MATHEMATICS NEWSLETTER

Volume 16 March 2007 No. 4

CONTENTS

Visit: www.ramanujanmathsociety.org

Typeset in LaTeX at Krishtel eMaging Solutions Pvt. Ltd., Chennai - 600 017 Phone: 2434 55 16 and printed at United Bind Graphics, Chennai - 600 004. Phone: 2498 7562, 2466 1807

Representations of a Finite Group in Positive Characteristic

Amritanshu Prasad
The Institute of Mathematical Sciences
CIT campus, Taramani, Chennai 600113
E-mail: amri@imsc.res.in
http://www.imsc.res.in/~amri

Abstract. An element x of a finite group G is said to be *p-regular* if its order is not divisible by p. Brauer gave several proofs of the fact that the number of isomorphism classes of irreducible representations of G over an algebraically closed field of characteristic p is the same as the number of conjugacy classes in G that consist of p-regular elements. One such proof is presented here.

Let G be a finite group, and K be any field. Then the group algebra $K[G]$ is a K-vector space with basis consisting of the elements of G:

$$K[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in K \right\}.$$

Multiplication is defined by linearly extending the product on basis elements

$$g \cdot h = gh \text{ for } g, h \in G$$

to $K[G]$. The group algebra was introduced by the German mathematician *Ferdinand Georg Frobenius* in 1897 to study the representations of finite groups.

Exercise 1. Let $n > 1$ be an integer. Let $\mathbf{Z}/n\mathbf{Z}$ denote the cyclic group with n elements. Show that $K[\mathbf{Z}/n\mathbf{Z}]$ is isomorphic to $K[t]/(t^n - 1)$.

In this article, the term $K[G]$-module will be used to refer to a vector space M over K, together with an algebra homomorphism $R: K[G] \to \mathrm{End}_K(M)$, where $\mathrm{End}_K(M)$ denotes the algebra of K-linear maps from M to itself. In practice, for any $a \in K[G]$ and $m \in M$, the element $R(a)(m)$ of M will be denoted simply by am. For any vector space M, let $GL(M)$ denote the group of invertible K-linear maps from M to itself. Recall that a representation of G over the field K consists of a vector space M over K and a function $r: G \to GL(M)$ such that $r(gh) = r(g)r(h)$ for all $g, h \in G$. Such a vector space becomes a $K[G]$-module under the action

$$\left(\sum_{g \in G} a_g g \right) m = \sum_{g \in G} a_g r(g) m \qquad \text{for all } m \in V.$$

The representation r can be recovered from the $K[G]$-module structure by restricting to the basis elements of $K[G]$ coming from G. In fact, the study of $K[G]$-modules is equivalent to the study of representations of G over K (in a category-theoretic sense, which will not be formulated here). This article takes the module-theoretic viewpoint.

Two modules (or representations) are said to be *isomorphic* if there is an isomorphism between their underlying vector spaces which preserves the actions of the algebra (or group). A module defined by R and M as above is called *simple* if M is non-trivial and does not admit a non-trivial proper subspace that is invariant under $R(a)$ for every $a \in K[G]$. Similarly, a representation defined by r and M as above is called *irreducible* if M is non-trivial and does not admit a non-trivial proper subspace that is invariant under $r(g)$ for every $g \in G$. Simple $K[G]$-modules correspond to irreducible representations of G over K. Irreducible representations may be considered to be the building blocks of all representations, a point of view which is partially justified by the Jordan–Hölder theorem.

Frobenius showed that the number of isomorphism classes of irreducible representations of a finite group G over an algebraically closed field K of characteristic zero (such as the field of complex numbers) is equal to the number of conjugacy classes in G. In many modern textbooks this is deduced from the fact that the characters of irreducible representations form a basis of the space of class functions (see e.g., [Art94]). This result fails when the characteristic of K divides the order of the group G, as was pointed out by the American mathematician *Leonard Eugene Dickson* in the first decade of the twentieth century. The determination of the number of isomorphism classes of irreducible representations in this case remained open for a long time and was finally solved by another German mathematician, *Richard Brauer*, in 1935.

Even though Brauer was already a leading representation theorist, he lost his position at the University of Königsberg in Germany in 1933, after Hitler assumed dictatorial powers and started implementing his anti-semitic policies. Brauer moved to the United States, and then to Canada, and went on to become the most influential figure in modern representation theory. The result of Brauer that is discussed here is only the first of many discoveries that he made on representations in positive characteristic by analysing the ring-theoretic properties of group algebras. The most striking of these is known as *the theory of blocks*, which has been applied with great success to the study of the structure and the classification of finite simple groups. The reader who is interested in Brauer's life and work is referred to Curtis's remarkable book [Cur99], from where the proof of Brauer's theorem given here (originally due to Brauer himself) has been adapted. The standard reference for results on non-semisimple algebras and modular representations is [CR62]. A picture of developments in the general theory of modular representations up to 1980 is found in [Fei82]. Many newer developments can be found in [Ben91a] and [Ben91b].

Brauer's theorem is easy to state: an element of G is called *p-regular* if its order is not divisible by p. A *p-regular conjugacy class* is a conjugacy class consisting of p-regular elements.

Theorem. **(Brauer).** *When K is algebraically closed of characteristic $p > 0$, the number of isomorphism classes of simple $K[G]$-modules is equal to the number of p-regular conjugacy classes in G.*

This theorem follows from Propositions 8 and 10 below. The reader who is already familiar with the basic theory of associative algebras may proceed directly to these statements and their proofs.

In what follows, K will always be an algebraically closed field and all K-algebras and all their modules will be assumed to be finite dimensional vector spaces over K. Every algebra A will be assumed to have a multiplicative unit $1 \in A$. For every module M it will be assumed that 1 acts on M as the identity (such a module is called *unital*). For two A-modules M and N, $\mathrm{Hom}_A(M, N)$ will denote the A-module *homomorphisms* from M to N, namely those linear maps $\phi\colon M \to N$ for which $\phi(am) = a\phi(m)$ for all $a \in A$ and $m \in M$. $\mathrm{End}_A(M)$ will denote the space $\mathrm{Hom}_A(M, M)$ of *endomorphisms* of M. A *submodule* of M will be a subspace M' of M such that $am' \in M'$ for every $a \in A$ and $m' \in M'$. Note that the image and the kernel of an A-module homomorphism is a submodule.

Definition. M is said to be a *simple* A-module if it is non-trivial and it contains no non-trivial proper submodules.

Theorem. **(Schur's lemma).**

(1) *If M is a simple A-module,* $\mathrm{End}_A(M) \cong K$.

(2) *If M and N are non-isomorphic simple A-modules then* $\mathrm{Hom}_A(M, N) = 0$.

Proof. Suppose M is simple and $\phi \in \mathrm{End}_A(M)$. Then, since K is algebraically closed, ϕ has an eigenvalue $\lambda \in K$. $\phi - \lambda I$ is singular and lies in $\mathrm{End}_A(M)$. Its kernel is a non-trivial A-submodule. By the simplicity of M, this kernel must be all of M. Therefore $\phi = \lambda I$. The proof of the second assertion is an easy exercise for the reader. □

Suppose M is a simple A-module. Pick $m \in M$ such that $m \neq 0$. The map $\phi_m : A \to M$ given by

$$\phi_m(a) = am \text{ for all } a \in A$$

is an A-module homomorphism (here A is viewed as a left A-module). Since the image of ϕ_m is a non-trivial submodule of M, it must be all of M. Therefore, ϕ_m is surjective. Its kernel is a left ideal in A.

Conclusion. Every simple A-module is isomorphic to a quotient of A by a left ideal.

Definition. A left ideal N of A is said to be *nilpotent* if there exists a positive integer k such that $N^k = 0$ (here N^k is the vector space spanned by products of k elements in N).

Exercise 2. Suppose that K is an algebraically closed field of characteristic p, and that $n = pm$ for some positive integer m. Show that $(t^m - 1)$ generates a nilpotent ideal in $K[t]/(t^n - 1)$.

Proposition 1. *Every nilpotent left ideal of A is contained in the kernel of ϕ_m.*

Proof. Suppose that N is a left ideal of A not contained in $\ker \phi_m$. Then Nm is a non-trivial submodule of M, hence

$Nm = M$. In particular, there exists $n \in N$ such that $nm = m$. It follows that $n^k m = m$ for every positive integer k. Therefore, every power of n is non-zero. N cannot, therefore, be nilpotent. □

It is not always the case that a finite dimensional A-module is a direct sum of simple modules.

Exercise 3. Take K to be any field of characteristic two. Take A to be $K[\mathbf{Z}/2\mathbf{Z}]$. Show that A has a unique non-trivial proper submodule, which is spanned by $0 + 1$ (here 0 and 1 are the basis vectors). Conclude that A can not be written as a direct sum of simple A-modules.

Definition. An A-module M is said to be *semisimple* if it can be written as a direct sum of simple modules. A is called a *semisimple algebra* if, as an A-module, A is semisimple. A is called a *simple algebra* if it has no proper two-sided ideals.

Exercise 4. Suppose that K is a algebraically closed and that the characteristic of K does not divide n. Show that the equation $t^n - 1 = 0$ has n distinct roots.

Exercise 5. Assume that K is as in Exercise 4. Show that $K[\mathbf{Z}/n\mathbf{Z}]$ is semisimple (Hint: use Exercises 1 and 4).

Example. *Maschke's theorem* (see, e.g., [Art94, p. 316]) states that $K[G]$ is semisimple when the characteristic of K does not divide the order of G.

Exercise 6. (see [Lan99, p. 656]) Show that the algebra $M_n(K)$ of $n \times n$ matrices is simple (for example, by showing that the two-sided ideal generated by any non-zero matrix is all of $M_n(K)$). Show that every simple module is isomorphic to K^n (which can be thought of as the space of column vectors on which $M_n(K)$ acts on the left by multiplication).

Proposition 2. *Every semisimple algebra is a direct sum of simple algebras.*

Proof. Let A_1 be a minimal two-sided ideal of A. Let A' be a complement of A_1 (as a left A-module), so that $A = A_1 \oplus A'$. Suppose that the decomposition of 1 under the above direct sum decomposition is $1 = \epsilon_1 + \epsilon'$. The decomposition of $a \in A$ is given by $a = a\epsilon_1 + a\epsilon'$. In particular, $\epsilon_1 = \epsilon_1 1 = \epsilon_1(\epsilon_1 + \epsilon') = \epsilon_1^2 + \epsilon_1\epsilon'$. Therefore, $\epsilon_1^2 = \epsilon_1$ and $\epsilon_1\epsilon' = 0$. Similarly, $\epsilon'\epsilon_1 = 0$. We can also write $A = \epsilon_1 A \oplus \epsilon' A$, where the decomposition of $a \in A$ is given by $a = \epsilon_1 a + \epsilon' a$. If $a_1 \in A_1$, then comparing its two decompositions shows that $a_1 = a_1\epsilon_1 = \epsilon_1 a_1$. More generally, if $a \in A$, then $a\epsilon_1 = a\epsilon_1^2 = (a\epsilon_1)\epsilon_1$. But $a\epsilon_1 \in A_1$. Therefore, $(a\epsilon_1)\epsilon_1 = \epsilon_1(a\epsilon_1)$. A similar argument can be used to show that $\epsilon_1 a = (\epsilon_1 a)\epsilon_1$. Therefore $a\epsilon_1 = \epsilon_1 a$. Since $\epsilon' = 1 - \epsilon_1$, it also follows that $\epsilon' a = a\epsilon'$ for every $a \in A$. Therefore, $A' = A\epsilon' = \epsilon' A$, so that A' is also a two sided ideal. Now repeat this argument replacing A by A'. Continuing in this manner, one obtains that $A = A_1 \oplus \cdots \oplus A_s$ for some s, where the summands are minimal two-sided ideals, hence simple algebras. □

We now discuss another characterization of semisimple algebras. Firstly note that

Lemma 3. *The sum of two nilpotent left ideals is nilpotent.*

Proof. Suppose that N_1 and N_2 are two nilpotent left ideals. Take k such that $N_1^k = N_2^k = 0$. Every element of $(N_1 + N_2)^{2k}$ is a linear combination of elements of the form $(n_1 + n_2)^k$, where $n_1 \in N_1$ and $n_2 \in N_2$. In each term of the expansion of the product $(n_1 + n_2)^k$, either n_1 or n_2 occurs at least k times, so that each term is either in N_1^k or N_2^k, and is therefore 0. □

Exercise 7. Show that $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ are nilpotent elements in $M_2(K)$, but their sum is not nilpotent. Why does this example not contradict Lemma 3?

Suppose that N is a nilpotent left ideal of A. If N is not maximal, then there exists a nilpotent ideal N' that is not contained in N. By Lemma 3, $N + N'$ is a nilpotent ideal, which is strictly larger than N. From the finite dimensionality of A, it now follows that A has a unique maximal nilpotent left ideal, which is called the *radical* of A, denoted $\mathrm{Rad} A$. By Proposition 1, $\mathrm{Rad} A \subset \ker \phi_m$. Now $(\mathrm{Rad} A)A$ is a two-sided ideal. It is nilpotent because

$$[(\mathrm{Rad} A)A]^2 \subset (\mathrm{Rad} A)^2 A, \ [(\mathrm{Rad} A)A]^3 \subset (\mathrm{Rad} A)^3 A, \ldots .$$

It follows that $(\mathrm{Rad} A)A \subset \mathrm{Rad} A$, and so $\mathrm{Rad} A$ is a two-sided ideal.

Exercise 8. If I is a two-sided ideal in A, show that the formulas $(a+I)+(b+I) = a+b+I$ and $(a+I)(b+I) = ab+I$ give rise to a well-defined algebra structure on the quotient space $\frac{A}{I}$.

This allows one to make sense of the quotient $\frac{A}{\mathrm{Rad} A}$ as an algebra.

Proposition 4. *A is semisimple if and only if* $\mathrm{Rad}A = 0$.

Proof. Suppose that A is semisimple. Then A, as a left A-module, can be written as a sum of simple A-modules:

$$A = M_1 \oplus \cdots \oplus M_k.$$

Suppose that $1 = e_1 + \cdots + e_k$ is the decomposition of 1. Then the identity map of A (which is right multiplication by 1) can be written as $\phi_{e_1} + \cdots + \phi_{e_m}$. By Proposition 1, $\mathrm{Rad}A \subset \cap_{i=1}^k \ker \phi_{e_i}$. On the other hand

$$\cap_{i=1}^k \ker \phi_{e_i} = \ker(\phi_{e_1} + \cdots + \phi_{e_k}) = 0.$$

Therefore, $\mathrm{Rad}A = 0$.

Conversely, suppose that $\mathrm{Rad}A = 0$. Then A has no non-trivial nilpotent left ideals. Let N be a minimal non-zero left ideal of A (as an A-module, N is simple). Then N^2 is a left ideal contained in N. Since $N^2 \neq 0$, $N^2 = N$. Therefore, there exists $a \in N$ such that $Na \neq 0$. But Na itself is a left ideal contained in N. Therefore, $Na = N$. It follows that $B = \{b \in N \mid ba = 0\}$ is a left ideal properly contained in N. Therefore $B = 0$. Moreover, since $Na = N$, $a = ca$ for some $c \in N$. Also $ca = c^2 a$, so that $(c - c^2)a = 0$. In other words, $c - c^2 \in B$. Therefore $c - c^2 = 0$. Therefore c is a non-zero idempotent in N. By the minimality of N, $Ac = N$. Moreover, $A = Ac \oplus A(1 - c)$. If $A(1 - c)$ is not simple, then take a minimal left ideal in $A(1 - c)$ and repeat the above process. Since A is a finite dimensional vector space, this process will end after a finite number of steps, resulting in a decomposition of A into a direct sum of simple modules. Therefore A is semisimple. □

Corollary 5. $\frac{A}{\mathrm{Rad}A}$ *is semisimple.*

Proof. Since $\mathrm{Rad}A$ is a maximal nilpotent ideal, $\frac{A}{\mathrm{Rad}A}$ has no nilpotent ideals. By Proposition 4, $\frac{A}{\mathrm{Rad}A}$ is semisimple. □

Exercise 9. Suppose that K has characteristic p, and let $n = pm$ for some positive integer m. Show that $K[\mathbf{Z}/n\mathbf{Z}]$ is not semisimple (Hint: use Exercises 1 and 2).

Corollary 6. *Every simple algebra is semisimple.*

Proof. Since $\mathrm{Rad}A$ is a proper two-sided ideal, the simplicity of A implies that $\mathrm{Rad}A = 0$. Therefore A is semisimple. □

Theorem. **(Wedderburn).** *Every simple algebra is isomorphic to* $M_n(K)$ *for some positive integer* n.

Proof. Since A is semisimple, A (viewed as a left A-module) can be decomposed into a direct sum of simple A-modules. Let

$$A = M_1^{\oplus m_1} \oplus \cdots \oplus M_k^{\oplus m_k} \tag{7}$$

be such a decomposition where $M_1, \ldots, M_k$ are pairwise non-isomorphic. For each $a \in A$, the map $\phi_a : A \to A$ defined by $\phi_a(x) = xa$ is an A-module homomorphism $A \to A$. Moreover, $\phi_a \circ \phi_b = \phi_{ba}$. Conversely, every A-module homomorphism $\phi : A \to A$ is of the form ϕ_a, where $a = \phi(1)$. Therefore the A-module homomorphisms $A \to A$ form an algebra A^* whose elements are the same as those of A, but multiplication is reversed. A two-sided ideal of A is also a two sided ideal of this A^*. Therefore A^* is also simple. Schur's lemma can be used to show that $A^* = M_{m_1}(K) \oplus \cdots \oplus M_{m_k}(K)$. $M_{m_1}(K)$ is proper two-sided ideal of A^*. Therefore, by the simplicity A^* we must have $k = 1$ in (7) and $A^* = M_{m_1}(K)$. □

Proposition 8. *Let A be a finite dimensional algebra over an algebraically closed field K of characteristic* $p > 0$. *Let*

$$S = \mathrm{Span}\,\{ab - ba \mid a, b \in A\},$$

$$T = \{r \in A \mid r^q \in S \text{ for some power } q \text{ of } p\}.$$

Then T is a subspace of A, and the number of isomorphism classes of simple A-modules is $\dim_K(A/T)$.

Proof. In the expansion

$$(a + b)^p = \sum_{\epsilon_i \in \{a,b\}} \epsilon_1 \cdots \epsilon_p,$$

all the terms except a^p and b^p can be grouped into sets of p summands of the form

$$\epsilon_1 \cdots \epsilon_p + \epsilon_2 \cdots \epsilon_p \epsilon_1 + \cdots + \epsilon_p \epsilon_1 \cdots \epsilon_{p-1}.$$

All the terms in the above expansion are congruent modulo S, and so their sum vanishes modulo S. Therefore,

$$(a + b)^p \equiv a^p + b^p \mod S. \tag{9}$$

It follows T is closed under addition. It is clear that T is closed under multiplication by scalars in K. Hence T is a subspace of A.

Now take $u, v \in A$, and let $w = v(uv)^{p-1}$. Then

$$(uv - vu)^p \equiv (uv)^p - (vu)^p \equiv uw - wu \equiv 0 \mod S.$$

Therefore, the pth power of an element of S is again in S. Hence $S \subset T$.

Suppose now, that A is *simple*. By Wedderburn's theorem, one may assume that $A = M_n(K)$. Clearly every matrix in S has trace zero. The converse of this statement is also true: every matrix with trace zero lies in S. To see this for $n = 2$, note that

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right],$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right],$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right].$$

In the above equations, $ab - ba$ has been denoted $[a, b]$, which is customary. Similar identities can be used to obtain the result for arbitrary n. Since S consists of trace zero matrices, $\dim(A/S) = 1$, and since $S \subset T$, $\dim(A/T)$ must be 0 or 1. The matrix E_{11} for which the entry in the first row and first column is 1 and all other entries are 0 is never in T. Therefore T is a proper subspace of A. One must therefore have that $\dim(A/T) = 1$. On the other hand $M_n(K)$, being simple, has a unique simple module up to isomorphism. Therefore, Proposition 8 holds when A is simple.

For the general case, note that every nilpotent element of A is in T. Therefore, $\mathrm{Rad}A \subset T$. It follows from Proposition 8 that $\mathrm{Rad}A$ acts trivially on every simple A-module and that the number of isomorphism classes of simple modules is the same for A and $\frac{A}{\mathrm{Rad}A}$. Now $\frac{A}{\mathrm{Rad}A}$ is a direct sum $A_1 \oplus \cdots \oplus A_r$ of simple algebras by Proposition 2. A simple module for A_i becomes a simple module for $A_1 \oplus \cdots \oplus A_r$ when the other summands act trivially. Moreover, every simple module is obtained in this way. Therefore, $A_1 \oplus \cdots \oplus A_r$ (and hence A) has r isomorphism classes of simple modules. On the other hand, define T_i for A_i just as T was defined for A. Then A/T is a direct sum of the A_i/T_i's. Therefore, applying Proposition 8 in the simple case to A_i, we see that $\dim A/T = r$. □

Proposition 10. *Let K be an algebraically closed field of characteristic $p > 0$ and let $A = K[G]$. Then, the number of p-regular conjugacy classes in G is the same as* $\dim A/T$.

Proof. Every $x \in G$ can be written as $x = st$, where s is p-regular and the order of t is a power of p, for if the order of x is $n = n'p^e$, where n' is not divisible by p, then there exist integers a and b such that $ap^e + bn' = 1$, and one may take $s = x^{ap^e}$ and $t = x^{bn'}$. By (9),

$$(st - s)^p \equiv s^p t^p - s^p \mod S.$$

Consequently, if q is the order of t,

$$(st - s)^q \equiv s^q t^q - s^q \equiv 0 \mod S.$$

Therefore, $st - s \in T$, or $st \equiv s \mod T$. Therefore, every element of G (thought of as an element of $K[G]$) is congruent modulo T to a p-regular element. Furthermore, since T contains S, all elements in the same conjugacy class are equivalent modulo T. Therefore, the number of p-regular conjugacy classes in G is an upper bound for $\dim A/T$.

Suppose $R \subset G$ is a set of representatives of p-regular conjugacy classes. It remains to show that R is a linearly independent set in A/T. Suppose that $\sum a_r r \in T$ for some $a_r \in K$, $r \in R$. There exists a power q of p such that $r^q = r$ for every $r \in R$ (because p is a unit in $\mathbf{Z}/n'\mathbf{Z}$, where n' is the order of r), and such that $(\sum a_r r)^q \in S$. Therefore,

$$\left(\sum a_r r\right)^q \equiv \sum a_r^q r^q \equiv \sum a_r^q r \mod S,$$

and consequently, $\sum a_r^q r \in S$. But S consists of those elements of $K[G]$ with the property that the sum of the coefficients of all the elements in each conjugacy class of G is zero (prove this). Therefore, a_r^q, and hence a_r is zero for every $r \in R$. It follows that R is linearly independent in A/T. □

Acknowledgments: The author is grateful to S. Ponnusamy and K. N. Ragahavan for helpful comments on preliminary drafts of this article. He thanks the students of TIFR for inviting him to give the lecture on which this article is based.

References

[Art94] Michael Artin. *Algebra*. Prentice Hall of India Private Limited, New Delhi, 1994.

[Ben91a] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991. Basic representation theory of finite groups and associative algebras.

[Ben91b] D. J. Benson. *Representations and cohomology. II*, volume 31 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1991. Cohomology of groups and modules.

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[Cur99] Charles W. Curtis. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*, volume 15 of *History of Mathematics*. American Mathematical Society, Providence, RI, 1999.

[Fei82] Walter Feit. *The representation theory of finite groups*, volume 25 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1982.

[Lan99] Serge Lang. *Algebra*. Addison-Wesley (International Student Edition), third edition, 1999.

Dichotomy of the Riemann Sphere

Shrihari Sridharan
Department of Mathematics
Indian Institute of Science
Bangalore, India, PIN 560 012
E-mail: shrihari@math.iisc.ernet.in

Abstract. In this expository article, we study the dichotomy of the complex sphere into Fatou and Julia sets based on the family of iterates of an arbitrary nonlinear rational map. We shall then observe some important properties that these two sets satisfy and go on to prove the first milestone of this theory which states that the repelling periodic points are dense in the Julia set. For a better understanding, we conclude with a few examples.

1. Introduction

Complex Dynamics is today very much a focus of interest. Holomorphic, non-invertible dynamical systems of the Riemann sphere are surprisingly intricate and beautiful. A surprising discovery was made by Fatou, in 1906, when he observed that iterations of a very simple function like $f(z) = z^2/(z^2+2)$ lead naturally to the appearance of a Cantor set. This was then considered to be a very exotic subject. Later, Fatou and Julia undertook a thorough study of the dynamics of rational functions during the time of first world war. Both of them independently published a number of *Comptes Rendus* notes, and then wrote long *Memoires*: Julia in 1918 and Fatou in 1919 and 1920. In 1918, Julia was awarded the *Grand Prix des Sciences Mathématiques* by the Paris Academy of sciences for his work.

The theory developed by Fatou and Julia was based on Koebe-Poincaré uniformisation theorem, Montel's normality criterion and some earlier work on functional equations due to Böttcher, Koenigs, Leau, Poincaré and Schröder at the beginning of the 20th century. This field was then dormant for the next few decades. Then, the creation of the theory of hyperbolic dynamical systems in the 1960's and early 1970's in the papers by Anosov, Smale, Sinai and Bowen led to the fact that the intricate dynamics of rational endomorphisms ceased to be considered as something strange related to irreversibility. The papers by Yakobson and Guckenheimer, where the iterates of rational functions are studied by the methods of symbolic dynamics, date back to that time.

The study of the dynamics of rational endomorphisms was also very popular in the 1980's. Great enthusiasm was caused by the numerical experiments carried out by Brooks and Matelski, Hubbard and Mandelbrot, which resulted in the appearance of deep conjectures and beautiful pictures visually demonstrating the fact that the situation is non-trivial. Soon, papers by Douady and Hubbard, Sullivan and Thurston appeared, which related the dynamics of rational functions to the theory of

Kleinian groups and Teichmüller spaces. These relations cast a new light on the whole field and provided a key to many problems.

In the last 15 years, the theory of holomorphic dynamical systems has had a resurgence of activity, particularly concerning the fine analysis of the Julia sets associated with polynomials. It should be mentioned that there has been an explosion of interest in the subject and many mathematicians have made substantial contributions.

In this article, we shall study the dichotomy of the Riemann sphere into sets now bearing the names of the above mentioned eminent French mathematicians, the founders of this theory by considering the sequence of iterates of an arbitrary, non-linear, rational map. It was noted that the function was well-behaved on one set and chaotic on the other.

2. The Riemann Sphere

By a Riemann sphere or complex sphere, we mean the complex plane along with the point at ∞. Observe that this is possible through a well-known process called stereographic projection. We shall denote the Riemann sphere by $\widehat{\mathbb{C}}$. By a region, we shall mean a non-empty, connected, open subset of the complex sphere, $\widehat{\mathbb{C}}$. Let us call the union of disjoint regions by the term plane open set, and denote it by Ω. By a Riemann surface, we shall mean a connected, complex analytic manifold of complex dimension one. Two Riemann surfaces S_1 and S_2 are said to be conformally isomorphic if there exists a homeomorphism from S_1 onto S_2 which is holomorphic, with holomorphic inverse.

Lemma 1. (Schwarz). *Let $\mathbb{D}$ be the open unit disk in the complex sphere. Let $f\colon \mathbb{D} \longrightarrow \mathbb{D}$ be a holomorphic function with $f(0) = 0$. Then $|f'(0)| \leq 1$. If equality holds, then $f(z) = kz$, for some constant k with $|k| = 1$. In particular, it follows that f is a conformal automorphism of $\mathbb{D}$. Otherwise, $|f(z)| < |z|$ for all $z \neq 0$ and f is not a conformal automorphism.*

A proof of the Schwarz's lemma can be found in many standard books on Complex Analysis, say [1], [7]. If a function f defined on a surface Ω is differentiable everywhere in its domain, then f is said to be holomorphic (or analytic) in Ω. Let $H(\Omega)$ denote the class of all holomorphic functions in Ω.

Definition. Suppose $\mathcal{F} \subset H(\Omega)$, for some region Ω. $\mathcal{F}$ is said to be a *normal family* in Ω, if every sequence $\{f_n\}$ of functions; $f_n \in \mathcal{F}$, contains a subsequence $\{f_{n_k}\}$ which converges uniformly on every compact subset of Ω. The limit function is not required to belong to $\mathcal{F}$.

Consider the family of functions, $\{f(z) = z^n\}_{n=1}^{\infty}$ on the open unit disk. Then observe that the limit of this sequence is simply the constant function 0, which is not a member of this family. We shall now proceed to state a result due to Montel which throws some light on normal families.

Theorem 1. (Montel). *Let $\mathcal{F}$ be a family of holomorphic functions from a Riemann surface S to the Riemann sphere $\widehat{\mathbb{C}}$. If there are three distinct points of $\widehat{\mathbb{C}}$ that never occur as values, then this family $\mathcal{F}$ is normal.*

A proof of the Montel's theorem can be found in [4]. It is based on the Schwarz's lemma, the fact that the unit disc covers the 3-punctured sphere, and uses the hyperbolic geometry due to Lobachevsky extensively. Now, let us move on to define rational maps and their degree.

Definition. A *rational map* $f(z)$ is the quotient of two relatively prime polynomials, $P(z)$ and $Q(z)$.

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n} \tag{1}$$

with $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in \mathbb{C}$, $a_0 \neq 0$ and $b_0 \neq 0$. And its *degree* (d) is defined to be the maximum among the degrees of P and Q, $d = \max(\deg P, \deg Q) = \max(m, n)$.

We shall assume henceforth that the degree of the rational function f is strictly greater than 1.

Definition. The *Fatou set* $\mathbb{F} = \mathbb{F}(f)$ of a rational function f is the maximal subset of $\widehat{\mathbb{C}}$, the domain of f, on which the family of iterates of f i.e., $\mathcal{F} = \{f^n\colon n = 1, 2, \ldots\}$ is normal. The complement of the Fatou set is known as the *Julia set* $\mathbb{J} = \mathbb{J}(f)$.

3. Properties of the Fatou and Julia sets

Having dichotomised the Riemann sphere $\widehat{\mathbb{C}}$ into the Fatou set $\mathbb{F}(f)$ and the Julia set $\mathbb{J}(f)$, we shall in this section look at some simple properties these two sets satisfy.

Property 1. *The Fatou set is open.*

By the definition of the Fatou set, if $z \in \mathbb{F}$, we can find an open neighbourhood U of z such that $f^n|_U$ is normal. Thus for any $\zeta \in U$, we can use the same definition to show immediately that $\zeta \in \mathbb{F}$.

Property 2. *If $z \in \mathbb{F}(f)$, then the family $\{f^m\}_{m=0}^{\infty}$ is equicontinuous in a neighbourhood of z.*

Consider a compact set $K \subset \mathbb{F}$ containing z. Then, by definition we know that the family $\{f^m\}$ is normal in K. Now, for any open δ-neighbourhood of z in K containing ζ, we have $|f^m(z) - f^m(\zeta)| < \epsilon$ for $m = 1, 2, \ldots$.

The orbit of a point z with regard to a function f is said to be *Lyapunov stable* if for every $\epsilon > 0$, there exists a $\delta > 0$, such that $|f^m(z) - f^m(\zeta)| < \epsilon$, whenever $|z - \zeta| < \delta$ for $m = 0, 1, 2, \ldots$.

Property 3. *The orbit of any point in the Fatou set is Lyapunov stable.*

This is fairly obvious from property (2). The next one describes the property of complete invariance satisfied by the Fatou set. We first recall that a set X is said to be *invariant* if $f(X) \subset X$. It is said to be *completely invariant* if, in addition, $f^{-1}(X) = X$.

Property 4. *The Fatou set is completely invariant.*

Consider $z_0 \in f^{-1}(\mathbb{F})$. And let $f(z_0) = w_0$. Hence, $w_0 \in \mathbb{F}$. Now, the orbit of w_0 is Lyapunov stable. So, for every $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|z - z_0| < \delta_1 \implies |f(z) - f(z_0)| < \delta_2 \implies |f^{n+1}(z) - f^{n+1}(z_0)| < \epsilon$. Hence, $\{f^{n+1}\colon n \geq 1\}$ is equicontinuous at z_0 and so on $f^{-1}(\mathbb{F})$. Since, $f^{-1}(\mathbb{F})$ is open, $f^{-1}(\mathbb{F}) \subseteq \mathbb{F}$. To get the other inclusion, let $z_0 \in \mathbb{F}$ and $f(z_0) = w_0$. Then for an open δ-neighbourhood, say N, of z_0, $f(N)$ is an open neighbourhood of w_0. And for any $w \in f(N)$, there corresponds a $z \in N$. So, $|f^n(w) - f^n(w_0)| = |f^{n+1}(z) - f^{n+1}(z_0)| < \epsilon$. Hence, $w_0 \in \mathbb{F}$. So, $\mathbb{F} \subseteq f^{-1}(\mathbb{F})$.

The following properties describe the Julia set. These are, in part, a consequence of the above properties of the Fatou set.

Property 5. *The Julia set is non empty.*

If the Julia set is empty, $\mathbb{F}(f) = \widehat{\mathbb{C}}$, meaning the family $\{f^m\}_{m=0}^{\infty}$ is normal in $\widehat{\mathbb{C}}$. In such a case, we can find a sequence $\{m_k\}$ of numbers with $m_k \to \infty$ and a rational function g to claim that $f^{m_k} \to g$ uniformly on $\widehat{\mathbb{C}}$. But this is impossible since deg $f^{m_k} \to \infty$. Hence, $\mathbb{J}(f) \neq \phi$.

Property 6. *The Julia set is closed and completely invariant. And the orbit of any point in the Julia set is not Lyapunov stable.*

The first two properties are simple observations from properties (1) and (4) while the orbit of any point in the Julia set is not Lyapunov stable because the family $\{f^m\}$ is not normal in the Julia set.

Property 7. *The Julia set of f coincides with that of f^n for any $n \geq 1$.*

The Fatou sets of f and f^n are the same since the family $\{f^m\}$ is normal on an open set U if and only if the family $\{f^{mn}\}$ is normal on U. Hence the Fatou set of f and f^n are the same and so are their Julia sets.

Let $z \in \mathbb{J}$ and let U be any neighbourhood of z. Then by Montel's theorem, the family $\{f^n\}$ on U can afford to omit a set E_z containing atmost two points only. Such points are called *exceptional points*.

Consider a polynomial $P(z)$. Then the point at ∞ is very special because it is a fixed point whose only inverse image is itself. Very few fixed points have this property. In fact, if a rational map f fixes a point ζ and $f^{-1}(\zeta) = \{\zeta\}$, then f can be conjugated by a Möbius map that would send ∞ to ζ.

Property 8. *The set E_z is independent of z (so shall be denoted by E). If E is a singleton, then $f(z)$ can be conjugated to a polynomial. If E consists of two points, they could be conjugated to 0 and ∞, and $f(z)$ can be conjugated to the form cz^d or cz^{-d}. In all cases, E is contained in the Fatou set.*

By definition, $f^{-1}(E_z) \subset E_z$. If E_z contains one point ζ only, $f(\zeta) = \zeta$. Conjugating by a Möbius map, let $\zeta = \infty$. Since $\bar{f}^{-1}(\infty) = \infty$, there are no other poles, and hence, $\bar{f}$ is a polynomial. Clearly, E_z is independent of z. If E consists of two points, assume 0 and ∞ then either $\bar{f}(0) = 0$ and $\bar{f}(\infty) = \infty$, or $\bar{f}(0) = \infty$ and $\bar{f}(\infty) = 0$. In the first case, $\bar{f}$ is a polynomial with 0 as its only zero and so, $\bar{f}(z) = cz^d$. Similarly, in the second case, $\bar{f}(z) = cz^{-d}$.

For any function f,the zeroes of its derivative f' and its own multiple poles are called the *critical points*.

Property 9. *If V is a non-empty set completely invariant under the rational map f, then V contains one, two or infinitely many points.*

Assume V contains finitely many points, i.e., $V = \{z_1, z_2, \ldots, z_k\}$ such that $k > 2$. Since V is completely invariant under the rational map f, there exists some number, say m such that f^m fixes every point in V. In other words, each point in V has only itself as its preimage under f^m. Therefore, each of these is a critical point and is of multiplicity $d^m - 1$. But, the total number of critical points counted with multiplicities is $2d^m - 2$ for f^m, implying $k \leq 2$ a contradiction to our assumption.

Property 10. *The Julia set is infinite.*

We know that $\mathbb{J}$ is a non-empty set completely invariant under f. Now applying property (9), we conclude that $\mathbb{J}$ must be infinite. We rule out the case of $\mathbb{J}$ having 1 or 2 points because if a set contains at most two points, then it is indeed the exceptional set, $E \subset \mathbb{F}$.

Property 11. *The backward iterates of any $z \in \mathbb{J}$ are dense in $\mathbb{J}$.*

From the definition of E, it is clear that if $z \notin E$, $\mathbb{J}$ is in the closure of the inverse orbit $\bigcup_{n\geq 1} f^{-n}(z)$. Moreover, E is disjoint from $\mathbb{J}$.

Definition. A point $z \in \widehat{\mathbb{C}}$ is called *periodic* if $f^p(z) = z$, for some p. The least such number p is called the *period* of the periodic point z. The *multiplier* λ of a point $z_0 \in \widehat{\mathbb{C}}$ is the derivative of f calculated at this point, $\lambda(z_0) = f'(z_0)$.

The point z_0 is classified as super-attracting, attracting, neutral or repelling with regard to the value of its multiplier, $\lambda(z_0)$. If $\lambda(z_0) = 0$ then we call z_0, a super-attracting point, if $0 < |\lambda(z_0)| < 1$, then z_0 is known as an attracting point, if $|\lambda(z_0)| = 1$, then we name z_0 a neutral point and if $|\lambda(z_0)| > 1$, z_0 is classified as a repelling point.

Property 12. *For every $z_1 \in \mathbb{J}$, there exists a point $z_2 \in \mathbb{J}$ such that $z_1 \in \mathcal{O}^+(z_2)$ but $z_2 \notin \mathcal{O}^+(z_1)$. Here, by $\mathcal{O}^+$ we describe the forward orbit of the point, under the rational map f.*

Let z_1 be a non-periodic point. Then z_2 can be any inverse image of z_1. Now let z_1 be a periodic point of period p; $f^p(z_1) = z_1$. Consider $h = f^p$ and the equation $h(z) = z_1$. If z_1 is the only solution to this equation, we can conjugate h to a polynomial. And since $\infty \in \mathbb{F}$ of the polynomial, we have $z_1 \in \mathbb{F}(h)$, a contradiction to our assumption. Consequently, there exists another solution z_2 to the above equation and $z_2 \notin \mathcal{O}^+(z_1)$ because z_1 is the only solution to that equation in $\mathcal{O}^+(z_1)$.

Property 13. *The Julia set is perfect.*

Consider any point $z \in \mathbb{J}$. Then choose a point w as in property (12). Let D be a neighbourhood of z. Since $w \in \mathbb{J}$, it is clear that $w \notin E$. Hence, there exists an integer n such that $w \in f^n(D)$. Let ζ denote a point in D such that $f^n(\zeta) = w$. Then, $\zeta \neq z$ because $w \notin \mathcal{O}^+(z)$ and $\zeta \in \mathbb{J}$ because $\mathbb{J}$ is f-invariant.

Property 14. *If the Julia set $\mathbb{J}$ has a non empty interior, then it coincides with the extended complex plane.*

Suppose there is an open $U \subset \mathbb{J}$. Then, $f^n(U) \subseteq \mathbb{J}$ by property (6). But, $\bigcup f^n(U) = \widehat{\mathbb{C}} \setminus E$ (by the definition of E) is dense in $\widehat{\mathbb{C}}$ and since $\mathbb{J}$ is closed, $\mathbb{J} = \widehat{\mathbb{C}}$.

Property 15. *Every attracting periodic orbit is contained in the Fatou set. In fact the entire basin of attraction Ψ for an attracting periodic orbit is contained in the Fatou set. However, the boundary $\partial\Psi$ is contained in the Julia set, and every repelling periodic orbit is contained in the Julia set.*

In view of the property (7), we need to consider the case of a fixed point $f(z_0) = z_0$ only. If z_0 is attracting, then it follows from the Taylor's theorem that $\{f^m\}_{m=1}^{\infty}$ restricted to a small neighbourhood of z_0 converge uniformly to the constant function $g(z) = z_0$. The corresponding statement for any compact subset of the basin Ψ then follows. But, around a point on $\partial\Psi$ no sequence from $\{f^m\}_{m=1}^{\infty}$ can converge to a continuous limit. However, if z_0 is repelling, no sequence of elements in $\{f^m\}_{m=1}^{\infty}$ can converge uniformly near z_0, since the derivative $\frac{d}{dz} f^n(z)$ at z_0 takes the value λ^n which diverges to ∞ as $n \to \infty$.

A periodic point $f^p(z_0) = z_0$ is called *parabolic* if the absolute value of the multiplier $|\lambda|$ at z_0 is equal to 1, yet f^p is not the identity map. Consider the rational map $f\colon \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ defined by $f(z) = z/z - 1$. Note that the two fixed points of this function are 0 and 2 and that the absolute value of the multiplier calculated at both these points is 1. However, these points do not count as parabolic points since $f \circ f(z)$ is identically equal to z. This exclusion is necessary so that the following assertion will be true.

Property 16. *Every parabolic periodic point belongs to the Julia set.*

Let w be a local uniformising parameter, with $w = 0$ corresponding to the periodic point. Then some iterate f^m corresponds to a local mapping of the w-plane with power series expansion of the form $w \longmapsto w + a_k w^k + a_{k+1} w^{k+1} + \cdots$ where $k \geq 2,\ a_k \neq 0$. It follows that f^{mp} corresponds to a power series $w \longmapsto w + pa_k w^k + \cdots$. Thus the k-th derivatives of f^{mp} diverge as $p \to \infty$. Hence, no subsequence can converge locally uniformly, by Weierstrass theorem, as in [1] which states "*If a sequence of holomorphic functions converges uniformly, then their derivatives also converge uniformly, and the limit function is itself holomorphic.*"

4. An alternate definition of the Julia set

We begin this section by defining what is meant by a meromorphic function. A rational function $f(z) = P(z)/Q(z)$ defined on Ω is said to be a *meromorphic function* if both the functions $P(z)$ and $Q(z)$ are holomorphic in Ω. Though this is not the standard definition of a meromorphic function, we urge the reader to observe that this is no different from the usual one. Interested readers are referred to [7].

Lemma 2. *If a family $\{f_m\}$ of meromorphic functions is normal in a neighbourhood of $\zeta \in \widehat{\mathbb{C}}$ and if there exists a positive constant c_1 such that $|f_m(\zeta)| \leq c_1$, then there exists a positive constant c_2 such that $|f'_m(\zeta)| \leq c_2$.*

Lemma 3. *Consider a family $\{f_m\}$ of meromorphic functions in the open unit disc, $\mathbb{D}$. Suppose there are three different meromorphic functions g_n $(n = 1, 2, 3)$ in D with the property that the equations $f_m(z) = g_n(z),\ g_r(z) = g_n(z)$ where $n \neq r, n = 1, 2, 3$ and $r = 1, 2, 3$ have no roots in $\mathbb{D}$. Then the family $\{f_m\}$ is normal.*

This lemma reduces to the Montel's theorem for the family

$$\left\{ \frac{(f_m - g_2)(g_1 - g_3)}{(f_m - g_3)(g_1 - g_2)} \right\}$$

which has three exceptional values namely 0, 1 and ∞.

Theorem 2. *The Julia set $\mathbb{J}(f)$ is the closure of the set of all repelling periodic points of the rational map f.*

Proof: Let z be a repelling periodic point of order p, i.e., $f^p(z) = z$. Without loss of generality, let us assume $z \neq \infty$. Let λ be its multiplier. Then, $(f^{pm})'(z) = \lambda^m \to \infty$ as $m \to \infty$ and $|f^{pm}(z)| = |z| < \infty$. Hence, by lemma (2), $\{f^{pm}\}_{m=1}^{\infty}$ is not normal in a neighbourhood of z. Hence, all repelling cycles lie on $\mathbb{J}$.

Now, let $\zeta \in \mathbb{J}(f)$. Since $\mathbb{J}$ is perfect, we may assume that ζ is not periodic and ζ is not a critical value of f. So, ζ has d different inverse images under f. Let them be ζ_i ; $1 \leq i \leq d$. Here, $\zeta_i \neq \zeta$ for any i. Let D_1 be a neighbourhood of ζ. Then in D_1 there exist d univalent branches of f^{-1}, namely g_i ; $1 \leq i \leq d$ such that $g_i(\zeta_i) = \zeta$. Moreover, $g_i(D_1) \cap g_j(D_1) = \phi$, for $i \neq j$. Define a function $g_0(z) = z$ on D_1. Note that D_1 can be chosen sufficiently small such that $g_i(D_1) \cap g_0(D_1) = \phi\ \forall\ i \neq 0$. Hence, the equation $g_i(z) = g_j(z)$ has no roots in D_1. Now consider another neighbourhood D_2 of ζ such that $D_2 \subset D_1$. Then by lemma (3) there should be a root z_0 of some equation $f^p(z) = g_i(z)$. The point z_0 is periodic (with period p for $j = 0$ and $p+1$ for $j = 1, 2$). And since there are only finitely many neutral and attracting periodic points, z_0 is repelling provided the neighbourhood D_2 is chosen sufficiently small. Hence, the theorem follows. •

5. Examples

We conclude our article by studying some examples.

Example 1. *Consider the polynomial map $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = z^d$ for $d > 1$.*

Consider the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon\ |z| < 1\}$. Then it is clear that $z = 0$ is the only super-attracting point. The points in $\mathbb{D}$ are the attracting points. However, the points on the boundary of $\mathbb{D}$ are neutral while the rest are repelling points. Hence, the Julia set for this function is

$$\mathbb{J} = \{z \in \mathbb{C}\colon\ |z| = 1\},$$

while the rest of the points in the complex plane lie in $\mathbb{F}$.

Example 2. *Consider the quadratic map $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = z^2 - 2$.*

Consider on the real axis, the interval $I = [-2, 2]$. Note that f maps I onto itself. For any point $z_0 \in I$, both the solutions of the equation $f(z) = z_0$ lies in I. And since I contains a repelling fixed point $z = 2$, property (11) tells us that I contains the Julia set. On the other hand, the basin of attraction $\Psi(\infty)$ is a neighbourhood of ∞, whose boundary is contained in $\mathbb{J}(f) \subset I$, by property (15). Hence, every point outside $\mathbb{J}$ belongs to this basin. Since $f(I) \subset I$, it is clear that the Julia set for this function is the above considered interval $\mathbb{J} = I$ and $\mathbb{F} = \mathbb{C} \setminus I$.

Though the Julia set in both the above examples have a smooth boundary, in general, it is not the case. The next example is typical. A totally disconnected, perfect, compact metric space is called a Cantor set.

Example 3. *Consider the rational function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $f(z) = 2z - \frac{1}{z}$.*

Note that the upper half plane, the lower half plane and the exterior of the unit disc ($\mathbb{C} \setminus \mathbb{D}$) are invariant under f. Hence, $\mathbb{J}(f) \subset [-1, 1]$. Now consider the interval $I_0 = (-\frac{1}{2}, \frac{1}{2})$. Then $f(I_0) = \{x \in \mathbb{R} : |x| > 1\}$. Hence, $I_0 \subset \mathbb{F}$. Let $K_1 = [-1, -\frac{1}{2}]$ and $K_2 = [\frac{1}{2}, 1]$. Then, $\mathbb{J} \subset K_1 \cup K_2$. And since f maps K_i monotonically onto I, there should exist an interval $I_i \subset K_i$ such that $f(I_i) = I_0$. Hence, if we cut off I_i from K_i we obtain 4 intervals $K_{i,j}$ such that $\mathbb{J}(f) = \bigcup_{i,j=1,2} K_{i,j}$. And f maps $K_{i,j}$ monotonically onto K_i. If we continue this construction, we obtain a family of 2^n intervals, namely $K_{i,j}$ with $i = 1, 2, \ldots, n-1$ and $j = 1, 2$. And $\mathbb{J} \subset \bigcup_{i=1}^{n-1} \bigcup_{j=1}^{2} K_{i,j} \equiv K_n$. Moreover, f maps $K_{i,j}$ monotonically onto K_i. Since $|f'(z)| \geq 3$ on $[-1, 1]$, the lengths of these intervals $K_{i,j}$ do not exceed $\frac{2}{3^n}$. So, $K_\infty = \bigcap K_n$ is a Cantor set. The Julia set $\mathbb{J}(f)$ is contained in K_∞. Conversely, if $x \in K_\infty$, then, $|f^n x| \leq 1$ and $|(f^n)' x| \geq 3^n$. Hence, by lemma (2), $x \in \mathbb{J}(f)$. Thus, in this example, $\mathbb{J} = K_\infty$ is a Cantor set.

Property (5) tells us that the Julia set is non-empty. However, there are maps with empty Fatou set. One familiar example of such kind is the Lattes' example.

Example 4. *Consider the rational map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ defined by*

$$f(z) = \frac{(z^2+1)^2}{4z(z^2-1)}.$$

This rational function has for its Julia set all of $\widehat{\mathbb{C}}$. This example was constructed by S.Lattes, shortly before his death in 1918. The estimation of Fatou set and Julia set for this map is done with the help of Riemann–Hurwitz formula and the Weierstrass $\wp$ function. We do not go into the details, as it may get too technical. Interested readers are referred to [2] and [3] for more details.

References

[1] L. Ahlfors, *Complex Analysis*, McGraw Hill Book Co, New York, 1978.

[2] A. F. Beardon, *Iteration of Rational Functions*, Graduate Texts in Mathematics, 132, Springer-Verlag, New York, 1991.

[3] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York, 1993.

[4] G. M. Golusin, “Geometric Theory of functions of a complex variable”, *Transl. Amer. Math. Soc.*, 1969.

[5] Lyubich and M. Yu., “The dynamics of rational transforms: the topological picture”, *Russian Math. Surveys*, 1986, 43-117.

[6] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*, Friedr. Vieweg and Sohn, Braunschweig, 1999.

[7] W. Rudin, *Real and Complex Analysis, Third Edition*, McGraw Hill, 1987.

On Deviations from Expected Hardy–Weinberg Proportions

Yaping Liu and Josh Collins
Department of Mathematics
Pittsburg State University
Pittsburg, Kansas 66762, U.S.A.
Phone: 620-235-0353
E-mail: yliu@pittstate.edu

Abstract. In population genetics, the simplest model involves two alleles at a single locus. Under ideal conditions, the proportions of the genotypes in the population are determined by the Hardy–Weinberg law. But in nature, it is frequently observed that values of the genotypes differ from those predicted by the Hardy–Weinberg law. An unusual case is when the number of heterozygotes is far below its expected value, although they are believed to be more viable than the homozygotes. In this paper we will examine a few theoretical models for this case and try to supply a satisfactory explanation for the excess of homozygotes.

Key words and phrases: population genetics, Hardy–Weinberg law, positive equilibrium, stability, selection, non-random mating, multiple population model.

1. Introduction

Natural selection is a fundamental tenet of the biological sciences. The underlying mechanism of natural selection is genetic heredity, and it is this concept that serves as the foundation for population genetics. At the heart of population genetics lies a mathematical model developed independently by the British mathematician Godfrey H. Hardy and the German physician Wilhelm Weinberg ([3]). This model is known today as the Hardy–Weinberg law.

Under a set of assumptions, the Hardy–Weinberg law predicts that genotype and allele frequencies will remain constant from generation to generation. The following conditions are assumed ([3], [6]):

1. The population is large enough that sampling errors and random effects are negligible.
2. Mating within a population occurs at random.
3. All genotypes produced by random mating are equally viable and fertile.
4. There is an absence of mutation, migration, and random genetic drift.
5. There is equal allele frequency among females and males.

The basic model for the Hardy–Weinberg law assumes a single gene with only two alleles, a dominant allele A and a recessive allele a. Assuming allele frequencies of p and q, respectively, the probability of an individual having a particular genotype will be given by p^2, $2pq$, and q^2 for AA, Aa, and aa respectively. Clearly, $p^2+2pq+q^2 = (p+q)^2 = 1$. In general the Hardy–Weinberg law extends to any number of alleles. For an elementary mathematical treatment of the Hardy–Weinberg law, see [7].

In nature, the frequencies predicted by the Hardy–Weinberg law are often violated. Consider an example cited by John Maynard Smith, which, for convenience, is going to be referred to as Example E.

Example E. Abdomen color in *Drosophila polymorpha* is determined by two alleles at a single locus. A dark colored abdomen is represented by AA, an intermediate color by Aa, and a pale color by aa. Data on 8070 flies sampled in Brazil was originally collected by Da Cunha in 1949 ([2]) and is summarized as follows ([6]):

	Genotype			
	dark (AA)	intermediate (Aa)	pale (aa)	Total
Observed numbers of flies	3969	3174	927	8070
Expected from H-W ratio	3825	3462	783	

One can easily calculate the allele frequencies using the observed genotypes. Two A alleles were observed in 3,969 flies while one A allele was observed in 3,174 flies. This gives a total of 11,112 A alleles within the sample population. There were 8070 flies, each with two alleles, giving a total of 16,140 alleles. Therefore, the frequency for the A allele is $p = \frac{11112}{16140} \approx 0.6885$, and the frequency for the a allele is $q = 1 - p \approx 0.3115$. The Hardy–Weinberg proportions are $p^2 = 0.4740$, $2pq = 0.4289$, and $q^2 = 0.0970$. Thus, the expected numbers of sampled flies having the AA, Aa, and aa genotypes are respectively $(0.4740)(8070) = 3825$, $(0.4289)(8070) = 3462$, and $(0.0970)(8070) = 783$. Notice that the observed number of heterozygous flies is significantly lower than its expected value while the observed numbers of homozygous flies are significantly higher than their expected values.

Martinez and Cordeiro ([5]) obtained experimental results on *Drosophila polymorpha* that suggest the existence of color modifying alleles that segregate independently from the major locus. The modifying alleles do not determine abdomen color by themselves, but modify the expression of gene products thereby altering the expected phenotypes from those predicted by the Hardy–Weinberg law. The so called modifier alleles, if they do exist, may account for some of the discrepancies in Example E. However, in Da Cunha's same study ([2]), it was reported that dark × dark resulted in only dark phenotypes, light × light resulted in only light phenotypes, and dark × light resulted in only intermediate phenotypes. These results are consistent with the view that abdomen color is controlled by one pair of alleles. Further, crosses of dark × light, intermediate × light, and intermediate × dark resulted in an *excess* of intermediate phenotypes, sharply contrasting the results observed in natural populations.

Regardless of the experimental results obtained thus far, to our knowledge, an adequate explanation has not been provided for the unusual excess of homozygotes from that predicted by the Hardy–Weinberg law. In this paper we will consider three possible reasons that can explain the type of deviations from the Hardy–Weinberg law as observed in Example E. They are selection, non-random mating, and multiple population composition. Our focus is to provide a theoretical explanation for such deviations.

2. Evolution Operators

Consider the evolution of the population from one generation to the next. Let x, y, z be the proportions of the three genotypes in the parent population. We can define an evolution operator based on the proportions of the alleles in the gene pool,

$$\text{i.e., } \mathcal{E}_1 : (p_n, q_n) \longrightarrow (p_{n+1}, q_{n+1}).$$

Here $p_n = x + \frac{1}{2}y$ and $q_n = \frac{1}{2}y + z$ are the proportions of the alleles A and a respectively in the parent population. Without selection (but with random mating), the proportions in the next generation are

$$p_{n+1} = p_n^2 + p_n q_n = p_n \text{ and } q_{n+1} = p_n q_n + q_n^2 = q_n$$

respectively. It follows that the evolution operator $\mathcal{E}_1$ is an identity. This is the so called Gene Conservation Law, in which the genes neither arise nor disappear.

To keep track of the proportions of the genotypes, we employ the following notations, termed as the Mendel Diallelic Zygote Algebra (see [4] and [7]).

First note that during meiosis,

$$AA \leftrightarrow A, \; aa \leftrightarrow a, \; Aa \leftrightarrow \frac{1}{2}A + \frac{1}{2}a.$$

Then during fertilization,

$$\begin{aligned}
AA \times AA &= A \cdot A = AA, \\
AA \times aa &= A \cdot a = Aa, \\
aa \times aa &= a \cdot a = aa, \\
AA \times Aa &= A \cdot \left(\frac{1}{2}A + \frac{1}{2}a\right) = \frac{1}{2}AA + \frac{1}{2}Aa, \\
Aa \times aa &= \left(\frac{1}{2}A + \frac{1}{2}a\right) \cdot a = \frac{1}{2}Aa + \frac{1}{2}aa, \\
Aa \times Aa &= \left(\frac{1}{2}A + \frac{1}{2}a\right) \cdot \left(\frac{1}{2}A + \frac{1}{2}a\right) = \frac{1}{4}AA + \frac{1}{2}Aa + \frac{1}{4}aa.
\end{aligned}$$

It follows that given proportions x, y, z of the three genotypes in the parent population, with random mating, the following algebra governs the proportions of the genotypes in the next generation:

$$\begin{aligned}
&(xAA + yAa + zaa)^2 \\
&= \left[xA + y\left(\frac{1}{2}A + \frac{1}{2}a\right) + za\right]^2 \\
&= \left[\left(x + \frac{1}{2}y\right)A + \left(\frac{1}{2}y + z\right)a\right]^2 \\
&= \left(x + \frac{1}{2}y\right)^2 AA + 2\left(x + \frac{1}{2}y\right)\left(\frac{1}{2}y + z\right)Aa + \left(\frac{1}{2}y + z\right)^2 aa.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&(xAA + yAa + zaa)^2 \\
&= x^2 AA \times AA + 2xyAA \times Aa + y^2 Aa \times Aa + 2xzAA \times aa + 2yzAa \times aa + z^2 aa \times aa \\
&= \left(x^2 + xy + \frac{1}{4}y^2\right)AA + \left(xy + \frac{1}{2}y^2 + 2xz + yz\right)Aa + \left(\frac{1}{4}y^2 + yz + z^2\right)aa.
\end{aligned}$$

Let Δ be the subset of $\boldsymbol{R}^3$ defined by

$$\Delta = \{(x, y, z) | x, y, z \geq 0, x + y + z = 1\}$$

and let x_n, y_n, z_n be the proportions of AA, Aa, and aa in the nth generation respectively. Then the evolution operator $\mathcal{E}_2 : \Delta \longrightarrow \Delta$ is defined by

$$\mathcal{E}_2(x_n, y_n, z_n) = (x_{n+1}, y_{n+1}, z_{n+1}).$$

The proportions $x_{n+1}, y_{n+1}, z_{n+1}$ in the next generation are given by the following evolution equations

$$\begin{cases}
x_{n+1} = x_n^2 + x_n y_n + \frac{1}{4}y_n^2 \\
y_{n+1} = x_n y_n + \frac{1}{2}y_n^2 + 2x_n z_n + y_n z_n \\
z_{n+1} = \frac{1}{4}y_n^2 + y_n z_n + z_n^2
\end{cases} . \tag{4}$$

Note that with random mating, the population is stationary in the sense that $\mathcal{E}_2$ is an idempotent: $\mathcal{E}_2^2 = \mathcal{E}_2$, and the system reaches Hardy–Weinberg equilibrium in one step.

From the above we see that with random mating, but without selection, the evolution operators are trivial. In the following two sections, we will study how the population evolves when there is selection or non-random mating.

3. Selection

When selection is present in the population, only a fraction of the offspring survive to the next generation. These fractions are represented by the so called fitness parameters. Let $\lambda_1, \lambda_2, \lambda_3$ be the fitness parameters for the three genotypes AA, Aa, and aa respectively, with $\lambda_i = 1$ being neutral. Let p_n and q_n be the proportions of alleles A and a respectively in the gene pool at generation n. Then the evolution operator $\mathcal{E}_1$ can be represented by the following evolution equation

$$p_{n+1} = \frac{\lambda_1 p_n^2 + \lambda_2 p_n q_n}{T_n}$$

where $T_n = \lambda_1 p_n^2 + 2\lambda_2 p_n q_n + \lambda_3 q_n^2$. Using the relation $p_n + q_n = 1$ for all n, this equation can be written as $p_{n+1} = f(p_n)$ with

$$f(p) = \frac{\lambda_1 p^2 + \lambda_2 p(1-p)}{\lambda_1 p^2 + 2\lambda_2 p(1-p) + \lambda_3 (1-p)^2}, \quad \text{for } p \in [0, 1].$$

The stability results on the equilibria of the evolution equation are well known, see for example, [1] and [4]. We will briefly summarize the results here and examine their implications. There are two boundary equilibria, $p_1^* = 0$ and $p_2^* = 1$, and one possible interior equilibrium

$$p_3^* = \frac{\lambda_3 - \lambda_2}{\lambda_1 - 2\lambda_2 + \lambda_3}.$$

The stability of an equilibrium p^* can be determined by $f'(p^*)$. More exactly, if $|f'(p^*)| < 1$ then p^* is stable, while if $|f'(p^*)| > 1$ then p^* is unstable. It turns out that as far as the long term proportion of population is concerned, it is the relative size, not the absolute size, of the λ_i's that are important. Specifically, it has been shown that if $\lambda_1 > \lambda_2 > \lambda_3$, then p_1^* is unstable, p_2^* is stable, and p_3^* does not exist because the value is not in [0, 1]. Thus AA is the fittest and will survive, but Aa and aa will become extinct in the long run. Conversely, if $\lambda_1 < \lambda_2 < \lambda_3$, then p_1^* is stable, p_2^* is unstable, and again p_3^* does not exist. Thus aa is the fittest and will survive, but Aa and AA will become extinct in the long run. In the case of the so called *superrecessivity*, that is, $\lambda_2 < \lambda_1$ and $\lambda_2 < \lambda_3$, p_1^* and p_2^* are stable, while p_3^* exists but is unstable. Thus there is bistability. Obviously none of the three cases above can explain the deviations in Example E. Finally, in the case of the so called *superdominance*, i.e., $\lambda_2 > \lambda_1$ and $\lambda_2 > \lambda_3$, p_1^* and p_2^* are unstable, while p_3^* is stable. Thus when the heterozygotes are more viable, we reach the polymorphism case where all alleles survive in the long run. Can this case explain the deviations in Example E? Our hope is quickly dashed by noting that it corresponds to the situation where there is a higher proportion of Ea than in the Hardy–Weinberg equilibrium, which is the opposite of Example E.

Conclusion: Selection alone does not explain the type of deviations in Example E.

4. Non-random Mating

It is known that random mating is a sufficient but not a necessary condition for the Hardy–Weinberg law. A pseudo-random mating population is one in which nonrandom mating results in Hardy–Weinberg proportions ([10]). In a single population, when the Hardy–Weinberg law is violated, under the condition that no other factor is at work, such as selection, mutation, and migration, there must be nonrandom mating. The question is, can the type of deviation in Example E be explained by nonrandom mating?

The laboratory work of Da Cunha ([2]) and Martinez and Cordeiro ([5]) do not provide a clear indication on the mating preference of the flies. If non-random mating is present in the natural environment, it is not unreasonable to assume that the abdomen color plays an important role, as is the case for certain other species of fruit flies. For simplicity, consider a model in which the mating rates between phenotypes is largely determined by the presence of the dominant allele. Then there will be three different mating rates:

- the same rate α between AA and AA, AA and Aa, or Aa and Aa.
- the same rate β between AA and aa, or Aa and aa.
- the same rate γ between aa and aa.

Then the evolution equations are

$$\begin{cases} x_{n+1} = \dfrac{\alpha}{T_n}\left[x_n^2 + x_n y_n + \dfrac{1}{4}y_n^2\right], \\ y_{n+1} = \dfrac{1}{T_n}\left[\alpha\left(x_n y_n + \dfrac{1}{2}y_n^2\right) + \beta(2x_n z_n + y_n z_n)\right], \\ z_{n+1} = \dfrac{1}{T_n}\left[\dfrac{1}{4}\alpha y_n^2 + \beta y_n z_n + \gamma z_n^2\right]. \end{cases} \tag{5}$$

Here $T_n = \alpha(x_n^2 + 2x_n y_n + y_n^2) + \beta(2x_n z_n + 2y_n z_n) + \gamma z_n^2$. With random mating, we have $\alpha = \beta = \gamma$, and the evolution equations in (5) will be the same as those in (4).

By setting $(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n)$, we can find two boundary equilibria $(0, 0, 1)$, $(1, 0, 0)$, and possibly one interior equilibrium (x^*, y^*, z^*). If $\alpha, \gamma > \beta$, then

$$x^* = \frac{1}{\alpha(\alpha - 2\beta + \gamma)}\left[\alpha\beta + \alpha\gamma - 2\beta^2 - 2\sqrt{\beta(\alpha - \beta)(\alpha\gamma - \beta^2)}\right],$$

$$y^* = \frac{2}{\alpha(\alpha - 2\beta + \gamma)}\left[-\beta(\alpha - \beta) + \sqrt{\beta(\alpha - \beta)(\alpha\gamma - \beta^2)}\right],$$

$$z^* = \frac{\alpha - \beta}{\alpha - 2\beta + \gamma}.$$

If $\alpha, \gamma < \beta$, then

$$x^* = \frac{1}{\alpha(\alpha - 2\beta + \gamma)}\left[\alpha\beta + \alpha\gamma - 2\beta^2 + 2\sqrt{\beta(\alpha - \beta)(\alpha\gamma - \beta^2)}\right],$$

$$y^* = \frac{2}{\alpha(\alpha - 2\beta + \gamma)}\left[-\beta(\alpha - \beta) - \sqrt{\beta(\alpha - \beta)(\alpha\gamma - \beta^2)}\right],$$

$$z^* = \frac{\alpha - \beta}{\alpha - 2\beta + \gamma}.$$

In all other cases, there is no interior equilibrium. It's easy to see that $\alpha(y^*)^2 = 4\beta x^* z^*$. This fact will be used below.

We can measure the degree of departure of the proportions in (x^*, y^*, z^*) from the Hardy–Weinberg Law. One such measure was introduced by Smith as

$$\eta = x^* z^* - \frac{1}{4}(y^*)^2$$

(see [8] and [9]). The sign of η indicates the type of mating. If $\eta > 0$, it is assortative mating; if $\eta < 0$, it is dissortative mating; while if $\eta = 0$, it can be shown that the population must be in Hardy–Weinberg Equilibrium. Check directly that for the equilibrium (x^*, y^*, z^*) above, the Smith measure is

$$\eta = \frac{(\alpha - \beta)^2}{\alpha(\alpha - 2\beta + \gamma)}x^*.$$

Clearly, $\eta > 0$ if $\alpha, \gamma > \beta$, and $\eta < 0$ if $\alpha, \gamma < \beta$.

To study the stability of the equilibria, we can use the relation $x_n + y_n + z_n = 1$ for $n \geq 1$, and rewrite the equations in (5) as

$$\begin{cases} x_{n+1} = f(x_n, y_n), \\ y_{n+1} = g(x_n, y_n), \end{cases} \tag{6}$$

where

$$f(x, y) = \frac{\alpha}{T(x, y)}\left[x^2 + xy + \frac{1}{4}y^2\right],$$

$$g(x, y) = \frac{1}{T(x, y)}\left[\alpha\left(xy + \frac{1}{2}y^2\right) + \beta(2x + y)(1 - x - y)\right].$$

with $T(x, y) = \alpha(x + y)^2 + 2\beta(x + y)(1 - x - y) + \gamma(1 - x - y)^2$. We then compute the Jacobian $\boldsymbol{J}(x, y)$ of system (6) and evaluate it at each of the equilibria $(0, 0)$, $(1, 0)$, and (x^*, y^*). It can be shown that at $(0, 0)$,

$$\boldsymbol{J}(0, 0) = \begin{bmatrix} 0 & 0 \\ \frac{2\beta}{\gamma} & \frac{\beta}{\gamma} \end{bmatrix}.$$

It has two eigenvalues $\lambda_1 = 0$, $\lambda_2 = \dfrac{\beta}{\gamma}$. At $(1, 0)$,

$$\boldsymbol{J}(1, 0) = \begin{bmatrix} \frac{2\beta}{\alpha} & \frac{2\beta}{\alpha} - 1 \\ -\frac{2\beta}{\alpha} & 1 - \frac{2\beta}{\alpha} \end{bmatrix}.$$

It has two eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$. While $(0, 0)$ can be stable or unstable depending on whether $\beta < \gamma$ or $\beta > \gamma$, $(1, 0)$ is always neutrally stable. At the interior equilibrium (x^*, y^*), we have

$$\boldsymbol{J}(x^*, y^*) = \frac{1}{\alpha(\alpha\gamma - \beta^2)}\begin{bmatrix} 2\alpha(\alpha\gamma - \beta^2 - \sigma Q) & \alpha(\alpha\gamma - \beta^2 - \sigma Q) \\ 2[-\beta(\alpha\gamma - \beta^2) + (\alpha + \beta)\sigma Q] & (\alpha\gamma - \beta^2)(\alpha - 2\beta) + 2\beta\sigma Q \end{bmatrix}$$

where $Q = \sqrt{\beta(\alpha - \beta)(\alpha\gamma - \beta^2)}$ and $\sigma = 1$ or -1 depending on whether $\alpha, \gamma > \beta$ or $\alpha, \gamma < \beta$. It is easy to see that the long term proportions of the genotypes only depend on the relative size of the three parameters α, β, γ. So we set $\alpha = s\beta$, $\gamma = t\beta$, and consider the relative size of the parameters s and t to 1. Then

$$\boldsymbol{J}(x^*, y^*) = \frac{1}{s(st - 1)}\begin{bmatrix} 2s(st - 1 - \sigma Q) & s(st - 1 - \sigma Q) \\ 2[-(st - 1) + (s + 1)\sigma Q] & (st - 1)(s - 2) + 2\sigma Q \end{bmatrix}$$

with $Q = \sqrt{(s - 1)(st - 1)}$.

Let the two eigenvalues of $\boldsymbol{J}(x^*, y^*)$ be $\lambda_1 < \lambda_2$. Long calculation shows that for $s, t > 1$, $0 < \lambda_1 < 1 < \lambda_2$, so (x^*, y^*) is unstable while $(0, 0)$ is stable. This does not explain Example E. On the other hand, if $s, t < 1$, then $-1 < \lambda_1 < 0 < \lambda_2 < 1$, so (x^*, y^*) is stable while $(0, 0)$ is unstable. In this case, using the relation $\alpha(y^*)^2 = 4\beta x^* z^*$, and letting

$$p = x^* + \frac{1}{2}y^* \text{ and } q = \frac{1}{2}y^* + z^*,$$

we have

$$p^2 = (x^*)^2 + x^*y^* + \frac{1}{4}(y^*)^2 = (x^*)^2 + x^*y^* + \frac{\beta}{\alpha}x^*z^* > (x^*)^2 + x^*y^* + x^*z^* = x^*,$$

$$q^2 = \frac{1}{4}(y^*)^2 + y^*z^* + (z^*)^2 = \frac{\beta}{\alpha}x^*z^* + y^*z^* + (z^*)^2 > x^*z^* + y^*z^* + (z^*)^2 = z^*.$$

Similar to the case of selection, the predictions for the homozygotes are higher than the observations. This is exactly the opposite of Example E.

Conclusion: Non-random mating does not explain the type of deviations in Example E.

5. Multiple Population Models

Consider two single populations N_i $(i = 1, 2)$, each perfectly following the Hardy–Weinberg law. In population N_i the proportions of the alleles A and a are p_i and q_i respectively, with $p_i + q_i = 1$. The proportions for the three genotypes AA, Aa, and aa are then p_i^2, $2p_iq_i$, and q_i^2 respectively. When the two populations are mixed, the total population is $N = N_1 + N_2$. The proportions of alleles A and a are

$$p = \frac{p_1N_1 + p_2N_2}{N_1 + N_2}, \quad q = \frac{q_1N_1 + q_2N_2}{N_1 + N_2},$$

respectively.

The actual proportions of the three genotypes are given by

$$x^* = \frac{p_1^2N_1 + p_2^2N_2}{N_1 + N_2}, \quad y^* = \frac{2(p_1q_1N_1 + p_2q_2N_2)}{N_1 + N_2}, \quad z^* = \frac{q_1^2N_1 + q_2^2N_2}{N_1 + N_2}.$$

These can be compared with their expected values if the combined population were to satisfy the Hardy–Weinberg Law:

$$p^2 = \frac{(p_1N_1 + p_2N_2)^2}{(N_1 + N_2)^2}, \quad 2pq = \frac{2(p_1N_1 + p_2N_2)(q_1N1 + q_2N_2)}{(N_1 + N_2)^2}, \quad q^2 = \frac{(q_1N_1 + q_2N_2)^2}{(N_1 + N_2)^2}.$$

We have

$$x^* - p^2 = (p_1 - p_2)^2 \frac{N_1N_2}{(N_1 + N_2)^2} \geq 0,$$

$$z^* - q^2 = (q_1 - q_2)^2 \frac{N_1N_2}{(N_1 + N_2)^2} \geq 0,$$

and

$$y^* - 2pq = 2(p_1 - p_2)(q_1 - q_2) \frac{N_1N_2}{(N_1 + N_2)^2} \leq 0.$$

Also note that $p_1 - p_2 = q_2 - q_1$, so we have the symmetric relation $x^* - p^2 = z^* - q^2$. Thus, in a two population model, the heterozygote Aa always has a lower proportion, while the homozygotes AA and aa always have higher proportions than what will be predicted by the Hardy–Weinberg law. This agrees with Example E.

In particular, consider the special case $N_1 = N_2$. We have

$$x^* - p^2 = \frac{1}{4}(p_1 - p_2)^2 = z^* - q^2, \quad y^* - 2pq = -\frac{1}{2}(p_1 - p_2)^2.$$

To fit Example E, we let $N_1 = N_2 = 4035$ so the total population is $N = 8070$. Also let

$$x^* = \frac{3969}{8070}, \quad y^* = \frac{3174}{8070}, \quad z^* = \frac{927}{8070}.$$

Then

$$p = x^* + \frac{1}{2}y^* = \frac{5556}{8070}.$$

Using

$$p = \frac{p_1 + p_2}{2} \text{ and } x^* - p^2 = \frac{(p_1 - p_2)^2}{4}$$

we can solve for the proportions in the original individual populations to get $p_1 = 0.8221$ and $p_2 = 0.5549$. These values give the number of alleles as follows:

$$AA\colon\ p_1^2N_1 + p_2^2N_2 \approx 3969,$$

$$Aa\colon\ 2p_1q_1N_1 + 2p_2q_2N_2 \approx 3173,$$

$$aa\colon\ q_1^2N_1 + q_2^2N_2 \approx 927.$$

These values match that of Example E exactly.

Conclusion: The deviations in Example E can be perfectly explained by a two population model. It's easy to see that any multiple population model also can serve this purpose.

6. Discussion

In this paper we have considered three possible reasons that can explain the type of deviations from the Hardy–Weinberg Law as observed in Example E. They are selection, non-random mating, and multiple population composition. These have been suggested by Maynard Smith without supplying any detail. We quote him as follows ([6]).

> One possible explanation is that heterozygotes are less viable in the wild, but this is unlikely, because laboratory measurements suggested that heterozygotes had a higher viability than either homozygote. It is conceivable that abdomen color influences mate choice: if there is a tendency for like to mate with like this could explain the discrepancy. Perhaps the most plausible explanation is that the flies are not drawn from a single random-mating population.

Two of the possible explanations supplied by him, selection and random mating, are shown to be largely invalid. Each working alone cannot be the legitimate cause for such deviations. His last suggestion that the flies are not drawn from a single random-mating population is confirmed by our study. The multiple population model is elementary and only involves algebra. But we are able to show that a two population model can provide a perfect explanation to Example E. The observations recorded by Da Cunha strongly support the results of our mathematical models. The excess of homozygotes resulted from sampling multiple populations is called the Wahlund effect.

Other possible reasons can be supplied, such as mutation and migration. They can be studied along similar lines employed in this paper. Each of the factors can be formulated in more complicated ways, say, with age structures or sex differentials. In addition, a few factors can take effect at the same time, as it is typically the case in the natural world. Our study strongly suggests which of the factors can possibly be the main contributor to such deviations.

References

[1] W. L. Briggs, A. Chung, M. Grant and D. S. Robertson, “Differential and difference equations of population genetics", *Proc. of the 15th Annual Conference on Applied Math.*, Edmond, Oklahoma, (1999) 37–49.

[2] Da Cunha, A. Brito, “Genetic analysis of the polymorphism of color pattern in Drosophila Polymorpha", *Evolution*, **3**, (1949) 239–251.

[3] W. S. Klug, (1999), *Essential of Genetics*, 3rd ed. Prentice-Hall, Inc.

[4] Lyubich, I. Yuri, *Mathematical Structures in Population Genetics*, Springer-Verlag, (1992).

[5] M. N. Martinez and A. R. Cordeiro, “Modifiers of color pattern genes in Drosophila Polymorpha", *Genetics*, **64**, (1970), 573–587.

[6] J. Maynard Smith, *Evolutionary Genetics*, Oxford University Press, (1996).

[7] Preiss, Mitchell, “The Hardy–Weinberg law in population genetics", *New England Mathematics Journal*, Nov., (1997), 51–62.

[8] Price and R. George , “Extension of the Hardy–Weinberg Law to assortative mating", *Annals of Human Genetics*, **34**, (1971), 455–458.

[9] C. A. B. Smith, “A note on testing the Hardy–Weinberg Law", *Annals of Human Genetics*, **33**, (1970), 377–383.

[10] Tai and J. John, “On nonrandom mating systems for attaining Hardy–Weinberg equilibrium", *Biom. J.*, **32**(8), (1990), 1015–1018.

Seventh Asian Computational Fluid Dynamics Conference

November 26–30, 2007

Scope of the Conference

The Asian Computational Fluid Dynamics Conference (ACFD) is held once in every two years. The first conference (ACFD1) was held at Hong Kong in 1995, followed by the successive ones held at Tokyo (Japan), Bangalore (India), Miyanyang (China), Pusan (Korea) and Taipei (Taiwan) respectively. The major objective of ACFD is to provide a common forum for exchange of new ideas and experiences amongst the scientists and engineers from Asia as well as other parts of the globe, working on algorithms and applications of CFD.

The ACFD7 will be held at Bangalore, India during November 26–30, 2007. The programme includes invited keynote lectures by distinguished experts from across the globe. There will be a few plenary sessions and a large number of parallel technical sessions spread over five days. The conference is being jointly organized by National Aerospace Laboratories, Bangalore (www.nal.res.in), CFD Division of the Aeronautical Society of India (www.aesi.org), Indian Institute of Science, Bangalore (www.iisc.ernet.in), Centre for Development of Advanced Computing, Pune (www.cdac.in) and Indian Institute of Technology, Mumbai (www.iitb.ac.in).

Conference Venue:

J. N. Tata Auditorium
National Science Seminar Complex
Sir C. V. Raman Avenue
Bangalore 560 012, India

Important Dates:

Submission of Extended Abstract	May 31, 2007
Acceptance of Extended Abstract	July 31, 2007
Submission of Full Paper	September 30, 2007
ACFD7 Conference	November 26–30, 2007

For Further Details Contact:

J S Mathur
Organising Secretary ACFD7
Scientist CTFD Division
National Aerospace Laboratories
PB 1779, Bangalore 560 017, India
Tel: 91 80 25051613; Fax: 91 80 25220952
E-mail: jsm@ctfd.cmmacs.ernet.in

Avijit Chatterjee
Organising Secretary ACFD7
Department of Aerospace Engineering
Indian Institute of Technology
Powai, Mumbai 400 076, India
Tel: 91 22 25767128; Fax: 91 22 25722602
E-mail: avijit@aero.iitb.ac.in
Website: http://acfd7.cdac.in

Indian Institute of Science

Bangalore, India
IISc Centenary Post-Doctoral Fellowship

The Indian Institute of Science, Bangalore is pleased to announce the IISc Centenary POST-DOCTORAL FELLOWSHIP to encourage bright scientists and engineers (preferably below 35 years) of all nationalities to work in the stimulating environment of the Institute.

Duration: The fellowship will be for a period of 2 years (renewable for one more year)

Qualification: Candidates with a Ph.D. degree (in Science or Engineering) or those who have recently submitted their doctoral theses can apply.

Fellowship:

(1) Rs. 25,000 per month for a Ph.D. with 2 years experience.
(2) Rs. 20,000 per month for a recent Ph.D.
(3) Rs. 15,000 per month for those yet to be awarded the Ph.D.

Applications can be submitted at any time of the year and applicants will be informed about the decision on their application within three months.

For Further Details Visit:
http://www.iisc.ernet.in/opportunities/

Summer Programme in Mathematics (SPIM)

Harish–Chandra Research Institute, Allahabad
18th June–6th July, 2007

Harish–Chandra Research Institute conducts the Summer Programme in Mathematics every year. The aim of this programme is to introduce interesting topics in Mathematics to the students and to give them an exposure to the world of Mathematics Research. The programme involves intensive training on selected topics in mathematics for a period of three weeks.

The programme is primarily intended for students from universities and colleges. Preference will be given to M.Sc. I year and II year students, however exceptional B.Sc. final year students are also encouraged to apply.

The students who do exceptionally well in the programme will be considered for a special visiting programme at HRI. The programme has, in the past, provided HRI with several doctoral students.

Important Dates

The deadline for receiving completed application form: 20th April, 2007.

The list of selected candidates will be available by: 27th April, 2007

This list will be put in the web site by 29th April, 2007.

Latest date on which selected candidates are expected to get postal intimation: 12th May, 2007.

For Further Information Visit:
http://www.mri.ernet.in/~spim/

Contact Address:
Ratnakumar P. K/Kalyan Chakraborty
Co-ordinator, SPIM 2007
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi Allahabad 211 019, Uttar Pradesh
spim@hri.res.in; spim@mri.ernet.in

Professor Srinivasa S. R. Varadhan to Receive 2007 Abel Prize

The Norwegian Academy of Science and Letters has announced that Srinivasa S. R. Varadhan of the Courant Institute of Mathematical Sciences is the winner of the 2007 Abel Prize.

Srinivasa S. R. Varadhan was born on January 2, 1940 in Madras (Chennai), India. He received his B.Sc. honours degree in 1959 and his M.A. the following year, both from Madras University. In 1963 he received his Ph.D. from the Indian Statistical Institute. He is currently Professor of Mathematics and Frank J. Gould Professor of Science at the Courant Institute.

Varadhan was awarded the prize "for his fundamental contributions to probability theory and in particular for creating a unified theory of large deviations." The Abel Committee also said that "Varadhan's theory of large deviations provides a unifying and efficient method for clarifying a rich variety of phenomena arising in complex stochastic systems, in fields as diverse as quantum field theory, statistical physics, population dynamics, econometrics and finance, and traffic engineering. . . . Varadhan's work has great conceptual strength and ageless beauty. His ideas have been hugely influential and will continue to stimulate further research for a long time."

The Abel Prize is awarded annually by the Norwegian Academy of Science and Letters. The prize amount is NOK 6,000,000 or close to USD 1,000,000. The Abel Prize for 2007 will be presented to Srinivasa S. R. Varadhan by HM King Harald at the award ceremony in Oslo on 22nd May.

The Abel Prize website has more information about Varadhan and the prize, see `http://www.abelprisen.no/en/`

Ramanujan Mathematical Society

National Instructional Workshop on Complex Analysis (June 1–5, 2007) and 22nd Annual Conference (June 6–8, 2007)

About Workshop (June 1–5, 2007): The International Congress of the Mathematicians (ICM) will be held in Hyderabad during August 2010. With the objective of training many researchers in modern areas of Mathematics (so that they will be able to take active part in ICM -– 2010), Ramanujan Mathematical Society has embarked on a series of workshops oriented towards thrust areas of various branches of Mathematics. Last year, Ramanujan Mathematical Society organized a 3-days workshop on Number Theroy at University of Hyderabad during July 3–5, 2006. This year, the Society will be organizing a 5-days instructional workshop on Complex Analysis (from the standpoint of Geometry and Topology). The programme will be directed by Prof. Ravi Kulkarni, I.I.T., Bombay. The other expected resource persons are Prof. S. Mitra (Cornell University, USA), Prof. S. S. Bhoosnurmath (Karnatak University, Dharwad), Prof. R. R. Simha (University of Mumbai) and Prof. I. Biswas (TIFR, Bombay). There will be three lectures everyday for five days followed by discussions and tutorials.

Applications (with brief C.V.) for the workshop may be sent by E-mail to Prof. Ravi Kulkarni [E-mail-Ids: `kulkarni@math.iitb.ac.in`, `punekulk@yahoo.com`] and the Local Secretary: Dr. Shyam S. Kamath [E-mail-Id: `shyam@nitk.ac.in`, `shyam.kamath@yahoo.com`]. Selected participants will be intimated (through email) by April 15, 2007. Upon attending the workshop and the conference, the participants will be paid to and fro 2nd class sleeper train fare/actual bus fare, as per the Central Govt. rules. Local hospitality will be provided with no additional charges. All the selected participants are also entitled to stay on to attend the 22nd Annual Conference of the Ramanujan Mathematical Society during 6th-–8th June 2007, where more advanced lectures will be delivered.

About the Venue:

National Institute of Technology of Karnataka, Surathkal, (established originally as Karnataka Regional Engineering College, Surathkal in 1961) has been one of the premier educational institutions owned by the Govt. of India and is a Deemed University

About the 22nd Annual Conference (June 6–8, 2007)

Academic Programme:

There will be two invited talks of one hour in the morning sessions. The speakers are:

1. S. M. Bhatwadekar	TIFR, Mumbai	Algebra
2. John Hubbard*	Cornell University, USA	Analysis
3. Ravi Kannan*	Yale University, USA	Computer Science/ Combinatorics
4. Somenath Mukherjee	NAL, Bangalore	Applied Maths, Finite Elements
5. Sharad Sane	University of Mumbai	Graph Theory/ Discrete Mathematics
6. Harish Sheshadri	I.I.Sc., Bangalore	Poincare's Conjecture

*to be confirmed.

In the afternoon sessions, there will be three parallel symposia each of 90 minutes duration everyday. The organizers of the symposia have made tentative list of speakers most of whom have confirmed their participation. In addition, there will be sessions for presentation of papers especially by young workers and teachers. There will also be a public lecture, perhaps on a topic related to L. Euler whose 300th Birth anniversary falls in 2007. The registration fee for the National Workshop and the 22nd Annual Conference of the Ramanujan Mathematical Society is:

Workshop only	Rs. 500.00
Workshop and Conference	Rs. 800.00
Conference only	Rs. 400.00

(DD in favour of "22 ND AC - RMS 2007" payable at Mangalore)

For further details, please visit our website:
http://www.ramanujanmathsociety.org

There will, however, be no registration fee for the Life Members of the Ramanujan Mathematical Society.

Local Secretary:
Dr. Shyam S. Kamath
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal
Post: Srinivasanagar 575 025
Mangalore – Karnataka State
E-mail: shyam@nitk.ac.in, shyam kamath@yahoo.com
Phone: +91-824-247-4000 Ext. 3254
Fax: +91-824-247-4048/ +91-824-247-4033

Symposium 1: Probability Theory – organized by Abhay Bhatt

Participants:

1. A. Krishnamoorthy, Cochin University of Science and Technology
2. Anindya Goswami, I.I.Sc., Bangalore
3. Bhupendra Gupta, I.I.T., Kanpur
4. Krishanu Maulik, I.S.I., Kolkata
5. Nabin Kumar Jana, Bijoy Krishna Girls' Collage, West Bengal
6. R. Vasudev, University of Mysore
7. S. Ravi, University of Mysore

Symposium 2: Complex Analysis and Teichmuller Theory – organized by Ravi Kulkarni

Participants:

1. S. S. Bhoosnurmath, Karnatak University, Dharwad
2. Sudeb Mitra, Cornell University, USA
3. Indranil Biswas, TIFR, Mumbai
4. R. R. Simha, University of Mumbai (retd.)
5. S. Ponnusamy, I.I.T., Madras
6. A. P. Singh, University of Jammu
7. 3 John Hubbard, Cornell University, USA

Symposium 3: Arithmetic and Geometry – organized by Kapil Paranjape

Participants:

1. Chandan S. Dalawal, HRI, Allahabd	Varieties over p-adic fields
2. Ramesh Srikantan	K-Theory of Arithmetic varieties
3. Arvind Nair, TIFR, Mumbai	The arithmetic of modular varieties
4. V. Suresh, University of Hyderabad	The arithmetic and geometry of quadratic forms
5. C. S. Rajan, TIFR	Varieties over global fields
6. Suryaramana, HRI, Allahabad	Grothendieck-Teichmuller Lego

A special symposium devoted to teaching of mathematics will be organised:

Symposium 4: Teaching of Functional Analysis – organized by Subhash Bhatt

Participants:

1. Tirthankar Bhattacharyya, ISI, Bangalore
2. B. V. Rajarama Bhat, ISI, Bangalore
3. Prof. S. H. Kulkarni, IIT, Chennai
4. Pradipta Bandopadhaya ISI, Kolkata
5. Rajendra Bhatia, ISI, Delhi
6. Subhash J. Bhatt, Sardar Patel Univ., Vallabh Vidyanagar
7. V. Muruganandam, University of Pondicherry

Call for Papers: Those intending to submit papers for presentation, may send the abstracts of the papers (preferably in PDF format) to Prof. V. Balaji, CMI, Chennai (E-mail-Id: `balaji@cmi.ac.in`) and Local Secretary, Dr. Shyam S. Kamath (E-mail-Id: `shyam@nitk.ac.in`, `shyam.kamath@yahoo.com`) latest by April 25, 2007. Authors of the selected papers will be intimated through E-mail by May 5, 2007.

Ramanujan Mathematical Society
National Instructional
Workshop on Complex Analysis
(June 1–5, 2007)
and
22nd Annual Conference
(June 6–8, 2007)
at
Department of Mathematical
and Computational Sciences
National Institute of Technology
Karnataka, Surathkal
Post: Srinivasnagar 575 025
Mangalore – Karnataka

Registration Form

Name:
(in BLOCK LETTERS)

Designation:

Institution:

Sex (M/F):

Address for Correspondence:

Telephone/
Mobile No.: STD Code Telephone No.

(off)

(Res)

(Mobile)

E-mail-Id:

Attending: Workshop/Conference/Workshop and Conference

Demand Draft Details:	Amount: Rs.
(DD should be drawn in	Number:
favour of	Date:
"22nd AC-RMS 2007"	Bank:
payable at Mangalore)	Branch:
Accommodation needed?:	YES/NO
Presenting Paper?:	YES/NO

Place:

Date: Name and Signature of the Applicant

6th International ISAAC Congress

The ISAAC (International Society for Analysis, its Applications and Computation) board, the Turkish Mathematical Society - Ankara Branch, the Local Organizing Committee and the Department of Mathematics at METU, are pleased to invite you to the 6th International ISAAC Congress to be held from August 13, 2007 to August 18, 2007 on the campus of Middle East Technical University (METU), Ankara, Turkey. The 6th International ISAAC congress will resume the style established during the previous meetings in USA (1997), Japan (1999), Germany (2001), Canada (2003) and Italy (2005).

Special Sessions:

(1) Analytic Function Spaces and their Operators, Organizers: R. Aulaskari (Finland) and H. T. Kaptanog (lu) (Turkey)

(2) Clifford and Quaternion Analysis,

Organizers: I. Sabadini (Italy), M. Shapiro (Mexico) and F. Sommen (Belgium)

(3) Complex Analysis and Potential Theory,

Organizers: T. A. Aliyev (Turkey), M. L. Cristoforis (Italy) and P. Tamrazov (Ukraine)

(4) Complex Analytic Methods for Applied Sciences,

Organizers: S. V. Rogosin (Belarus) and V. V. Mityushev (Belarus)

(5) Complex and Functional Analytic Methods in PDEs,

Organizers: H. G. W. Begehr (Germany), D. Dai (China), J. Du (China)

(6) Dispersive Equations,

Organizers: V. Georgiev (Italy) and M. Reissig (Germany)

(7) Fractional Differential Equations and their Applications,

Organizers: A. Kilbas (Belarus) and J. J. Trujillo (Spain)

(8) Inverse and Ill-Posed Problems: Analysis and their Applications,

Organizers: A. Hasanov (Turkey), M. Yamamoto (Japan) and S. I. Kabanikhin (Russia)

(9) Integral Geometry,

Organizer: M. Yamamoto (Japan)

(10) Modern Aspects of the Theory of Integral Transforms and their Applications,

Organizers: A. Kilbas (Belarus) and S. Saitoh (Japan)

(11) Oscillation of Functional-Differential and Difference Equations,

Organizers: L. Berezansky (Israel) and A. Zafer (Turkey)

(12) Pseudo-Differential Operators,

Organizers: L. Rodino (Italy) and M. W. Wong (Canada)

(13) Reproducing Kernels and Related Topics,

Organizers: D. Alpay (Israel), A. Berlinet (France), S. Saitoh (Japan) and D. X. Zhou (Hong Kong)

(14) Spaces of Differentiable Functions of Several Real Variables and Applications,

Organizers: V. Burenkov (UK) and S. Samko (Portugal)

(15) Numerical Functional Analysis,

Organizers: P. E. Sobolevskii (Brazil), A. Ashyralyev (Turkey)

(16) Integrable Systems,

Organizers: M. Gürses (Turkey), Ismagil Habibullin (Turkey)

(17) General Session (topics which are not suitable for the sessions listed above),

Organizer: A. O. Çelebi (Turkey)

Deadlines:

Abstract Submission:	May 15, 2007
Early Registration:	May 15, 2007

For Further Details Contact: info@isaac2007.org

XIII-TH Conference on Mathematics and Computer Science

July 1–4, 2007, in Chelm

The Conference Program Covers the Following Sections:

- Mathematical and functional analysis
- Probability theory and statistics
- Mathematical didactics
- Computer science and Applications of mathematics to economics

Address for Contact:

XIII th International Conference on Mathematics and Computer Science

The State University of Applied Science

Pocztowa 54 22 – 100 Chelm, Poland

Phone +48 82 565 88 95

Phone/Fax. +48 82 565 88 94

E-mail: konfmat@pwsz.chelm.pl

http://www.pwsz.chelm.pl/konfmat

Fourth International Conference of Applied Mathematics and Computing

Plovdiv, Bulgaria
(August 12–18, 2007)

The Fourth International Conference of Applied Mathematics and Computing will take place in Plovdiv, Bulgaria, August 12–18, 2007.

Address of the Organizing Committee:

Prof. Drumi Bainov
P. O. Box 45, 1504 Sofia, Bulgaria
Tel: +359-888-91 45 32
E-mail: drumibainov@yahoo.com
http://math.uctm.edu/conference2007/

First Joint International Meeting between the American Mathematical Society and Polish Mathematical Society

July 31, 2007–August 3, 2007

Venue: At the old campus of the University of Warsaw, located close to the Warsaw's Old Town and other sites of interest. Most activities will take place at the old campus (No. 10) of the University of Warsaw, located close to the Warsaw's Old Town and other sites of interest.

Plenary Speakers:

- Henryk Iwaniec (Rutgers University)
- Tomasz J. Luczak (Adam Mickiewicz University)
- Tomasz Mrowka (Massachusetts Institute of Technology)
- Ludomir Newelski (University of Wroclaw)
- Madhu Sudan (Massachusetts Institute of Technology)
- Anna Zdunik (Warsaw University)

For info on organizers and programs click on the session title from the web page at http://www.ams.ptm.org.pl/

Interested person for giving a talk in a session should contact directly session organizers.

International Symposium on Geometric Function Theory and Applications

August 20–24, 2007
Istanbul, Turkey

TC Istanbul Kultur University Faculty of Science and Letters
Department of Mathematics and Computer Science

Aim: We would like to draw your attention to the forthcoming International Symposium on Geometric Function Theory and Applications (GFTA 2007) dedicated to the 10th Anniversary of TC Istanbul Kultur University, which will be held at TC Istanbul Kultur University during August 20–24, 2007. The aim of the symposium is to bring together leading experts as well as young researchers working on topics mainly related to Univalent and Geometric Function Theory and to present their recent work to the mathematical community. English is the official language of the symposium.

Main Topics:

- Univalent Function Theory
- Differential Subordination
- Quasiconformal Mappings
- Fractional Calculus

Important Dates:

- June 01, 2007, early registration
- June 30, 2007, deadline for the extended abstract submission
- July 31, 2007, deadline for the audience

In order for the papers presented to be published in the proceedings of the symposium, full texts of papers should be sent to e.yavuz@iku.edu.tr by October 31, 2007.

For Further Details Visit: http://fen-edebiyat.iku.edu.tr/cfta2007/bildiri_ozetleri_en.htm

Contact Address:

Emel YAVUZ (Scientific Secretary)
Address: TC Istanbul Kultur University
Faculty of Science and Letters
Department of Mathematics and Computer Science
Atakoy Campus,
Bakirkoy, 34156 Istanbul, Turkey
Phone: +90 (212) 498 43 61
+90 (212) 498 43 00
Fax: +90 (212) 661 92 74
E-mail: e.yavuz@iku.edu.tr

International Symposium on Complex Function Theory "Lucian Blaga"

University of Sibiu, Romania
August 26–29, 2007.

Organizers: Istanbul Kultur University (Turkey), Lucian Blaga University of Sibiu (Romania) and Kinki Univeristy (Japan).

For detailed information see http://www.siac-sibiu.net/

25 Years of Collaboration between the Indian National Science Academy and the Hungarian Academy of Sciences

The Indian National Science Academy (INSA), New Delhi, and the Hungarian Academy of Sciences (HAS), Budapest, jointly broughtout a compendium to mark the 25^{th} Anniversary of the Scientific – cooperation (1980–2005). In this connection Prof. R. A. Mashelkar, FNA, FRS, the president of INSA expressed his views as follows: The future belongs to knowledge based society. As such support to Scientific activities and motivation of Scientists to exchange their views is extremely important. It is well demonstrated that with the advent of globalization, global partnerships in science and technology are going to play an increasingly important role in shaping the world. We are proud to have collaboration with HAS, Hungary in Scientifically & Culturally an advanced country with a great effinity towards India. The Hungarian Scientists have earned a great recognition in the world of Science.

In the book "25 years of collaboration between the Indian National Science Academy and the Hungarian Academy of Sciences" broughtout by INSA-HAS, the names of 92 Hungarian Scientists who visited India & 52 Indian Scientists who visited Hungary under this Scientific Collaborative programme, were listed. The workdone and the views of these scientists in brief, also included.

Professor R. Balasubrahmanian is one of the members of the INSA delegation (six men) to Hungary to discuss & identify the scientific areas to improve the scientific cooperation between INSA & HAS. Regarding Mathematics, the detailed visit report of Prof. R. Balasubrahmanian states that

(i) Theoritical computer Science;
(ii) Combinatorial & analytic number theory; and
(iii) Probability

are the three major areas identified in which meaningful scientific cooperation can be started. He further suggested that the programme be visit based for mathematicians. Encouragement for the youngsters is essential and will ensure that the exchange programme is well utilized.

Advanced Training in Mathematics Schools

Funded by National Board for Higher Mathematics

For Getting Complete Details of:

- Overview of ATM Schools
- Committees for ATM Schools
- Annual Foundation schools
- Advanced Instructional Schools
- Workshops linked to an AIS
- Schools for lecturers
- Notes of lectures in ATM Schools

Visit: http://www.math.iitb.ac.in/atm/

International Conference of Numerical Analysis and Applied Mathematics (ICNAAM 2007)

16–20 September, 2007

Venue:

Hotel Marbella (Agios Ioannis Peristeron)
Corfu., Corfu, Greece

Important Dates:

Early Registration ends: April 30, 2007
Normal Registration ends: June 15, 2007
Late Registration ends: July 20, 2007
Submission of Extended Abstract: July 15, 2007 (Final Date)
Notification of acceptance: July 20, 2007
Submission of the source files of the camera ready extended abstracts: July 31, 2007 – Final Date
Submission of the full paper: September 30, 2007 – January 31, 2008

Address For Contact:

Mrs Eleni Ralli-Simou
Secretary ICNAAM
10 Konitsis Street
Amfithea Paleon Faliro GR-175 64
Athens, Greece
Fax: +30210 94 20 091, +30 2710 237 397
E-mail: tsimos@mail.ariadne-t.gr
: tsimos.conf@gmail.com
Web page: http://www.icnaam.org/

The 32nd Conference on Stochastic Processes and their Applications

August 6–10, 2007

Venue:

Department of Mathematics
University of Illinois
Urbana-Champaign

Important Dates:

Registration Fees:
Regular: $ 150 (before April 30, 2007),
Late: $ 200 (after April 30, 2007)

Student: $ 50 (before April 30, 2007),
Late: $ 75 (after April 30, 2007)

Abstract Deadline: May 31, 2007

For Further Details Visit: http://www.math.uiuc.edu/SPA07/

Fifth Symposium on Nonlinear Analysis (SNA 2007)

10–14 September, 2007

Venue:

Nicolaus Copernicus University
Torun, Poland

Topics:

Topics in the topological and metric fixed point theory, Topological and variational methods in nonlinear analysis, Qualitative theory of ordinary and partial differential equations and inclusions, Nonsmooth and convex analysis, Critical point theory, Optimal control theory, Applications.

Deadlines:

Registration deadline: Thu May 31, 2007
Abstract deadline: Sat June 30, 2007

Contact Address:

Slawomir Plaskacz
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18
87-100 Torun, Poland
Fax: +48 56 622-89-79
E-mail: sna2007@mat.uni.torun.pl
Web page: http://www-users.mat.uni.torun.pl/∼sna2007/contact.html