MATH 3263 Hilbert Spaces: Some useful results and formulae (2)

J.R.P. *March 2001*

Please let me know of any errors, omissions or obscurities!

Dual spaces. A *linear functional* on a vector space X is a linear mapping $\alpha : X \to \mathbb{C}$ (or to $\mathbb{R}$ in the real case), i.e., $\alpha(ax + by) = a\alpha(x) + b\alpha(y)$. When X is a normed space, α is continuous if and only if it is *bounded*, i.e., $\sup\{|\alpha(x)| : \|x\| \le 1\} < \infty$. Then we define $\|\alpha\|$ to be this sup, and it is a norm on the space X^* of bounded linear functionals, making X^* into a Banach space.

Riesz–Fréchet. If $\alpha : H \to \mathbb{C}$ is a bounded linear functional on a Hilbert space H, then there is a unique $y \in H$ such that $\alpha(x) = \langle x, y\rangle$ for all $x \in H$; also $\|\alpha\| = \|y\|$.

Linear Operators. These are linear mappings $T : X \to Y$, between normed spaces. Defining $\|T\| = \sup\{\|T(x)\| : \|x\| \le 1\}$, finite, makes the bounded (i.e., continuous) operators into a normed space, $B(X, Y)$. When Y is complete, so is $B(X, Y)$. We get $\|Tx\| \le \|T\|\,\|x\|$, and, when we can compose operators, $\|ST\| \le \|S\|\,\|T\|$. Write $B(X)$ for $B(X, X)$, and for $T \in B(X)$, $\|T^n\| \le \|T\|^n$. *Inverse* $S = T^{-1}$ when $ST = TS = I$.

Adjoints. $T \in B(H, K)$ determines $T^* \in B(K, H)$ such that $\langle Th, k\rangle_K = \langle h, T^*k\rangle_H$ for all $h \in H$, $k \in K$. Also $\|T^*\| = \|T\|$ and $T^{**} = T$.

Unitary operators. Those $U \in B(H)$ for which $UU^* = U^*U = I$. Equivalently, U is surjective and an isometry (and hence preserves the inner product).

Self-adjoint or *Hermitian* operators. Those $T \in B(H)$ such that $T = T^*$.

Normal operators. Those $T \in B(H)$ such that $TT^* = T^*T$ (so including Hermitian and unitary operators).

Spectrum. $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible in } B(X)\}$. Includes all *eigenvalues* λ where $Tx = \lambda x$ for some $x \ne 0$, and often other things as well. *Spectral radius:* $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. Properties: $\sigma(T)$ is closed, bounded and nonempty. Proof: based on the fact that $(I - A)$ is invertible for $\|A\| < 1$. This implies that $r(T) \le \|T\|$.

Spectral radius formula. $r(T) = \inf_{n\ge1} \|T^n\|^{1/n} = \lim_{n\to\infty} \|T^n\|^{1/n}$.

Note that $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$ and $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$. The spectrum of a unitary operator is contained in $\{|z| = 1\}$, and the spectrum of a self-adjoint operator is real (proof by *Cayley transform*: $U = (T - iI)(T + iI)^{-1}$ is unitary).

Finite rank operators. $T \in F(X, Y)$ if $\operatorname{Im} T$ is finite-dimensional.

Compact operators. $T \in K(X, Y)$ if: whenever (x_n) is bounded, then (Tx_n) has a convergent subsequence. Now $F(X, Y) \subseteq K(X, Y)$ since bounded sequences in a finite-dimensional space have convergent subsequences (because when Z is f.d., Z is isomorphic to ℓ_2^n, i.e., $\exists S : \ell_2^n \to Z$ with S, S^{-1} bounded). Also limits of compact operators are compact, which shows that a diagonal operator $Tx = \sum \lambda_n \langle x, e_n\rangle e_n$ is compact iff $\lambda_n \to 0$.

Hilbert–Schmidt operators. T is H–S when $\sum \|Te_n\|^2 < \infty$ for some o.n.b. (e_n). All such operators are compact—write them as a limit of finite rank operators T_k with $T_k \sum_{n=1}^\infty a_n e_n = \sum_{n=1}^k a_n (Te_n)$. This class includes integral operators $T : L_2(a,b) \to L_2(a,b)$ of the form

$$(Tf)(x) = \int_a^b K(x,y) f(y)\, dy,$$

where K is continuous on $[a,b] \times [a,b]$.

Spectral properties of normal operators. If T is normal, then (i) $\operatorname{Ker} T = \operatorname{Ker} T^*$, so $Tx = \lambda x \implies T^* x = \overline{\lambda} x$; (ii) eigenvectors corresponding to distinct eigenvalues are orthogonal; (iii) $\|T\| = r(T)$.

If $T \in B(H)$ is compact normal, then its set of eigenvalues is either finite or a sequence tending to zero. The eigenspaces are finite-dimensional, except possibly for $\lambda = 0$. All nonzero points of the spectrum are eigenvalues.

Spectral theorem for compact normal operators. There is an orthonormal sequence (e_k) of eigenvectors of T, and eigenvalues (λ_k), such that $Tx = \sum_k \lambda_k \langle x, e_k \rangle e_k$. If (λ_k) is an infinite sequence, then it tends to 0. All operators of the above form are compact and normal.

Corollary. In the spectral theorem we can have the same formula with an orthonormal *basis*, adding in vectors from $\operatorname{Ker} T$.

General compact operators. We can write $Tx = \sum \mu_k \langle x, e_k \rangle f_k$, where (e_k) and (f_k) are orthonormal sequences and (μ_k) is either a finite sequence or an infinite sequence tending to 0. *Hence $T \in B(H)$ is compact if and only if it is the norm limit of a sequence of finite-rank operators.*

Integral equations. Fredholm equations on $L_2(a,b)$ are $T\phi = f$ or $\phi - \lambda T\phi = f$, where $(T\phi)(x) = \int_a^b K(x,y)\phi(y)\, dy$. Volterra equations similar, except that T is now defined by $(T\phi)(x) = \int_a^x K(x,y)\phi(y)\, dy$.

Neumann series. $(I - \lambda T)^{-1} = 1 + \lambda T + \lambda^2 T^2 + \ldots$, for $\|\lambda T\| < 1$.

Separable kernels. $K(x,y) = \sum_{j=1}^n g_j(x) h_j(y)$. The image of T (and hence its eigenvectors for $\lambda \neq 0$) lies in the space spanned by $g_1, \ldots, g_n$.

Hilbert–Schmidt theory. Suppose that $K \in C([a,b] \times [a,b])$ and $K(y,x) = \overline{K(x,y)}$. Then (in the Fredholm case) T is a self-adjoint Hilbert-Schmidt operator and eigenvectors corresponding to nonzero eigenvalues are continuous functions. If $\lambda \neq 0$ and $1/\lambda \notin \sigma(T)$, the the solution of $\phi - \lambda T\phi = f$ is

$$\phi = \sum_{k=1}^\infty \frac{\langle f, v_k \rangle}{1 - \lambda \lambda_k} v_k.$$

Fredholm alternative. Let T be compact and normal and $\lambda \neq 0$. Consider the equations (i) $\phi - \lambda T\phi = 0$ and (ii) $\phi - \lambda T\phi = f$. Then EITHER (A) The only solution of (i) is $\phi = 0$ and (ii) has a unique solution for all f OR (B) (i) has nonzero solutions ϕ and (ii) can be solved if and only if f is orthogonal to every solution of (i).