Mathematical fuzzy logic
– state of art.

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1 Introduction

Fuzzy logic – why should we, pure logicians care? Admittedly, for about three decades the field called "fuzzy logic" has been developed mainly by non-logicians – there was work on fuzzy convectives, but main interest was paid to approximate reasoning, the popular fuzzy IF-THEN rules and their applications, notably to fuzzy control. Even of the famous Ω-bibliography of mathematical logic [19] lists over 400 papers from 1965-85 under the heading "fuzzy logic". Till very recently the majority of logical community were skeptical, mostly due to tho fact that many papers declaring to deal with fuzzy logic were mathematicaly uninteresting. But the question has remained: can there really be a logic of inexact, imprecise, vague propositions? Propositions that may be more true or less true, thus a logic with a *comparative* notion of truth? In how far does this motivation give a new impulse and inspiration for developing systems of *many-valued logic*, whose history began in 1920's?

The time seems to be ripe to answer there question – for the benefit of both the pure logicians, finding new insights to formal logic and the developers and users of fuzzy logic (in the broad sense of the word), getting strict mathematical foundations for their activity.

In last few years I have tried hard to contribute to what could be called *mathematical fuzzy logic;*. My book "Metamathematics of fuzzy logic" [10] is the main outcome. (I note also two discusion papers [18] and [11].) Several first-class logicians have recently published or are preparing papers on systems of fuzzy logic; and at the Logic Colloquium'98 I had the honour to give a tutorial on fuzzy logic and there was a Special session chaired by D. Mundici) on Many-valued and fuzzy logic. Thus it seems that presently fuzzy logic has become one of established topics of mathematical logic or, in other words, that the idea of fuzzy logic has lead to a revival of interest in mathematical many-valued logics.

The present paper does not intend to be a written tutorial on fuzzy logic; such a tutorial was published as [15] and a shorter (updated) version is planned to appear in [12]. Here my plan is (1) to offer the reader a sort of self-review of the book [10], summarizing the *design choices* made, list the formal systems obtained, evaluate the present state of knowledge about them and their use as means of analysis of specific methods of fuzzy logic, as well as to underline debts and omissions of the book and (2) to survey new results of various authors closely related to the material of the book (thus describing the "state of art" of mathematical fuzzy logic). The list of references should also serve by updating and completing that of [10].

2 Metamathematics of Fuzzy Logic

2.1 Choices

The book [10] relies on the following main assumptions:

(1) The ordered real interval $[0, 1]$ is understood as the standard set of truth degrees (truth values).

(2) Continuous t-norms are taken to be (standard) truth functions of conjuction; particular interest is payed to the three famous continuous t-norms – Łukasiewicz ($\max(0, x + y - 1)$), Gödel (minimum) and product[1].

(3) Given a continuous t-norm, its residuum is taken to be the truth-function of implication. (These truth-functions are often called R-implications in the literature on fuzzy logic).

(4) Given such an R-implication $\Rightarrow$, the truth function of negation $(-)x$ is defined to be $x \Rightarrow 0$ (x implies falsity).

Thus choosing a continuous t-norm one chooses a semantics of propositional calculus – a t-norm logic. Main questions are: what is common to all these logics, and what is the logic of our outstanding t-norms Łukasiewicz, Gödel and product logic. For prodicate calculus, we add:

(5) A natural generalization of Tarskian semantics is taken to be the (standard) semantics of fuzzy predicate calculus: an interpretation consists of a non-empty (crisp) universe, predicates are interpreted by fuzzy relations and the quantifiers $\forall, \exists$ are interpreted by inf and sup respectively.

Similar questions are asked.

Clearly these are not the only possible choices; but the systems obtained seen to be of *central* importance. (For possible variants and extensions see section 2.)

[1]Note that giving these names to these t-norms is now common; in Chap. 10 (Historical remarks) of [10] I comment on this.

2.2 Results

To grasp what is to all t-norm logics, a set of 7 t-tautologies (having value 1 for each t-norm and each evaluation of propositional variables) is taken to be axioms of *basic (fuzzy) logic* BL. With modus ponens as the anly deduction rule some three dozens of formulas are shown to be provable (and hence to be 1−tautologies). The axioms naturaly determine a variety of algebras called BL-algebras. Each algebra can be taken as the algebra of truth functions of connectives (over the set of truth values given by the domain of the algebra). Axioms of BL are sound and strongly complete with respect to this generalized semantics; in particular, $\vdash_{BL} \varphi$ iff φ is a tautology over each BL-algebra (and, in general, for each theory T, $T \vdash_{BL} \varphi$ iff φ is true in each $\mathbf{L}$-model of T, for each BL-algebra $\mathbf{L}$).

For each of the three outstanding t-norms the following is mode: BL is extended by one or two additional axioms found for the respective t-norm; the resulting systems are $Ł, G, \Pi$ (Łukasiewicz, Gödel and product logic). This gives the corresponding varieties af algebras (MV-algebras, G-algebras, product algebras) and we get strong completeness w.r.t. models over these algebras for free from the strong completeness of BL. What does cost work is *standard completeness* – with respect to the respective standard algebra with the domain $[0,1]$ given by the respective t-norm. For G we get full strong standard completeness; for Ł and Π only strong standard completeness for finitely axiomatized theories.

For Łukasiewicz logic one obtains a well-behaving extension by rational truth-constants $\bar{r}$ (for each rational $r \in [0,1]$) called RPL – *Rational Pavelka logic.* For each theory T and formula φ, the *provability degree* is $|\varphi|_T = \inf\{r | T \vdash_{RPL} \bar{r} \to \varphi\}$ and the *truth degree* is $\|\varphi\|_T = \sup\{e(\varphi) | e \text{ is a model of } T\}$. Pavelka completeness says that $|\varphi|_T = \|\varphi|_T$ for each φ, T. (Here one works with models over the standard MV-algebra given by the Łukasiewicz t-norm.)

Let $\mathcal{C}$ be Ł, G, or Π, let $\mathrm{SAT}_1^{\mathcal{C}}$ be the set of formulas satisfiable over the standard $\mathcal{C}$-algebra (i.e. for some evaluation e of propositional variables in $[0,1], e_{\mathcal{C}}(\varphi) = 1$); similarly $\mathrm{SAT}_{pos}^{\mathcal{C}}$ (posivitely satisfiable, $e(\varphi) > 0$), $\mathrm{TAUT}_1^{\mathcal{C}}$, $\mathrm{TAUT}_{pos}^{\mathcal{C}}$ (tautologies, positive tautologies). The compositional complexity is "as it shoul be": $\mathrm{SAT}_1^{\mathcal{C}}$, $\mathrm{SAT}_{pos}^{\mathcal{C}}$ are NP-complete, $\mathrm{TAUT}_1^{\mathcal{C}}$, $\mathrm{TAUT}_{pos}^{\mathcal{C}}$, are co-NP-complete. Several equalities hold amony these classes: e.g. $\mathrm{SAT}_1^G = \mathrm{SAT}_{pos}^G = \mathrm{SAT}_1^{\Pi} = \mathrm{SAT}_{pos}^{\Pi}$.

Each propositional logic $\mathcal{C}$ as BL, G, Π, Ł together with its variety of algebras gives the corresponding predicate calculus $\mathcal{C}\forall$ (without functions symbols and equality) whose models are *safe* $\mathbf{L}$-*interpretations* for any algebra $\mathbf{L}$ from the variety. Predicates are interpreted by $\mathbf{L}$-fuzzy relations over a domain M (i. e. mappings $r : M^n \to \mathbf{L}$). The value $\|\varphi\|_{\mathbf{M},v}^{\mathbf{L}}$ of a formula depends on the algebra $\mathbf{L}$, the $\mathbf{L}$-interpretation $\mathbf{M}$ and an evaluation of object variables through truth functions of connections and the interpretation of $\forall, \exists$ by inf and sup. Clearly, $\|\varphi\|_{\mathbf{M},v}^{\mathbf{L}}$ may be undefined if same infinite sup or inf used in the definition of

$\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ does not exist in $\mathbf{L}$. $\mathbf{M}$ is *safe* if $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is defined for each φ and v. *Axioms* of $\mathcal{C}\forall$ are those of $\mathcal{C}$ plus five fixed axioms for quantifiers (two usual axiom for $\forall$ from Boolean predicate calculus, their "duals" for $\exists$ and one additional axiom).

The *strong completeness* says that $T \vdash_{\mathcal{C}\forall} \varphi$ iff φ is true in all $\mathbf{L}$-models of T for each $\mathcal{C}$-algebra $\mathbf{L}$. Here $\mathbf{M}$ is an $\mathbf{L}$-model of T if $\mathbf{M}$ is a safe $\mathbf{L}$-interpretation making all axioms of T true. *Strong standard completeness* holds for G$\forall$ (provable = true to all models over the standard G-algebra). This shows the set $\mathrm{TAUT}_1^{G\forall}$ (of tautologies of G$\forall$) to be Σ_1 and it is easy to show thet it is Σ_1-complete. On the other hand, $\mathrm{TAUT}_1^{\text{Ł}\forall}$ is Π_2-complete and $\mathrm{TAUT}_1^{\Pi\forall}$ is Π_2-hard ($\mathrm{TAUT}_1^{\text{Ł}\forall}$ is reducible to $\mathrm{TAUT}_1^{\Pi\forall}$). Note that RPL also generalizes to RPL$\forall$ and the analogous Pavelka completeness ($|\varphi|_T = \|\varphi\|_T$) holds for RPL$\forall$.

This has been a sketch of main results of pure mathematical fuzzy logic, say (chapters 2 - 6.). This logic may serve as a means of logical analysis of approximate reasoning in Zadeh's style as his generalized modus ponens and "fuzzy inference" in fuzzy control (using the notions of similarity and fuzzy functions) as well as logical theory of fuzzy quantifiens or modalities like "many", "probably", "possibly" etc. (Chapter 7 - 8.) Chapter 9 (Miscellanea) contains a logic $\Pi\forall$ (with an infinitary deduction rule) containing the three logics Ł$\forall$, G$\forall$, $\Pi\forall$, further an analysis of the liar's paradox in fuzzy logic and one more systems of Pavelka. References to previous works are given throughout the text and chapter 10 consists of Historical remarks. (Note the all references were omitted in this subsection.)

2.3 Debts and omissions

Needless to say, the above is a *selection* of topics and approaches; even if I belive to have covered main central topics. I feel obliged to mention important things not covered. (Some of then are least mentioned in the book.)

On the one land, there has been extensive research in "classical" many-valued logic, namely in Łukasiewicz propositional logic and MV-algebras (Mundici's school) and in Gödel logic (both propositional and predicate logic, Baaz's school) that could not be included. Second, from the point of view of traditional fuzzy logic our choice (4) can be questioned: independently of the choice of the truth functions of conjunction and implication one usualy insists on having Łukasiewicz negation $(1-x)$ (present in Ł but absent in G, Π). Having this (involutive) negation one can use de Morgan rule to introduce a dual of the starting conjunction – the corresponding disjunction. (Note that in each t-norm logic we do have a disjunction whose truth function is maximum and also we do have the min-disjunction.) Further, in traditional fuzzy logic one sometimes uses so-called S-implications. They are not elaborated in the book (but in 9.4.6, reference is given to the corresponding literature).

We shall go into more details in the next section.

3 New results in mathematical fuzzy logic

3.1 Łukasiewicz propositional logic

D. Mundici, together with various co-authors, has undertaken a long-term research in Łukasiewicz propositional logic and MV-algebras, important not only for fuzzy logic but also for various diverse branches as toric varieties [27, 29], abelian lattice-ordered groups [30, 20], error correcting codes [22, 23] and C^*-algebras [20, 21, 24, 25, 26, 31]. Full presentation will be contained in the forthcoming book [7]; one may consult its predecessor [28] in Portugiese as well as the paper [32] [2]

Here I shall comment on three most recent works. First, in the paper [33] one finds a general result on comlexity of satisfaction (generalizing Mundici's result of 1987): GENSAT_∞ is the problem to decide, for a given propositional formula φ and two integers $1 \leq c \leq d$ writen in binary, whether there is an evaluation e of propositional variables such that $e(\varphi) \geq \frac{c}{d}$. This problem is shown to be NP-complete.

A *positive literal* is a formula built from one propositional variable using connectives $\&, \underline{\vee} \wedge, \vee$ (of strong and weak conjuction and disjunction). A negative literal is a negated positive literal; a *clause* is a finite set of literals (understood as their $\vee$-disjunction). A set of clauses is understood as the $\wedge$-conjunction of its members. A sophisticated notion of resolution is introduced and shown to be refutation complete; and the restriction of GENSAT_∞ to inputs (φ, c, d) where φ is a finite set of Horn clauses or Krom clauses is shown to be in P.

[34] has the title "Ulam games, the logic of MAXSAT, and many-valued partitions". MAXSAT is the problem whether a set of boolean clauses contains a satisfable subset of cardinality ot least k. (MAXSAT is known to be NP-complete.) The authors present a polynomial-time reduction τ of MAXSAT to the set SAT_∞ of satisfable formulas of Ł for which the image $\tau(\varphi)$ of a set $(C_1, \ldots, C_m, k)$ has a natural interpretation related to Ulam games with lies (errors); in the final section these games are briefly related to partitions in MV-algebras. Note that both the relation of Ulam games to Ł and partitions in MV-algebras are topics that have been detailedly studied in Mundici's previous papers. The corresponding material and references may be found in [28]. Partitions of MV-algebras are also used in [35] for a study of imprecisely defined functions. This paper contains a nice Appendix giving a survey on MV-algebras, their partitions and Łukasiewicz propositional logic.

[2] I also used a short unpublished extented abstract "Present development in many-valued logic" by Mundici.

3.2 Gödel (and other) logics

Here I refer on some results of the Vienna school of M. Baaz. I shall disregard results of this group concerning finitely-valued logics; in the domain of infinite-valued logics studied by them Gödel logic plays a central role. In fact they investigate different Gödel logics, each given by a complete sublattice V of $[0,1]$, important examples being $V^0 = \{0\} \cup \{\frac{1}{k}, k > 0\}$ and $V^1 = \{1\} \cup \{1 - \frac{1}{k} | k > 0\}$. Whereas all infinite V give the same set of propositional tautologies, they give many different propositional logics with respect to entailment: a set Π of formulas entails a formula φ if each evaluation of variables making all members of Π true makes also φ true. This is studied in [5]; the authors show that between the logic $G^{\uparrow}$ with the truth set V^1 and the logic G (with [0,1]) there are infinitely many intermediate logics with pairwise different entailmemt relations. In the same paper the authors prove that logic over an infinite V is compact iff V contains a non-trivial densely ordered subset (in which case its entailment coincides with the entailment of G).

In [4] it is shown, among other things, that Gödel propositional logic G has interpolation (even uniform) but neither Ł nor Π has interpolation. Extensions of these logics by Baaz's connective $\triangle$ and some other unary connectives are studied from the point of view of interpolation (There are many other results in this paper.)

In [3] Gödel *predicate* logics over different truth sets are studied and it is shown that, in contradistinction to $G\forall$, the predicate logic $G_{\downarrow}\forall$ over V^0 is not recursively completely axiomatizable. Finally we mention [1]; from many results of this paper we note that the axiom (A4) of BL (commutativity of the min-conjunction) is not redundant (does not follow from the others; this was once asked by me as a problem.)

3.3 Some results on fuzzy predicate calculi

I am happy to have two joint papers with J. Paris and J. Shepherdson: the first [16] on the liar paradox and definability of truth (results of this paper are presented in [10] Chap. 9 Sec. 3) and the second [17], whose title "Rational Pavelka logic is a conservative extension of Łukasiewicz logic" describes the main result of the paper. But this paper contains another result of independent importance; we present its formulation here. Recall that the truth value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ of a (closed) formula in the $\mathbf{L}$-structure $\mathbf{M}$ may be undefined if $\mathbf{L}$ is not completely ordered (some sups or infs do not exist); $\mathbf{M}$ is safe if all $|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ are defined. Assume now that $\mathbf{L}$ is an MV-algebra, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}}$ is defined and Ł$\forall$ (Łukasiewicz predicate logic) proves φ. Does it imply $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} = 1$? Belluce and Chang [6] claim it does; but this is false: there is an MV-algebra $\mathbf{L}$, an $\mathbf{L}$-structure $\mathbf{M}$ (not safe) and a closed formula φ such that Ł$\forall \vdash \varphi$ but $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} < 1_{\mathbf{L}}$. An analogous example works for product logic: there is a product algebra $\mathbf{L}$, an $\mathbf{L}$-structure $\mathbf{M}$ and a formula φ

such that $\Pi\forall \vdash \varphi$ and $\|\varphi\|^{\mathbf{L}}_{\mathbf{M}} < 1_{\mathbf{L}}$. (For G∀ there is no such connterexample.)

Let me tell another result (see [13]). For $\mathcal{C}$ being Ł, G, Π we may ask what is the arithmetical complexity of the set f-TAUT$_1^{\mathcal{C}\forall}$ of formulas of $\mathcal{C}\forall$ true (having value 1) for each *finite* structure **M** (over the standard $\mathcal{C}$-algebra $[0,1]_{\mathcal{C}}$). Similarly for f-TAUT$_{pos}^{\mathcal{C}\forall}$ (formulas having never value 0), f-SAT$_1^{\mathcal{C}\forall}$ (formulas having value 1 in at least one model) and f-SAT$_{pos}^{\mathcal{C}\forall}$ (positively satisfable). The result is that everything is fully analogous to be situation on Boolean logic, described by the famous Trakhtenbrot theorem: both f-TAUT$_1^{\mathcal{C}\forall}$ and f-TAUT$_{pos}^{\mathcal{C}\forall}$ is Π_1-complete and both f-SAT$_1^{\mathcal{C}\forall}$ and f-SAT$_{pos}^{\mathcal{C}\forall}$ is $\sum_1$-complete. Interesting enough, we have various equalities, e.g. f-SAT$_1^{\mathcal{G}\forall}$ = f-SAT$_{pos}^{G}$ = f-SAT$_1^{\Pi}$ = f-SAT$_{pos}^{\Pi}$. The situation is fully analogous to that of satisfability/tautologicity in propositional logic).

3.4 Enriched systems

Recall that for Łukasiewicz logic the truth function $n(x)$ of negation is $1-x$, which is a particular involution ($n(n(x)) = x$ for all x); but if a logic is given by a t-norm without non-trivial zero divisors (as Gödel and product logic) then its $n(x)$ is Gödel negation ($n(0) = 1, n(x) = 0$ for $x > 0$).

In [8] (by Esteva, Godo, Hájek and Navara) one studies first an extension SBL of BL by an axiom sound for all t-norms without non-trivial zero divisors; then this system is expanded by a new negation and axiom expressing the fact that the truth function of the new negation os involutive. Completeness of axioms of this (propositional) system SBL with respect to the corresponding variety of algebras is proved in the standard manner. Adding axioms for specific t-norms we get $G_\sim$ and $\Pi_\sim$ – Gödel and product logic with involutive negation. We have the standard $G_\sim$-algebra and standard $\Pi_\sim$-algebra (which is the standard G-algebra or Π-algebra over $[0,1]$ expanded by the operation $1-x$) and may ask if we have standard completeness. For G∼ the answer is positive, for $\Pi_\sim$ not, but for $\Pi_\sim$ we get semi-standard completeness – completeness with respect to expansions of the standard Π-algebra by any involutive negation (not necessary $1-x$). The theory extends to predicate calculi SBL$\forall_\sim$, Ł$\forall_\sim$, G$\forall_\sim$, Π$\forall_\sim$: we get strong completeness w.r.t. safe models over the corresponding algebras and for G$\forall_\sim$ we get even strong standard completeness.

Esteva, Godo and Montagna are preparing a joint paper [9] in which they present a (finitary) complete axiomatization of a logic having both Łukasiewicz and product conjunction and both Łukasiewicz and product implication (in which case the logic defines also Gödel conjunction and implication). They find a very elegant representation of the underlying ŁΠ-algebras using ordered fields and prove also the corresponding standard completeness theorem.

4 Conclusion

There are various other papers; I mention at least [14], [2], [36]. Many open problems remain – see the list in [10]. I hope the reader will agree that the field of mathematical fuzzy logic is presently sound, lively and promising, worth to be studied and developed.

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