

defined on $[x_0, x_0 + T/2]$. Then either $g(x_0) = 0$ (in which case $g(x_0 + T/2) = 0$ as well) or $g(x_0) = -g(x_0 + T/2)$. In the latter case, by the intermediate value theorem, we get a point x' such that $g(x') = 0$ with $x_0 < x' < x_0 + T/2$. In either case we get a pair of points x_1 and y_1 such that $y_1 - x_1 = T/2$ and $f(x_1) = f(y_1)$. Iterating this procedure, we see that for any positive integer $n \geq 2$, there exist points x_n and $y_n - x_n = T/2^n$ such that $f(x_n) = f(y_n)$. This completes the proof.

Solution received. M. Suresh Kumar of NIT Surathkal sent in a correct solution to this problem.

4. **K. N. Raghavan, IMSc.** Given m distinct n -tuples of integers p_1, \dots, p_m , and m integers $\alpha_1, \dots, \alpha_m$, show that there exists a polynomial f in n variables (with rational coefficients) such that $f(p_1) = \alpha_1, \dots, f(p_m) = \alpha_m$ and f takes integral values at all integer n -tuples.

Solution. We may assume $\alpha_1 = 1, \alpha_2 = \dots = \alpha_m = 0$, for, if f_1, \dots, f_m are polynomials that take integer values on integer n -tuples and satisfy $f_j(p_k) = \delta_{jk}$ (Kronecker delta), then we need only set $f = \alpha_1 f_1 + \dots + \alpha_m f_m$. We may assume p_1 to be the origin $(0, \dots, 0)$, for, if g is a polynomial such that $g(0, \dots, 0) = 1$ and $g(p_2 - p_1) = \dots = g(p_m - p_1) = 0$, then we need only set $f(X) = g(X - p_1)$. Now choose N large enough to exceed moduli of all co-ordinates of p_2, \dots, p_m . Set

$$f := \prod_{i=1}^n \binom{X_i + N}{N} \binom{-X_i + N}{N}$$

Here $\binom{A}{N}$ denotes the binomial coefficient, to be interpreted as

$$\frac{A(A-1)\dots(A-N+1)}{N!} \quad \text{for any } A.$$

The polynomial $\binom{X+N}{N}$ takes integer values at integers, vanishes at $-N, -(N-1), \dots, -1$, and is 1 at 0; and $\binom{X+N}{N}$ takes integer values at integers, vanishes at $1, 2, \dots, N$, and is 1 at 0. Since each of the p_j has a non-zero co-ordinate in the range $[N, N]$, we see that $f(p_j) = 0$.

5. **B. Sury, ISI Bangalore.** If n is a natural number greater than 1 such that all its powers n, n^2, n^3, n^4, \dots have an odd number of digits, then n is a power of 100.

Solution. Clearly, powers of 100 have the asserted property and we need only prove the converse. If n has the mentioned property, then

$$10^{2k} \leq n < 10^{2k+1}$$

for some $k \geq 0$. Now

$$10^{4k} \leq n^2 < 10^{4k+2},$$

which actually means that

$$10^{4k} \leq n^2 < 10^{4k+1},$$

because n^2 has an odd number of digits. Similarly,

$$10^{2^d k} \leq n^{2^{d-1}} < 10^{2^d k+1}.$$

Thus $10^{2^k} \leq n < 10^{2^k+1/2^{d-1}}$ for all d which gives $n = 10^{2^k}$.

6. **B. Sury, ISI Bangalore.** Find all pairs of primes p, q , whose sum is a power of their difference.

Solution. Let $p + q = (p - q)^r$ where $p > q$ are primes. If l is a prime dividing $p - q$, then $p + q \equiv 2q \equiv 0 \pmod{l}$ (the last because $p + q = (p - q)^r$). So, $l = 2$, or $l = q$. If $l = q$, then $p = l = q$, a contradiction.

Therefore $l = 2$, and p, q are odd primes. So $p - q = 2^k$ for some $k \geq 1$. Hence $p + q = 2^k + 2q = (p - q)^r = 2^{kr}$. So $q = 2^{kr-1} - 2^{k-1} = 2^{k-1}(2^{k(r-1)} - 1)$. As q is an odd prime, this gives $k - 1 = 0$, which gives $q = 2^{r-1} - 1$ and $p = q + 2 = 2^{r-1} + 1$. The fact that q is prime implies that $r - 1$ is prime. If $r - 1 = 2$, we get the solution $r = 3$, $q = 3$ and $p = 5$; that is

$$5 + 3 = (5 - 3)^3. \quad (*)$$

If $r - 1$ is an odd prime, then $p = 2^{r-1} + 1$ is a multiple of $2 + 1$, and is bigger than 3, and hence is not a prime. So (*) gives the unique solution.

Solution received. Aditi Phadke of Nowrosjee Wadia college sent in a correct solution to this problem by a different method.

7. **Amritanshu Prasad, IMSc.** Show that the probability that a monic polynomial of degree n in with coefficients in a

finite field of order q is square-free is $1 - q^{-1}$ for all integers $n > 1$.

Solution. Since there are q^n monic polynomials of degree n , we need to show that for $n > 1$, there are $q^n - q^{n-1}$ square-free monic polynomials.

By the unique factorization theorem for polynomials over a field, every monic polynomial can be uniquely written as the product of a monic square and a monic square-free polynomial. Therefore if $c_n(q)$ is the number of square-free polynomials of degree n then

$$q^n = \sum_{r=0}^{\lfloor n/2 \rfloor} q^{2r} c_{n-2r}(q),$$

whence

$$c_n(q) = q^n - \sum_{r=1}^{\lfloor n/2 \rfloor} q^{2r} c_{n-2r}(q). \quad (\dagger)$$

We now proceed inductively: we have the base cases $c_0(q) = 1$, $c_1(q) = q$ (since all monic polynomials of degree 0 and 1 are square-free). Using (\dagger) we find that

$$c_2(q) = q^2 - q \quad \text{and} \quad c_3(q) = q^3 - q^2.$$

We now proceed by induction. If n is even, (\dagger) gives

$$c_n(q) = q^n - c_{n-2}(q)q - c_{n-4}(q)q^2 - \dots - c_2(q)q^{\lfloor n/2 \rfloor - 1} - c_0(q)q^{\lfloor n/2 \rfloor}.$$

If we assume the result to be true for all $m < n$, then the above identity becomes

$$c_n(q) = q^n - (q^{n-2} - q^{n-3})q - (q^{n-4} - q^{n-5})q^2 - \dots - (q^2 - q)q^{\lfloor n/2 \rfloor - 1} - q^{\lfloor n/2 \rfloor},$$

which telescopes to $q^n - q^{n-1}$. A similar proof works for odd n .

at the Indian Institute of Science, Bangalore as an international joint research unit. IFCAM is designed as a platform for cooperation in mathematical sciences with the primary focus being the area of applied mathematics.

IFCAM has funds to support visits of Indian researchers, particularly from Universities and Colleges, working in applied mathematics (interpreted broadly to include mathematical aspects of engineering, physics, biology etc). Visitors can be hosted either at IFCAM, Indian Institute of Science, Bangalore or at a neighboring research institution (subject to consent of the institution). Visits can range from 1 month to 3 months and should aim to initiate or continue a research collaboration with a faculty member at the host institute. Interested researchers should apply online through the website given below. All visits should be completed by March 31, 2014. Selected visitors would be paid TA/DA for the duration of the visit. Further details on all of the above along with the application form can be obtained from:

<http://www.math.iisc.ernet.in/~ifcam/visitors.html>

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Indo-French Centre for Applied Mathematics (IFCAM)

Indian Institute of Science, Bangalore

Visitors Programme

The Indo-French Centre for Applied Mathematics (IFCAM) has been jointly set up by the Indian and French Governments

International Symposium on Complex Analysis and Conformal Geometry (ISCACG 2013)

28–30 December, 2013

Organizing Institution: Indian Institute of Technology Indore.

Aim: This symposium is mainly aimed at young researchers from all over the country who are interested in research in the following areas of Complex Analysis of current

relevance: univalent harmonic mappings, hyperbolic geometry and functions spaces. It will put special emphasis on exposure of Ph.D. students and post-doctoral fellows from India to the latest trends. In particular, young Indian research scholars will be able to learn about the recent developments in the above areas, new avenues in current research, and their connections to related fields. It also aims to bring ideas and inspiration to their ongoing research work, and to foster conversations between them and the senior researchers participating in the programme.

Further information on the symposium will be available shortly at www.iiti.ac.in/~iscacg2013

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**Call for Applications BMS
Dirichlet Postdoctoral Fellowship**

The Berlin Mathematical School (BMS) invites applications for the Dirichlet Postdoctoral Fellowship starting in the fall of 2014. This two-year position is open to promising young mathematicians holding a PhD who want to pursue their own research in any of the fields of mathematics represented in Berlin. The competitive full-year salary includes health insurance. Fellows are expected to teach one course per semester, typically in English and at the graduate level.

Completed applications are due by 1 December 2013, and should be submitted online at the BMS website:

[http://www.math-berlin.de/about-bms/
dirichlet-fellowship](http://www.math-berlin.de/about-bms/dirichlet-fellowship)

Applications from all well-qualified individuals, especially women, are highly encouraged. The Berlin Mathematical School (BMS) is a joint graduate school of the mathematics

departments of the three major Berlin universities: Freie Universität (FU), Humboldt-Universität (HU) and Technische Universität (TU). The BMS has been funded under the German “Excellence Initiative” since October 2006.

Contact:

Phone: +49 30 314 78651
E-mail: office@math-berlin.de
Web: <http://www.math-berlin.de>

**Indian Statistical Institute,
Chennai Centre**

A Course on Composition Operators

by

Prof. Hervé Queffelec
University of Lille, France

26 November, 2013 to 13 December, 2013

The theory of composition is by definition border-line between abstract operator theory and more concrete analytic function theory. Thus, the course contains the following aspects:

- A reminder of analytic functions
- A reminder of Hilbert spaces
- Complements on discrete dynamics
- Complements on operator theory on a Hilbert space
- Hilbert spaces of analytic functions
- Composition operators

Venue:

Indian Statistical Institute
Chennai Centre
SETS Campus
MGR Knowledge City
C.I.T. Campus
Taramani, Chennai 600 113
Web: www.isichennai.res.in

Participation & Registration: Registration is free. The request for participation can be sent to the following address on or before November 22, 2013, by specifying your background in the above topics.

samy@isichennai.res.in

or

karthick@isichennai.res.in

Other Details: The number of participants is limited.

Selected outstation participants will be provided accommodation with nominal rates based on the availability.

Selected candidates will be informed via email soon after.

For further information, please visit us at:

www.isichennai.res.in/~cco

School of Mathematics, TIFR

School of Mathematics of the Tata Institute is actively seeking applications from strong researchers in all areas of pure mathematics. TIFR has a long tradition of research in Algebra, Algebraic Geometry, Combinatorics, Number Theory, Lie Theory and Ergodic Theory. We seek applications in these areas as well as other areas of pure mathematics.

TIFR has a beautiful campus next to the Arabian sea where accommodation is provided for the scientific staff. The library of TIFR subscribes to most of the important journals in mathematics and has a well stocked reference library.

TIFR has recently become a Deemed University and runs a Ph.D. program but teaching is voluntary.

Candidates will be considered for one of the following positions:

- Post doctoral Position: For 1 or 2 years for candidates with Ph.D. with atmost 3 years experience.
- Fellow: For 5 years for candidates with Ph.D. and 3 to 5 years post-doctoral experience.
- Reader and Professor are tenured appointments. Though appointments to these positions are generally made after a few years as Fellow, this may be waived for deserving candidates.

Applications at all levels and enquiries should be directed to Search Committee: (sc/at/math.tifr.res.in).

There is no deadline for applications.

School of Mathematics has a Centre in Bangalore where research in Applicable Mathematics is carried out. For more information about that program, see the website of TIFR Bangalore.

Seoul ICM 2014

The next International Congress of Mathematicians will take place at COEX in Seoul, Korea, from Wednesday August 13, through Thursday August 21, 2014. The pre-registration process for the ICM 2014 is underway. If you have not yet pre-registered, please do so by following the simple instructions at the homepage: <http://www.icm2014.org>. The ICM e-News is being circulated to the people who pre-registered for the congress. We strongly recommend that you visit the homepage regularly for updated information and ICM related activities.

* NANUM 2014: Application Period Extended 1,000 mathematicians from developing countries will be invited to Korea during ICM 2014 under the ICM 2014 signature travel grant program, called "NANUM 2014". NANUM means 'generoussharing' in Korean. Travel grants awarded will be in the range of approximately US\$1,500-US\$2,500 per person according to the regional groups, making the total sum of US\$2 million. See

<http://icm2014.org/en/participants/mathematicians>

for more details, e.g., the prerequisites to be eligible for a travel grant. Applications can only be submitted online at <http://nanum2014.org>. By August 31, 2013, about 2,800 NANUM applications were received. The online submission system was deactivated by August 31, 2013. However, by popular demand, the submission site will be re-opened on October 16, 2013 and will remain open until October 31, 2013. We hope that this grace period may enable the people who missed the deadline to submit their applications.

Important Dates:

- *June 1, 2013 – August 31, 2013: Applications received
- *October 16, 2013 – October 31, 2013: Submission site re-opened
- *December 31, 2013: Review of Applications completed
- *January 2014: Notification of acceptance

TWAS: Free Access to Mathematical Literature for Developing Countries

TWAS, the World Academy of Sciences for the advancement of science in developing countries (<http://twas.ictp.it/>) offers access to a lot of scientific resources (including mathematics), which are either 1. “Open Access”, 2. freely accessible for developing countries, or 3. accessible at reduced price for developing countries:

<http://twas.ictp.it/links/open-access-scientific-information>.

The special access to scientific literature for developing countries is through <http://www.research4life.org/>, which consists of the four networks HINARI, AGORA, OARE and ARDI. The ARDI network is the most relevant for mathematicians. On their website (<http://www.wipo.int/ardi/en/>) one can click in the left column on “Journals” and find the list of journals concerned by this program (see also <http://www.wipo.int/ardi/en/journals.html>). These journals are either freely accessible or accessible at reduced price (depending on the country’s GBP) to the scientists of the countries appearing on the list approved by the World Intellectual Property Organization (WIPO). This list can be consulted at

<http://www.research4life.org/institutions/>

For the researchers who do not have access to sufficient bandwidth to download material from the Internet in a timely manner and/or cannot afford the connection, the Abdus Salam International Center for Theoretical Physics (ICTP) offers to send articles in mathematics and physics as email attachments. This is offered through the electronic Journals Delivery Service (eJDS): see <http://ejds.ictp.it/ejds/>, which requires registration and is cost-free.

ABEL Visiting Scholar Program

The Niels Henrik Abel Board and the International Mathematical Union are pleased to announce the “Abel Visiting Scholar Program”.

The program invites mathematicians professionally based in developing countries to visit an international research collaborator for a period of one month. The period is extendable for up to three month in the case of matching support from the host institution. Up to three mathematicians per year will be supported.

The program is designed for post doctoral mathematicians in the early stages of their professional careers. It is designed to offer the opportunity for a ‘research sabbatical’, a necessary complement to teaching and other academic duties for mathematicians desiring to also sustain a viable research program.

Basic Requirements: Applicants must hold a PhD in Mathematics and be in the early stages of their professional careers. This means that the applicant should be under 40 years on July 1st 2013 and not yet of full professorial rank. The maximum age may be increased by up to three years in the case of an individual with a broken career pattern. Applications from women mathematicians are strongly encouraged. The applicant should already have initiated research contact with the proposed international research partner. The collaboration should take place at the international partner’s home institution

Financial Support: The grant will cover health insurance, visa cost, all travel (economy flights or equivalent) and living expenses including accommodation for one month for up to a total maximum amount of USD 5,000 per mathematician.

More Information can be Found Here:

<http://www.mathunion.org/cdc/grants/abel-visiting-scholar-program/>

The Abel Visiting Scholar Program is administered by the Commission for Developing Countries of the International Mathematical Union. In case of questions, please contact the CDC Administrator:

icmi.cdc.administrator@mathunion.org

Science Academies’ Summer Research Fellowship Programme for Students and Teachers 2014

The three national Science Academies offer several two-month Summer Fellowships to enable students/teachers

(studying/teaching in India) to work with scientists associated with the three Academies during 2014. A copy of the application format, instructions to applicants including eligibility criteria, and a list of names of scientists/faculty who have consented to guide students/teachers to work on short-term projects is displayed online.

Applications are invited from interested students and teachers from all universities and colleges affiliated to UGC/AICTE/MCI/Accredited Institutions of State Universities for these Fellowships. The application should include: (a) the application form in the prescribed format; (b) a write-up (in about 150–250 words) as to what the applicant wants to learn and achieve; (c) the guide with whom the applicant would like to work. Student applicants should include a recommendations letter (in the prescribed format) familiar with their work, in a sealed envelope. The selected candidate may work with the assigned guide for two months any time during the calendar year, preferably during the summer.

Applications should be submitted Online through one of our websites <http://www.ias.ac.in>, <http://www.insaindia.org> or <http://www.nasi.org.in>; however, a copy of the application together with enclosures must be sent by speed post or courier to The Coordinator, Science Education Programmes, Indian Academy of Sciences, CV Raman Avenue, Near Mekhri Circle, Sadashivanagar, Bangalore 560 080. The registration number assigned soon after online submission must be quoted both in the hard copy of the application to be sent and in the letter of recommendation to be forwarded by the teacher in the case of student applicants.

The last date for receipt of applications is 30 November 2013.

Information of selection along with concurrence of the guide will be despatched around February–March 2014. The selected students/teachers will be provided appropriate round trip train fare and a monthly fellowship to meet their living expenses at the place of work.

Details of Workshop/Conferences in India

For details regarding Advanced Training in Mathematics Schools

Visit: <http://www.atmschools.org/>

Name: National Symposium on Mathematical Methods and Applications

Date: December 22, 2013

Location: I.I.T Madras, Chennai, Tamil Nadu

Visit: <http://mat.iitm.ac.in/nsmma2013>

Name: National Conference On Mathematical Techniques and its Applications

Date: January 6–7, 2014

Location: SRM University, Kattankulathur, Chennai, Tamil Nadu

Contact: hod.maths@ktr.srmuniv.ac.in

Name: Third International Science Congress

Date: December 8–9, 2013

Location: Karunya University, Coimbatore, Tamil Nadu

Contact: dipaksharma07@yahoo.com

Name: National Programme on Differential Equations: Theory, Computation & Applications

Date: December 2–7, & 9–14, 2013

Location: National Institute of Technology, Calicut, Kerala

Visit: <http://nitc.ac.in/modelling> or www.iitb.ac.in/~npde-tca

Name: IWM2013-Teachers Training Programme

Date: December 23–28, 2013

Location: Department of Mathematics, Mumbai University

Visit: <https://sites.google.com/site/iwm2013tftp/>

Name: International Conference on Recent Advances in Statistics and Their Applications

Date: December 26–28, 2013

Location: Dr. Babasaheb Ambedkar Marathwada University, Aurangabad

Visit: <http://www.bamu.ac.in/icrastat2013/>

Name: International Conference on Mathematics and Computing – 2013

Date: December 26–29, 2013

Location: Haldia Institute of Technology, Haldia, West Bengal

Visit: <http://hithaldia.in/icmc2013/>

Name: 22nd National and 11th ISHMT ASME Heat and Mass transfer Conference

Date: December 28–31, 2013

Location: IIT Kharagpur, Kharagpur

Visit: <http://ishmt2013iitkgp.in/>

Name: 42nd Annual Conference of OMS & International Conference on Industrial Mathematics and Scientific Computing

Date: January 4–5, 2014

Location: KIIT University, Bhubaneswar, Odisha.

Visit: <http://icimsc2014.org/>

Name: 8th International conference on matrix analytic methods in stochastic models

Date: January 6–10, 2014

Location: National Institute of Technology Calicut (NITC), Kerala

Visit: <http://mam8.nitc.ac.in/>

Name: International Conference on Recent Advances in Mathematics (ICRAM 2014)

Date: January 20–23, 2014

Location: Rashtrasant Tukadoji Maharaj Nagpur University, Nagpur

Visit: <http://icram2014.com/>

For more details and updates regarding workshop/conferences in India please

Visit: <http://www.conference-service.com/conferences/in/>

Details of Workshop/Conferences in Abroad

For details regarding ICTP (International Centre for Theoretical Physics)

Visit: <http://www.ictp.it/>

Name: 3rd Annual International Conference on Computational Mathematics, Computational Geometry & Statistics (CMCGS 2014)

Date: February 3–4, 2014

Location: Hotel Fort Canning, 11 Canning Walk, Singapore, Singapore 17881.

Visit: <http://www.mathsstat.org/>

Name: Introductory Workshop: Model Theory, Arithmetic Geometry and Number Theory

Date: February 3–7, 2014

Location: Mathematical Sciences Research Institute, Berkeley, California.

Visit: <http://www.msri.org/workshops/688>

Name: ICERM Semester Program on “Network Science and Graph Algorithms”

Date: February 3–May 9, 2014

Location: ICERM, Providence, Rhode Island

Visit: <http://icerm.brown.edu/sp-s14>

Name: Function Theory on Infinite Dimensional Spaces XIII

Date: February 4–7, 2014

Location: ICMAT, Campus de Cantoblanco, Madrid, Spain

Visit: <http://www.icmat.es/congresos/2014/ftida/>

Name: Connections for Women: Model Theory and its interactions with number theory and arithmetic geometry

Date: February 10–11, 2014

Location: Mathematical Sciences Research Institute, Berkeley, California

Visit: <http://www.msri.org/web/msri/scientific/workshops/all-workshops/show/-/event/Wm9548>

Name: ICERM Workshop: Semidefinite Programming and Graph Algorithms

Date: February 10–14, 2014

Location: ICERM, Providence, Rhode Island.

Visit: <http://icerm.brown.edu/sp-s14-w1>

Name: Translating Cancer Data and Models to Clinical Practice

Date: February 10–14, 2014

Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Visit: <http://www.ipam.ucla.edu/programs/cdm2014/>

Name: Higher Structures in Algebraic Analysis

Date: February 10–21, 2014

Location: University of Padova, Department of Mathematics, Padova, Italy

Visit: <http://events.math.unipd.it/hxaa/>

Name: Hot Topics: Perfectoid Spaces and their Applications

Date: February 17–21, 2014

Location: Mathematical Sciences Research Institute, Berkeley, California

Visit: <http://www.msri.org/workshops/731>

Name: Stochastic Gradient Methods

Date: February 24–28, 2014

Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Visit: <http://www.ipam.ucla.edu/programs/sgm2014/>

Name: XIX SIMMAC – International Symposium on Mathematical Methods Applied to the Sciences

Date: February 25–28, 2014

Location: University of Costa Rica, San Jose, Costa Rica

Visit: <http://www.cimpa.ucr.ac.cr/simmac/en/>

Name: AIM Workshop: Postcritically finite maps in complex and arithmetic dynamics

Date: March 3–7, 2014

Location: American Institute of Mathematics, Palo Alto, California

Visit: <http://www.aimath.org/ARCC/workshops/finitedynamics.html>

Name: 11th German Probability and Statistics Days 2014 – Ulmer Stochastik-Tage

Date: March 4–7, 2014

Location: University Ulm, Ulm, Germany

Visit: <http://www.gpsd-ulm2014.de>

Name: International Workshop on Discrete Structures (IWODS)

Date: March 5–7, 2014

Location: Centre for Advanced Mathematics and Physics, National University of Sciences and Technology, H-12 Islamabad, Pakistan

Visit: <http://www.camp.nust.edu.pk/IWODS2014/>

Name: School and Workshop on Classification and Regression Trees

Date: March 10–26, 2014

Location: Institute for Mathematical Sciences, National University of Singapore, Singapore

Visit: <http://www2.ims.nus.edu.sg/Programs/014swclass/index.php>

Name: Algebraic Techniques for Combinatorial and Computational Geometry

Date: March 10–June 13, 2014

Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Visit: <http://www.ipam.ucla.edu/programs/ccg2014>

Name: Algebraic Techniques for Combinatorial and Computational Geometry: Tutorials

Date: March 11–14, 2014

Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Visit: <http://www.ipam.ucla.edu/programs/ccgtut/>

Name: IAENG International Conference on Operations Research 2014

Date: March 12–14, 2014

Location: Hong Kong, China

Visit: <http://www.iaeng.org/IMECS2014/ICOR2014.html>

Name: 48th Annual Spring Topology and Dynamical Systems Conference

Date: March 13–15, 2014

Location: University of Richmond, Richmond, Virginia

Visit: <http://math.richmond.edu/resources/topology-conference/index.html>

Name: Representation Theory and Geometry of Reductive Groups

Date: March 14–28, 2014

Location: KlosterHeiligkreuztal, a Monastery in Germany, Altheim, Germany

Visit: <http://www2.math.uni-paderborn.de/konferenzen/conferencespring-school.html>

Name: ICERM Workshop: Stochastic Graph Models

Date: March 17–21, 2014

Location: ICERM, Providence, Rhode Island

Visit: <http://icerm.brown.edu/sp-s14-w2>

Name: 2nd Annual International Conference on Architecture and Civil Engineering (ACE 2014)

Date: March 24–25, 2014

Location: Hotel Fort Canning, 11 Canning Walk, Singapore, Singapore 17881

Visit: <http://www.ace-conference.org/>

Name: Combinatorial Geometry Problems at the Algebraic Interface

Date: March 24–28, 2014

Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Visit: <http://www.ipam.ucla.edu/programs/ccgws1/>

Name: Mathematical, Statistical and Computational Aspects of the New Science of Metagenomics

Date: March 24–April 17, 2014

Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom

Visit: <http://www.newton.ac.uk/programmes/MTG/index.html>

Name: 38th Annual SIAM Southeastern Atlantic Section (SEAS) Conference

Date: March 28–30, 2014

Location: Florida Institute of Technology, Melbourne, Florida

Visit: <http://my.fit.edu/~abdulla/SIAMSEAS-2014>

Name: SIAM Conference on Uncertainty Quantification (UQ14)

Date: March 31–April 3, 2014

Location: Hyatt Regency Savannah, Savannah, Georgia, USA

Visit: <http://www.siam.org/meetings/uq14/>

Name: Ischia Group Theory 2014

Date: April 1–5, 2014

Location: Grand Hotel delle Terme Re Ferdinando, Ischia, Naples, Italy

Visit: <http://www.dipmat.unisa.it/ischiagrouptheory/>

Name: 13th New Mexico Analysis Seminar

Date: April 3–4, 2014

Location: University of New Mexico, Albuquerque, New Mexico

Visit: <http://www.math.unm.edu/conferences/13thAnalysis/>

Name: AIM Workshop: The many facets of the Maslov index

Date: April 7–11, 2014

Location: American Institute of Mathematics, Palo Alto, California

Visit: <http://www.aimath.org/ARCC/workshops/maslov.html>

Name: ICERM Workshop: Electrical Flows, Graph Laplacians, and Algorithms: Spectral Graph Theory and Beyond

Date: April 7–11, 2014

Location: ICERM, Providence, Rhode Island

Visit: <http://icerm.brown.edu/sp-s14-w3>

Name: Reimagining the Foundations of Algebraic Topology

Date: April 7–11, 2014

Location: Mathematical Sciences Research Institute, Berkeley, California

Visit: <http://www.msri.org/web/msri/scientific/workshops/programmatic-workshops/show/-/event/Wm9550>

Name: Tools from Algebraic Geometry

Date: April 7–11, 2014

Location: Institute for Pure and Applied Mathematics (IPAM), UCLA, Los Angeles, California

Visit: <http://www.ipam.ucla.edu/programs/ccgws2/>

Name: Advanced Monte Carlo Methods for Complex Inference Problems

Date: April 22–May 16, 2014

Location: Isaac Newton Institute for Mathematical Sciences, Cambridge, United Kingdom

Visit: <http://www.newton.ac.uk/programmes/MCM/index.html>

Name: International arab conference on mathematics and computations

Date: April 23–25, 2014

Location: Zarqa University, Zarqa, Jordan

Visit: <http://www.iacmc.org>

Name: AIM Workshop: Exact crossing numbers

Date: April 28–May 2, 2014

Location: American Institute of Mathematics, Palo Alto, California

Visit: <http://www.aimath.org/ARCC/workshops/exactcrossing.html>

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www.ramanujanmathsociety.org**