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On interpolation of bilinear operators

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In loving memory of my mother Anna and father Konstanty

Abstract

In this paper we study interpolation of bilinear operators between products of Banach spaces generated by abstract methods of interpolation in the sense of Aronszajn and Gagliardo. A variant of bilinear interpolation theorem is proved for bilinear operators from corresponding weighted c_0 spaces into Banach spaces of non-trivial the periodic Fourier cotype. This result is then extended to the spaces generated by the well-known minimal and maximal methods of interpolation determined by quasi-concave functions. In the case when a maximal construction is generated by Hilbert spaces, we obtain a general variant of bilinear interpolation theorem. Combining this result with the abstract Grothendieck theorem of Pisier yields further results. The results are applied in deriving a bilinear interpolation theorem for Calderón–Lozanovsky, for Orlicz spaces and an embedding interpolation formula for (E, p)-summing operators.

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1. Introduction

Multilinear operators arise naturally in many areas of classical, harmonic analysis as well as functional analysis, including the theory of Banach operator ideals. Fundamental multilinear operators arising in Euclidean harmonic analysis include convolutions, paraproducts as well as multilinear Fourier multiplier operators. For an extended discussion of important multilinear operators and their interesting applications, one may consult [8] and references therein. Recently, various singular multilinear operators have been investigated intensively. The latest progress (see [19]) in the study of the bilinear Hilbert transform has stimulated the need of the development of a systematic analysis of bilinear operators. It should be noticed that in the last few years, the attention to L_p analysis of certain bilinear extensions of standard linear operators increased. We mention here only the remarkable paper of Grafakos and Kalton [13] in which bilinear multipliers of Marcinkiewicz type are studied. A natural question that arises is whether one could extend those results for operators between general spaces using interpolation techniques. We note that interpolation of bilinear operators is a classical problem in interpolation theory. The situation for the real and complex method of interpolation is well understood however few results are known for other interpolation methods.

The purpose of this article is to prove some new abstract results on interpolation of bilinear operators that involve minimal and maximal interpolation construction in the sense of Aronszajn and Gagliardo [1] (see also [7]).

We now describe the main results of this paper. In Section 2 we fix the notation and recall basic facts that will be needed in this paper. In Section 3 we prove some vector-valued preliminary results related to double sequences and series generated by bounded bilinear operators defined on the product $c_0 \times c_0$ with values in Banach spaces with nontrivial Fourier cotype. Then, via abstract interpolation, these results are used to state and prove the key bilinear interpolation Theorem 2.2 of the paper.

In Section 3 we discuss the abstract interpolation of bilinear interpolation between Banach couples. The general results are shown for minimal and maximal methods of interpolation in the sense of Aronszajn and Gagliardo. Using these results and the main result of Section 2, we prove a general result on interpolation of bilinear operators involving the well-known interpolation methods of interpolation determined by quasi-concave functions. We should point out here that using the abstract Grothendieck theorem of Pisier, we prove a theorem which seems to be interesting on its own. Applications of this result are shown to Calderón–Lozanovsky spaces. Applying this result to bilinear operators between products of L_p spaces, we obtain new bilinear interpolation theorems for Orlicz spaces.

Finally, in Section 4, we present applications to the problem of interpolation of spaces of operators. We use the bilinear interpolation theorems proved in the paper to show certain continuous inclusions for interpolation spaces between spaces of operators. For the finite-dimensional couples satisfying geometrical assumptions involving cotype 2 and 2-concavity, we show similar inclusions for certain Banach operator ideals. In particular, for 2-summing operators, we obtain a variant of Pisier result for the complex method of interpolation for the Gustavsson–Peetre method ($\pm$method) of interpolation.

2. Preliminaries

This section contains notation, most notions and basic facts necessary in the whole paper.

We shall use standard notation and notions from interpolation theory, as presented, e.g. in [3,7,28]. Throughout the paper we will let (Ω, Σ, μ) be a complete σ-finite measure space. Let $L_0(\mu)$ denote, as usual, the space of equivalence classes of real-valued measurable functions on Ω, equipped with the topology of convergence in measure μ on sets of finite measure. By a Banach lattice on Ω we mean a Banach space X which is a subspace of $L_0(\mu)$ such that there exists $u \in X$ with $u > 0$ and if $|f| \leqslant |g|$ a.e., where $g \in X$ and $f \in L_0(\mu)$, then $f \in X$ and $\|f\|_X \leqslant \|g\|_X$.

In the special case when $\Omega = \mathbb{J}$ and μ is the counting measure, where $\mathbb{J} = \mathbb{Z}$ (resp., $\mathbb{J} = \mathbb{N}$) denote the set of all integers (resp., the set of all positive integers), a Banach lattice on Ω is called a Banach sequence space on $\mathbb{J}$.

If X is a Banach lattice on (Ω, μ) and $w \in L^0(\mu)$ with $w > 0$ a.e., then the weighted Banach lattice $X(w)$ is defined by $\|x\|_{X(w)} := \|xw\|_X < \infty$.

If $\bar{X} = (X_0, X_1)$ and $\bar{Y} = (Y_0, Y_1)$ are couples of Banach spaces, we let $\mathscr{L}(\bar{X}, \bar{Y})$ be the Banach space of all linear operators $T : \bar{X} \to \bar{Y}$ (which means, as usual, that $T : X_0 + X_1 \to Y_0 + Y_1$ is linear and the restrictions $T|_{X_j}$ are bounded operators from X_j to Y_j for $j = 0, 1$). This space is equipped with the norm

$$\|T\|_{\bar{X} \to \bar{Y}} := \max\{\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}\}.$$

The K-functional is defined on $X_0 + X_1$ by

$$K(s, t, x; \bar{X}) := \inf\{s\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x = x_0 + x_1\}, \quad s, t > 0.$$

In what follows, we write in short $K(t, x; \bar{X})$ instead of $K(1, t, x; \bar{X})$.

Let (X_0, X_1) be any Banach couple, E a Banach sequence space on $\mathbb{Z}$ which is an intermediate space with respect to $(\ell_\infty, \ell_\infty(2^{-n}))$. We denote by $(X_0, X_1)_E$ the Banach space of all $x \in X_0 + X_1$ such that $\{K(2^n, x; \bar{X})\} \in E$ equipped with the norm

$$\|x\| = \|\{K(2^n, x; \bar{X})\}\|_E.$$

Let Φ denote the set of all functions $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that φ is homogeneous of degree one and $\rho = \varphi(1, \cdot)$ is a quasi-concave function on $(0, \infty)$ (i.e., $t \mapsto \rho(t)$ is a non-decreasing and $t \mapsto \rho(t)/t$ is non-increasing positive function on $(0, \infty)$).

If $\varphi \in \Phi$, then the function φ^* is defined by $\varphi^*(s, t) = 1/\varphi(1/s, 1/t)$ for $s, t > 0$.

The subset of all $\varphi \in \Phi$ for which $\varphi(s, 1) \to 0$ and $\varphi(1, t) \to 0$ as $s \to 0$ and $t \to 0$ is denoted by Φ_0.

A function $\varphi \in \Phi$ is called *non-degenerate* if the ranges of the functions $t \mapsto \varphi(1, t)$ and $t \mapsto \varphi(t, 1)$ where $t > 0$, coincide with $(0, \infty)$.

A quasi-concave function ρ is called a *quasi-power* function if the dilatation indices δ_ρ and γ_ρ of the function ρ satisfy $0 < \delta_\rho \leqslant \gamma_\rho < 1$ (see, e.g., [16] or [15]). If $\varphi \in \Phi$ and $\rho = \varphi(1, \cdot)$ is a quasi-power function, we write in short that $\varphi \in \Phi^{+-}$.

We will deal with vector-valued Banach sequence spaces. Let E be a Banach sequence lattice on $\mathbb{J}$ and let X be a Banach space. The vector sequence $x = \{x_n\}_{n \in \mathbb{J}}$ in X is called *strongly E-summable* if the corresponding scalar sequence $\{||x_n||_X\}$ is in E. We denote by $E(X)$ the set of all such sequences in X. This is a Banach space under pointwise operations and a natural norm given by

$$||x||_{E(X)} \coloneqq ||\{||x_n||_X\}||_E.$$

Throughout the paper, for any scalar vector-valued sequence $\{x_n\}_{n \in \mathbb{Z}}$ the series with unspecified range of summation $\sum_{n \in \mathbb{Z}} x_n$ will always be interpreted as

$$\sum_{n \in \mathbb{Z}} x_n = \lim_{M,N \to \infty} \sum_{k=-N}^{M} x_k,$$

where the double limit exists in the Pringsheim sense.

Let (X_0, X_1) be any Banach couple, ρ a given quasi-power function and $1 \leqslant p_j \leqslant \infty$ with $j = 0, 1$. We denote by $(X_0, X_1)_{\rho, p_0, p_1}$ the Banach space of all $x \in X_0 + X_1$ which can be represented in the form

$$x = \sum_{n \in \mathbb{Z}} u_n \quad \text{(convergence in } X_0 + X_1)$$

with $\{u_n/\rho(2^n)\} \in \ell_{p_0}(X_0)$ and $\{2^n u_n/\rho(2^n)\} \in \ell_{p_1}(X_1)$. This space is equipped with the norm

$$||x|| = \inf \max\{||\{u_n/\rho(2^n)\}||_{\ell_{p_0}(X_0)}, ||\{2^n u_n/\rho(2^n)\}||_{\ell_{p_1}(X_1)}\},$$

where the infimum is taken over all the above representations of x as in above.

If $\varphi \in \Phi^{+-}$ and $\rho = \varphi(1, \cdot)$, we also write $\varphi(X_0, X_1)_{p_0, p_1}$ instead of $(X_0, X_1)_{\rho, p_0, p_1}$.

Note that in the case when $\rho(s) = s^\theta$ and $0 < \theta < 1$ these spaces were introduced by Lions and Peetre and were called the spaces of means (see [21]).

We will use the following theorem (see [24,26]).

Theorem 2.1. *If (X_0, X_1) is a Banach couple and $\varphi \in \Phi^{+-}$ and $1 \leqslant p_j \leqslant \infty$ for $j = 0, 1$, then*

$$\varphi(X_0, X_1)_{p_0, p_1} = (X_0, X_1)_{\varphi(\ell_{p_0}, \ell_{p_1}(2^{-n}))}.$$

In particular, if $(L_{p_0}(w_0), L_{p_1}(w_1))$ is any couple of the weighted spaces L_p-spaces, then

$$(L_{p_0}(w_0), L_{p_1}(w_1))_{\varphi, p_0, p_1} = \varphi(L_{p_0}(w_0), L_{p_1}(w_1)).$$

Here, as usual, if $\bar{E}=(E_0,E_1)$ is a couple of Banach lattices on (Ω,μ) and $\varphi\in\Phi$ is a concave function (in each variable). Then the Calderón–Lozanovsky space $\varphi(\bar{E})=\varphi(E_0,E_1)$ consists of all $x\in L^0(\mu)$ such that $|x|=\varphi(|x_0|,|x_1|)$ for some $x_j\in E_j$, $j=0,1$. The space $\varphi(\bar{E})$ is a Banach lattice equipped with the norm (see [22])

$$\|x\|:=\inf\{\max\{\|x_0\|_{E_0},\|x_1\|_{E_1}\};\ |x|=\varphi(|x_0|,|x_1|),\ x_0\in E_0,\ x_1\in E_1\}.$$

Here and throughout the paper for positive functions f and g, we write $f\asymp g$ whenever $f\prec g$ and $g\prec f$, where $f\prec g$ means that there is some $c>0$ such that $f\leqslant c\,g$.

Let $\{r_k\}$ be the sequence of Rademacher functions, and assume that $1\leqslant p\leqslant 2\leqslant q<\infty$. A Banach space X is said to have (*Rademacher*) *type p* (resp., (*Rademacher*) *cotype q*) if there exists a constant $C>0$ such that, for any choice of $m\in\mathbb{N}$ and $x_1,\dots,x_m\in X$,

$$\int_0^1\left\|\sum_{k=1}^m r_k(t)x_k\right\|_X dt\leqslant C\left(\sum_{k=1}^m\|x_k\|_X^p\right)^{1/p}$$

(resp., $\int_0^1\|\sum_{k=1}^m r_k(t)x_k\|_X\,dt\geqslant C^{-1}(\sum_{k=1}^m\|x_k\|_X^q)^{1/q}$). We denote by $T_p(X)$ (resp., $C_q(X)$) the smallest constant C for which this holds.

Let $2\leqslant q<\infty$ and $\{\omega_n\}_{n\in\mathbb{Z}}$ be the sequence of functions defined on $\mathbb{T}:=[0,2\pi]$ by $\omega_n(t)=e^{int}$ for any $t\in\mathbb{T}$ and $n\in\mathbb{Z}$. A (complex) Banach space X is said to have the *periodic Fourier cotype q* (cf. [12]) if there exists a constant $K>0$ such that for all sequences $\{x_n\}_{n\in\mathbb{Z}}$ of elements in X with only finitely many non-zero elements,

$$\left(\sum_{n\in\mathbb{Z}}\|x_n\|_X^q\right)^{1/q}\leqslant K\left(\frac{1}{2\pi}\int_{\mathbb{T}}\left\|\sum_{n\in\mathbb{Z}}\omega_n(t)x_n\right\|_X^{q'}dt\right)^{1/q'},$$

where as usual q' is the conjugate exponent to q defined by $1/q+1/q'=1$. We denote $K_q(X)$ the smallest constant K for which the above holds.

Let us recall that if $1<p\leqslant 2$, then the notion of the periodic Fourier cotype p' of the Banach space X coincides with the well-known notion of Fourier type p which is equivalent to the validity of an X-valued Hausdorff–Young inequality. For details we refer to the interesting survey paper [12].

Let us note that on the basis of Contraction Principle (see, e.g., [11, p. 231]) the periodic Fourier cotype q implies the Rademacher cotype q. It is also well-known that the periodic Fourier cotype q implies the Rademacher type q' (see, e.g., [5]). Thus, the result due to Kwapień [18], implies that any Banach space which has the periodic Fourier cotype 2 is isomorphic to a Hilbert space. We see immediately that any Hilbert space H has the periodic Fourier cotype 2 with $K_2(H)=1$. We note that

Bourgain [6] proved that if a Banach space X is of type p for some $p>1$, then it has the periodic Fourier cotype q with $q = 18\, T_p(X)^{p'}$.

In the sequel we will need the following lemma. Before proving it we recall that the series $\sum_{n\in\mathbb{Z}} x_n$ of elements of a Banach space is said to be *weakly conditionally convergent* (resp., *unconditionally convergent*) if for every functional $x^* \in X^*$ the series $\sum_{n\in\mathbb{Z}} |x^*(x_n)|$ is convergent (resp., the series $\sum_{n\in\mathbb{Z}} \varepsilon_n x_n$ converges for all ε_n with $\varepsilon_n = \pm 1$ for $n\in\mathbb{Z}$).

For the basic properties of weakly conditionally and unconditionally convergent series we refer to [11] or [20]. As usual, let $\{e_n\}_{\mathbb{J}}$ be the standard unit vector basis in $c_0(\mathbb{J})$ defined on the set $\mathbb{J}$.

Lemma 2.1. *Let X be a Banach space. Then, for any bounded bilinear operator $T: c_0(\mathbb{Z}) \times c_0(\mathbb{Z}) \to X$ the following statements are true*:

- (i) *If X does not contain an isomorphic copy of c_0, then $\sum_{k\in\mathbb{Z}} T(e_k, e_{m-k})$ is unconditionally convergent series in X for every $m\in\mathbb{Z}$.*
- (ii) *If X has the periodic Fourier cotype q for some $2 \leqslant q < \infty$, then $\{x_m\}_{m\in\mathbb{Z}} \in \ell_q(X)$ where $x_m = \sum_{k\in\mathbb{Z}} T(e_k, e_{m-k})$ for $m\in\mathbb{Z}$.*

Proof. (i) Fix $x^* \in X^*$ and define a bounded bilinear form $A : c_0(\mathbb{Z}) \times c_0(\mathbb{Z}) \to \mathbb{K}$ by $A := x^* \circ T$. Then by Aron et al. [2], we conclude that

$$\sum_{k\in\mathbb{Z}} |A(e_k, e_{m-k})| < \infty,$$

i.e., the series $\sum_{k\in\mathbb{Z}} T(e_k, e_{m-k})$ is weakly conditionally convergent in X. If X does not contain an isomorphic copy of c_0, it follows by the well-known result of Bessaga–Pełczyński (see [4]) that every weakly conditionally convergent series is unconditionally convergent. As a consequence, it follows that the series

$$\sum_{k\in\mathbb{Z}} T(e_k, e_{m-k})$$

is unconditionally convergent in X for every $m\in\mathbb{Z}$.

(ii) Assume that X has the periodic Fourier cotype q for some $2 \leqslant q < \infty$. We have already mentioned that this implies that X has cotype q. Hence, X does not contain a copy of c_0. Further, we note that as an easy consequence of the Orlicz–Pettis Theorem (see, e.g., [11]) every weakly compact operator acting between Banach spaces is unconditionally convergent, i.e., it maps every weakly conditionally convergent series into unconditionally convergent series. By the well-known result of Pełczyński that every bounded linear operator from c_0 into a Banach space X containing no isomorphic copy of c_0 is compact, it follows that $A_k = T(e_k, \cdot) : c_0(\mathbb{Z}) \to X$ is a compact operator. Since the series $\sum_{m\in\mathbb{Z}} e_{m-k}$ converges weakly conditionally in $c_0(\mathbb{Z})$, the series

$$\sum_{m\in\mathbb{Z}} T(e_k, e_{m-k})$$

converges unconditionally in X for every $k\in\mathbb{Z}$.

If we combine the above remarks with the equality

$$\sum_{m\in\mathbb{Z}} \omega_m(t)\, T(e_k, e_{m-k}) = \sum_{m\in\mathbb{Z}} \omega_{k+m}(t)\, T(e_k, e_m)$$

which holds for any $t\in\mathbb{T}$ and $k\in\mathbb{Z}$, we obtain that the following series converge unconditionally in X for any $N\in\mathbb{N}$ and $t\in\mathbb{T}$:

$$\sum_{m\in\mathbb{Z}} \omega_m(t) \left(\sum_{|k|\leqslant N} T(e_k, e_{m-k}) \right) = \sum_{|k|\leqslant N} \left(\sum_{m\in\mathbb{Z}} \omega_m(t)\, T(e_k, e_{m-k}) \right) = \sum_{|k|\leqslant N} \left(\sum_{m\in\mathbb{Z}} \omega_{k+m}(t)\, T(e_k, e_m) \right).$$

Since $\omega_{k+m}(t) = \omega_k(t)\, \omega_m(t)$ for any $t\in\mathbb{T}$, it follows that

$$\left\| \sum_{|k|\leqslant N} \left(\sum_{|m|\leqslant M} \omega_{k+m}(t)\, T(e_k, e_m) \right) \right\|_X \leqslant \sup_{t\in\mathbb{T}} \sup_{M,N\geqslant 1} \left\| T\left(\sum_{|k|\leqslant N} \omega_k(t) e_k, \sum_{|m|\leqslant M} \omega_m(t) e_m \right) \right\|_X \leqslant \|T\|_{c_0(\mathbb{Z})\times c_0(\mathbb{Z})\to X}$$

for any positive integers M and N.

Now fixing $N\in\mathbb{N}$ and combining the previous relations with the Dominated Convergence Theorem yields

$$\left(\sum_{m\in\mathbb{Z}} \left\| \sum_{|k|\leqslant N} T(e_k, e_{m-k}) \right\|_X^q \right)^{1/q} \leqslant K_q(X) \lim_{M\to\infty} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \left\| \sum_{|m|\leqslant M} \omega_m(t) \left(\sum_{|k|\leqslant N} T(e_k, e_{m-k}) \right) \right\|_X^{q'} dt \right)^{1/q'} = K_q(X) \left(\frac{1}{2\pi} \int_{\mathbb{T}} \lim_{M\to\infty} \left\| \sum_{|m|\leqslant M} \omega_m(t) \left(\sum_{|k|\leqslant N} T(e_k, e_{m-k}) \right) \right\|_X^{q'} dt \right)^{1/q'}$$

$$= K_q(X)\left(\frac{1}{2\pi}\int_{\mathbb{T}} \lim_{M\to\infty}\left\|\sum_{|k|\leqslant N}\left(\sum_{|m|\leqslant M}\omega_{k+m}(t)\,T(e_k,e_{m-k})\right)\right\|_X^{q'} dt\right)^{1/q'}$$
$$\leqslant K_q(X)\,\|T\|_{c_0(\mathbb{Z})\times c_0(\mathbb{Z})\to X}$$

for any $N\in\mathbb{N}$. Since the series $\sum_{k\in\mathbb{Z}} T(e_k,e_{m-k})$ converges for every $m\in\mathbb{Z}$, we immediately deduce that

$$\left(\sum_{m\in\mathbb{Z}}\|x_m\|_X^q\right)^{1/q}\leqslant K_q(X)\,\|T\|_{c_0(\mathbb{Z})\times c_0(\mathbb{Z})\to X}$$

and the result follows. □

We will need one further preliminary technical lemma.

Lemma 2.2. *Let X be a Banach space and let $\{x_{k,m}\}$ be an infinite matrix in X with $k,m\in\mathbb{Z}$. If the series $\sum_{k\in\mathbb{Z}} x_{k,m-k}$ is unconditionally convergent for every m in $\mathbb{Z}$ and $\sum_{m\in\mathbb{Z}}\|\sum_{k\in\mathbb{Z}} x_{k,m-k}\|_X<\infty$, then the double limit $\lim_{M,N\to\infty}\sum_{|k|\leqslant M}\sum_{|j|\leqslant N}x_{k,j}$ exists in X and*

$$\lim_{M,N\to\infty}\sum_{|k|\leqslant M}\sum_{|j|\leqslant N}x_{k,j}=\sum_{m\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}x_{k,m-k}\right).$$

Proof. Let $u_m := \sum_{k\in\mathbb{Z}} x_{k,m-k}$ for $m\in\mathbb{Z}$. Fix $\varepsilon>0$. Since $\sum_{m\in\mathbb{Z}}\|u_m\|_X<\infty$ and the series $\sum_{k\in\mathbb{Z}} x_{k,m-k}$ converges unconditionally, there exists $m_0\in\mathbb{N}$ such that

$$\left\|\sum_{|m|\leqslant m_0}u_m-\sum_{m\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}x_{k,m-k}\right)\right\|_X<\varepsilon/2$$

and

$$\sum_{|m|>m_0}\left\|\sum_{k\in F_m}x_{k,m-k}\right\|_X<\varepsilon/4,$$

where F_m is any finite subset of $\mathbb{Z}$. Further, there exists $k_0\in\mathbb{N}$ such that for any $k_m\geqslant k_0$ with $|m|\leqslant m_0$, we have

$$\sum_{|m|\leqslant m_0}\left\|\sum_{|k|>k_m}x_{k,m-k}\right\|_X<\varepsilon/4.$$

Let M and N be positive integers with $M > k_0$ and $N > m_0 + k_0$. Let us define two subsets A and B of $\mathbb{Z} \times \mathbb{Z}$ by

$$A = \{(k,j);\ |k| \leqslant M,\ |j| \leqslant N\}$$

and

$$B = \{(k,j);\ |k| \leqslant k_0,\ |j+k| \leqslant m_0\}.$$

For $m \in \mathbb{Z}$ we let $F_m = \{k \in \mathbb{Z};\ (k, m-k) \in A\}$ and $k_m = \max\{k \in \mathbb{Z};\ k \in F_m\}$. Since $B \subset A$, we conclude that $k_m \geqslant k_0$, whenever $|m| \leqslant m_0$. This implies that the above inequalities hold for F_m and k_m defined as in above. It is easy to verify that

$$\sum_{|k| \leqslant M} \sum_{|j| \leqslant N} x_{k,j} = \sum_{|m| \leqslant m_0} u_m - \sum_{|m| \leqslant m_0} \sum_{|k| > k_m} x_{k,m-k} + \sum_{|m| > m_0} \sum_{k \in F_m} x_{k,m-k}.$$

If we combine this equality with the three inequalities above, we obtain

$$\left\| \sum_{|k| \leqslant M} \sum_{|j| \leqslant N} x_{k,j} - \sum_{m \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} x_{k,m-k} \right) \right\|_X < \varepsilon$$

for $M > k_0$ and $N > m_0 + k_0$. This proves the assertion. $\square$

Theorem 2.2. *Let (X_0, X_1) be a couple of Banach spaces such that X_j for $j = 0, 1$ has the periodic Fourier cotype for some $2 \leqslant q_j < \infty$. Assume that the quasi-concave functions ρ_0, ρ_1 and ρ with ρ a quasi-power function are such that for some $C > 0$, we have $C\rho(st) \geqslant \rho_0(s)\,\rho_1(t)$ for all $s, t > 0$. If $T_0 : c_0 \times c_0 \to X_0$ and $T_1 : c_0(2^{-n}) \times c_0(2^{-n}) \to X_1$ are bounded bilinear operators such that $T_0(x,y) = T_1(x,y)$ for any finitely supported sequences x and y on $\mathbb{Z}$, then for any $x = \{\xi_n\}_{n \in \mathbb{Z}} \in \ell_\infty(1/\rho_0(2^n))$ and $y = \{\eta_n\}_{n \in \mathbb{Z}} \in \ell_\infty(1/\rho_1(2^n))$ there exists a double limit*

$$S(x,y) := \lim_{M,N \to \infty} \sum_{|k| \leqslant N} \sum_{|m| \leqslant M} \xi_k\, \eta_m\, T_0(e_k, e_m)$$

in $X_0 + X_1$ and it defines a bounded bilinear operator S from $\ell_\infty(1/\rho_0(2^n)) \times \ell_\infty(1/\rho_1(2^n))$ into $(X_0, X_1)_{\rho, q_0, q_1}$ with the norm

$$\|S\| \leqslant C \max\{K_{q_0}(X_0)\|T_0\|_{c_0 \times c_0 \to X_0}, K_{q_1}(X_1)\|T_1\|_{c_0(2^{-n}) \times c_0(2^{-n}) \to X_1}\}.$$

Proof. Fix norm one elements $x = \{\xi_n\} \in E_0 := \ell_\infty(1/\rho_0(2^n))$ and $y = \{\eta_n\} \in E_1 := \ell_\infty(1/\rho_1(2^n))$. We prove that the double limit

$$\lim_{M,N \to \infty} \sum_{|k| \leqslant N} \sum_{|j| \leqslant M} \xi_k\, \eta_j\, T_0(e_k, e_j)$$

exists in $X_0 + X_1$. First, we show that

$$\sum_{m \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} \xi_k \, \eta_{m-k} \, T_0(e_k, e_{m-k}) \right\|_{X_0+X_1} < \infty .$$

To see this, we first observe that for every $m \in \mathbb{Z}$ the series

$$\sum_{k \in \mathbb{Z}} \xi_k \, \eta_{m-k} \, T_0(e_k, e_{m-k})$$

converges in both X_0 and X_1. In fact, it follows immediately from $||x||_{E_0} \leqslant 1$, $||y||_{E_1} \leqslant 1$ and our hypothesis $C\rho(st) \geqslant \rho_0(s)\rho_1(t)$ for $s, t > 0$, that $|c_{k,m}| \leqslant C$ where

$$c_{k,m} := \frac{\xi_k \eta_{m-k}}{\rho(2^m)} \quad \text{for any } (k,m) \in \mathbb{Z}^2,$$

while we can apply Lemma 2.1 to obtain that the series $\sum_{k \in \mathbb{Z}} T_0(e_k, e_{m-k})$ is unconditionally convergent in X_0 for every $m \in \mathbb{Z}$. Since

$$\frac{1}{\rho(2^m)} \sum_{k \in \mathbb{Z}} \xi_k \eta_{m-k} T_0(e_k, e_{m-k}) = \sum_{k \in \mathbb{Z}} c_{k,m} T_0(e_k, e_{m-k}),$$

we get the required assertion. Combining the above remarks with Lemma 2.1(ii) yields

$$\begin{aligned}
&\left(\sum_{m \in \mathbb{Z}} \left(\frac{1}{\rho(2^m)} \left\| \sum_{k \in \mathbb{Z}} \xi_k \eta_{m-k} T_0(e_k, e_{m-k}) \right\|_{X_0} \right)^{q_0} \right)^{1/q_0} \\
&\quad = \left(\sum_{m \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} c_{k,m} T_0(e_k, e_{m-k}) \right\|_{X_0}^{q_0} \right)^{1/q_0} \\
&\quad \leqslant C \left(\sum_{m \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}} T_0(e_k, e_{m-k}) \right\|_{X_0}^{q_0} \right)^{1/q_0} \leqslant C K_{q_0}(X_0) ||T_0||_{c_0 \times c_0 \to X_0}.
\end{aligned}$$

Further, we note that a bounded bilinear operator $U : c_0 \times c_0 \to X_1$ defined by

$$U(\{\xi_n\}, \{\eta_n\}) := T_1(\{2^n \xi_n\}, \{2^n \eta_n\})$$

for $(\{\xi_n\}, \{\eta_n\}) \in c_0 \times c_0$ satisfies

$$||U||_{c_0 \times c_0 \to X_1} \leqslant ||T_1||_{c_0(2^{-n}) \times c_0(2^{-n}) \to X_1}.$$

Thus repeated use the above arguments gives the following estimates:

$$\begin{aligned}
&\left(\sum_{m\in\mathbb{Z}}\left(\frac{2^m}{\rho(2^m)}\left\|\sum_{k\in\mathbb{Z}}\xi_k\eta_{m-k}T_1(e_k,e_{m-k})\right\|_{X_1}\right)^{q_1}\right)^{1/q_1}\\
&\leqslant\left(\sum_{m\in\mathbb{Z}}\left\|\sum_{k\in\mathbb{Z}}c_{k,m}T_1(2^ke_k,2^{m-k}e_{m-k})\right\|_{X_1}^{q_1}\right)^{1/q_1}\\
&\leqslant C\left(\sum_{m\in\mathbb{Z}}\left\|\sum_{k\in\mathbb{Z}}U(e_k,e_{m-k})\right\|_{X_1}^{q_1}\right)^{1/q_1}\\
&\leqslant CK_{q_1}(X_1)\|T_1\|_{c_0(2^{-n})\times c_0(2^{-n})\to X_1}.
\end{aligned}$$

Since $T_0(e_k,e_{m-k})=T_1(e_k,e_{m-k})$ for all $(k,m)\in\mathbb{Z}^2$, the above estimates show that the sequence $\{u_m\}_{m\in\mathbb{Z}}$ with $u_m:=\sum_{k\in\mathbb{Z}}\xi_k\eta_{m-k}\,T_0(e_k,e_{m-k})$ satisfies the following inequalities:

$$\|\{u_m/\rho(2^m)\}\|_{\ell_{q_0}(X_0)}\leqslant C_0$$

and

$$\|\{2^m\,u_m/\rho(2^m)\}\|_{\ell_{q_1}(X_1)}\leqslant C_1,$$

where $C_j:=C\,K_{q_j}(X_j)\|T_j\|_{c_0(2^{-jn})\times c_0(2^{-jn})\to X_j}$ for $j=0,1$. Our hypothesis that ρ is a quasi-power function easily implies that

$$\sum_{m\in\mathbb{Z}}\|u_m\|_{X_0+X_1}<\infty.$$

Recall that for any $m\in\mathbb{Z}$ the series $\sum_{k\in\mathbb{Z}}\xi_k\eta_{m-k}T_0(e_k,e_{m-k})$ converges unconditionally in both X_0 and X_1, and thus also in X_0+X_1. Applying Lemma 2.2, we conclude that the double limit

$$S(x,y):=\lim_{M,N\to\infty}\sum_{|k|\leqslant M}\sum_{|j|\leqslant N}\xi_k\,\eta_j\,T_0(e_k,e_j)$$

exists in X_0+X_1. This immediately implies that S is a bilinear map from $E_0\times E_1$ into X_0+X_1. Since

$$S(x,y)=\sum_{m\in\mathbb{Z}}u_m\quad\text{(convergence in } X_0+X_1),$$

we obtain by the above obtained estimates that $S(x,y)\in(X_0,X_1)_{\rho,q_0,q_1}$ for any $(x,y)\in E_0\times E_1$ with $\|x\|_{E_0}\leqslant 1$ and $\|y\|_{E_1}\leqslant 1$. In consequence, we have proved that

$$S\colon \ell_\infty(1/\rho_0(2^n))\times\ell_\infty(1/\rho_1(2^n))\to(X_0,X_1)_{\rho,q_0,q_1}$$

is a bounded bilinear operator with $\|T\|\leqslant\max\{C_0,C_1\}$. This finishes the proof. □

3. Abstract bilinear interpolation

In this section we prove general results on interpolation of bilinear operators. We will need some further interpolation space concepts. Recall that the mapping F from the category of all couples of Banach spaces into the category of all Banach spaces is said to be an *interpolation functor* if for any Banach couple $\bar{X}$, $F(\bar{X})$ is an intermediate Banach space with respect to $\bar{X}$, and for any $T\in\mathscr{L}(\bar{X},\bar{Y})$ it follows that $T:F(\bar{X})\to F(\bar{Y})$. If additionally

$$\|T\|_{F(\bar{X})\to F(\bar{Y})}\leqslant\max\{\|T\|_{X_0\to Y_0},\|T\|_{X_1\to Y_1}\}$$

holds, then F is called an *exact* interpolation functor.

We will consider particular cases of the following two constructions by Aronszajn and Gagliardo [1] of exact interpolation functors. Given A and $\bar{A}$, such that A is an intermediate space with respect to the couple $\bar{A}$, two interpolation functors are defined by

$$G(\bar{X})=G_A^{\bar{A}}(\bar{X}):=\left\{\sum_{n=1}^{\infty}T_na_n;\ \sum_{n=1}^{\infty}\|T_n\|_{\bar{A}\to\bar{X}}\|a\|_A<\infty\right\}$$

and

$$H(\bar{X})=H_A^{\bar{A}}(\bar{X}):=\{x\in X_0+X_1;\ Tx\in A \text{ for every } T:\bar{X}\to\bar{A}\}.$$

The norms are given by

$$\|x\|_{G(\bar{X})}=\inf\left\{\sum_{n=1}^{\infty}\|T_n\|_{\bar{A}\to\bar{X}}\|a\|_A;\ x=\sum_{n=1}^{\infty}T_na_n\right\}$$

and, respectively,

$$\|x\|_{H(\bar{X})}=\sup\{\|Tx\|_A;\ \|T\|_{\bar{X}\to\bar{A}}\leqslant 1\}.$$

Note that G is the *minimal* interpolation functor satisfying $A\hookrightarrow G(\bar{A})$ and H is the *maximal* interpolation functor satisfying $H(\bar{A})\hookrightarrow A$.

The interpolation functor $G_A^{\bar{A}}$ is called *approximable* (cf. [15]) on a Banach couple $\bar{X}$ with a constant $c>0$ if for any $x \in X_0 \cap X_1$, we have

$$x = \sum_{n=1}^{N} S_n a_n$$

for some positive integer N, $a_n \in A_0 \cap A_1$, $n = 1, \dots, N$ and $S_n : \bar{A} \to \bar{X}$ satisfying

$$\sum_{n=1}^{N} ||S_n||_{\bar{A} \to \bar{X}} ||a_n||_A \leqslant c||x||_{G_A^{\bar{A}}(\bar{X})}.$$

For a study of approximable interpolation functor we refer to [17].

In what follows, we need a definition of interpolation bilinear theorems. Let $\bar{X} = (X_0, X_1)$, $\bar{Y} = (Y_0, Y_1)$ and $\bar{Z} = (Z_0, Z_1)$ be Banach couples. We will say that (T_0, T_1) is a bilinear operator from $\bar{X} \times \bar{Y}$ into $\bar{Z}$, and write $T = (T_0, T_1) \in \mathscr{B}(\bar{X}, \bar{Y}; \bar{Z})$ if $T_j : X_j \times Y_j \to Z_j$ is a bounded bilinear operator $(j = 0, 1)$ and $T_0(x, y) = T_1(x, y)$ for any $x \in X_0 \cap X_1$ and $y \in Y_0 \cap Y_1$.

It is easy to check that $\mathscr{B}(\bar{X}, \bar{Y}; \bar{Z})$ is a Banach space equipped with the norm

$$||(T_0, T_1)||_{\bar{X} \times \bar{Y} \to \bar{Z}} \coloneqq \max_{j=0,1} ||T_j||_{X_j \times Y_j \to Z_j}.$$

Note that any $(T_0, T_1) \in \mathscr{B}(\bar{X}, \bar{Y}; \bar{Z})$ defines a bilinear map T which is called in the sequel a *natural bounded bilinear extension* of (T_0, T_1) from $\Delta(\bar{X}) \times \Sigma(\bar{Y}) \to \Sigma(\bar{Z})$ by

$$T(x, y) \coloneqq T_0(x, y_0) + T_1(x, y_1)$$

for any $x \in \Delta(\bar{X})$ and $y = y_0 + y_1 \in \Sigma(\bar{Y})$ with $y_0 \in Y_0$ and $y_1 \in Y_1$. It is easy to see that T does not depend on the representation of $y \in \Sigma(\bar{Y})$ and T is bounded with

$$||T||_{\Delta(\bar{X}) \times \Sigma(\bar{Y}) \to \Sigma(\bar{Z})} \leqslant ||(T_0, T_1)||_{\bar{X} \times \bar{Y} \to \bar{Z}}.$$

Assume that X, Y and Z are Banach spaces intermediate with respect to the couples $\bar{X}$, $\bar{Y}$ and $\bar{Z}$. We say that a triple of Banach spaces $(X, Y; Z)$ is a bilinear interpolation with respect to $(\bar{X}, \bar{Y}; \bar{Z})$, and write $(X, Y; Z) \in \mathrm{Int}(\bar{X}, \bar{Y}; \bar{Z})$ if for every $(T_0, T_1) \in \mathscr{B}(\bar{X}, \bar{Y}; \bar{Z})$ there is a constant C_T such that a natural bounded bilinear extension T of (T_0, T_1) satisfies

$$||T(x, y)||_Z \leqslant C_T ||x||_X ||y||_Y$$

for any $(x, y) \in \Delta(\bar{X}) \times Y$. If $C_T = C$ does not depend on T, we write $(X, Y; Z) \in \mathrm{Int}_C(\bar{X}, \bar{Y}; \bar{Z})$ and say that C is a constant of the bilinear interpolation.

It is shown in [25] that if $(X, Y; Z) \in \mathrm{Int}(\bar{X}, \bar{Y}; \bar{Z})$, then there is a constant $C>0$ such that $(X, Y; Z) \in \mathrm{Int}_C(\bar{X}, \bar{Y}, \bar{Z})$.

For simplicity, we shall use the following notation for interpolation functors F_0, F_1 and F_2:

$$F_0(\bar{X}) \otimes F_1(\bar{Y}) \to F_2(\bar{Z}),$$

and say that a bilinear interpolation theorem holds, provided that

$$(F_0(\bar{X}), F_1(\bar{Y}); F_2(\bar{Z})) \in \mathrm{Int}(\bar{X}, \bar{Y}; \bar{Z}).$$

If the above relation holds for all Banach couples $\bar{X}$, $\bar{Y}$ and $\bar{Z}$, we write in short

$$F_0 \otimes F_1 \to F_2.$$

We are now ready to prove the following general theorem. We note that a variant of multilinear result holds (for details we refer to [25]).

Theorem 3.1. *Assume that $(A, B; C) \in \mathrm{Int}_M(\bar{A}, \bar{B}; \bar{C})$. If the interpolation functor $G_A^{\bar{A}}$ is approximable on a Banach couple $\bar{X}$ with a constant $c > 0$, then the following bilinear interpolation theorem*:

$$G_A^{\bar{A}}(\bar{X}) \otimes G_B^{\bar{B}}(\bar{Y}) \to H_C^{\bar{C}}(\bar{Z})$$

with the constant equals cM holds for any Banach couples $\bar{Y}$ and $\bar{Z}$.

Proof. Assume that $(T_0, T_1) \in \mathscr{B}(\bar{X}, \bar{Y}; \bar{Z})$ with $\|(T_0, T_1)\|_{\bar{X} \times \bar{Y} \to \bar{Z}} \leqslant 1$. Fix operators $S_0 : \bar{A} \to \bar{X}$, $S_1 : \bar{B} \to \bar{Y}$ and $R : \bar{Z} \to \bar{C}$ with $\|R\|_{\bar{Z} \to \bar{C}} \leqslant 1$. Define operators U_j by

$$U_j(a, b) := RT_j(S_0 a, S_1 b)$$

for $(a, b) \in A_j \times B_j$ and $j = 0, 1$. We have $(U_0, U_1) \in \mathscr{B}(\bar{A}, \bar{B}; \bar{C})$ with

$$\|(U_0, U_1)\|_{\bar{A} \times \bar{B} \to \bar{C}} \leqslant \|S_0\|_{\bar{A} \to \bar{X}} \|S_1\|_{\bar{B} \to \bar{Y}}.$$

Let $U : \Delta(\bar{A}) \times \Sigma(\bar{B}) \to \Sigma(\bar{C})$ be a natural bounded bilinear extension of (U_0, U_1). It is easy to see that

$$U(a, b) = RT(S_0 a, S_1 b)$$

for any $(a, b) \in \Delta(\bar{A}) \times \Sigma(\bar{B})$, where $T : \Delta(\bar{X}) \times \Sigma(\bar{Y}) \to \Sigma(\bar{Z})$ is a natural bounded bilinear extension of (T_0, T_1). Since $(A, B; C) \in \mathrm{Int}_M(\bar{A}, \bar{B}; \bar{C})$, we have

$$\|U(a, b)\|_C \leqslant M \|S_0\|_{\bar{A} \to \bar{X}} \|S_1\|_{\bar{B} \to \bar{Y}} \|a\|_A \|b\|_B$$

for any $(a, b) \in (A_0 \cap A_1) \times B$.

Fix $\varepsilon > 0$. Then, by our hypothesis on $G_A^{\bar{A}}$ it follows that for any $x \in X_0 \cap X_1$ and $y \in G_B^{\bar{B}}(\bar{Y})$ we have

$$x = \sum_{n=1}^{N} S_{0n} a_n, \quad y = \sum_{m=1}^{\infty} S_{1m} b_m$$

for some positive integer N and $a_n \in A_0 \cap A_1$, $n = 1, \dots, N$ and, $b_m \in B$, $S_{0n} : \bar{A} \to \bar{X}$, $S_{1m} : \bar{B} \to \bar{Y}$ satisfying

$$\sum_{n=1}^{N} ||S_{0n}||_{\bar{A} \to \bar{X}} ||a_n||_A \leqslant c||x||_{G_A^{\bar{A}}(\bar{X})}$$

and

$$\sum_{m=1}^{\infty} ||S_{0m}||_{\bar{B} \to \bar{Y}} ||b_m||_B \leqslant (1 + \varepsilon)||y||_{G_B^{\bar{B}}(\bar{Y})}.$$

This implies that for any $R : \bar{Z} \to \bar{C}$ with $||R||_{\bar{Z} \to \bar{C}} \leqslant 1$, we have

$$RT(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{\infty} RT(S_{0n} a_n, S_{1m} b_m) = \sum_{n=1}^{N} \sum_{m=1}^{\infty} U_{nm}(a_n, b_m),$$

where $U_{nm} : \Delta(\bar{A}) \times \Sigma(\bar{B}) \to \Sigma(\bar{C})$ is a natural bounded bilinear extension of (U_{0n}, U_{1m}) (defined in a similar way as U in above) satisfying

$$||U_{nm}(a, b)||_C \leqslant M||a||_A ||b||_B$$

for any $a \in A_0 \cap A_1$ and $b \in B$.

In consequence, the above remarks and estimates implies

$$\begin{aligned}
||RT(x, y)||_C &\leqslant \sum_{n=1}^{N} \sum_{m=1}^{\infty} ||U_{nm}(a_n, b_m)||_C \\
&\leqslant (1 + \varepsilon) M \sum_{n=1}^{N} \sum_{m=1}^{\infty} ||S_{0n}||_{\bar{A} \to \bar{X}} ||a_n||_A ||S_{1m}||_{\bar{B} \to \bar{Y}} ||b_m||_B \\
&= (1 + \varepsilon) M \left(\sum_{n=1}^{N} \sum_{m=1}^{\infty} ||S_{0n}||_{\bar{A} \to \bar{X}} \, ||a_n||_A \right) \left(\sum_{m=1}^{\infty} ||S_{1m}||_{\bar{B} \to \bar{Y}} \, ||b_m||_B \right) \\
&\leqslant (1 + \varepsilon) c M ||x||_{G_A^{\bar{A}}(\bar{X})} ||y||_{G_B^{\bar{B}}(\bar{Y})}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\begin{aligned}||T(x,y)||_{H_C^{\bar{C}}(\bar{Z})} &= \sup\{||R(T(x,y))||_C;\, ||R||_{\bar{Z}\to\bar{C}} \leqslant 1\}\\ &\leqslant cM||x||_{G_A^{\bar{A}}(\bar{X})}||y||_{G_B^{\bar{B}}(\bar{Y})}\end{aligned}$$

for $x \in X_0 \cap X_1$ and $y \in G_B^{\bar{B}}(\bar{Y})$. This completes the proof. □

We present some applications of the obtained results for special interpolations functors. Throughout the rest of the paper, for a given function $\varphi \in \Phi_0$ we let $G_\varphi := G_A^{\bar{A}}$ with $A := \ell_\infty(\varphi^*(1, 2^{-n}))$ and $\bar{A} := (c_0, c_0(2^{-n}))$ defined on the set of integers $\mathbb{Z}$. We note that G_φ coincides with the so called $\pm$ method of interpolation introduced by Gustavsson–Peetre [14] (see [15] or [28, p. 468]).

In the sequel, we write that the functions φ_0, φ_1 and φ in Φ satisfy the (generalized super-multiplicativity) condition (SM) if there exists a constant $C > 0$ such that

$$\varphi_0(1,s)\,\varphi_1(1,t) \leqslant C\varphi(1,st)$$

holds for all $s > 0$ and $t > 0$. If the above condition (SM) holds with $\varphi = \varphi_0 = \varphi_1$, then φ is called *super-multiplicative*.

Theorem 3.2. *Let the functions φ_0, φ_1 and φ in Φ_0 with φ_0 being non-degenerate and $\varphi \in \Phi^{+-}$ being a concave function satisfy condition* (SM). *Assume that $\bar{Z} = (Z_0, Z_1)$ is a Banach couple such that Z_j has the periodic Fourier cotype q_j for some $2 \leqslant q_j < \infty$, $j = 0, 1$. Then the following bilinear interpolation theorem*:

$$G_{\varphi_0}(\bar{X}) \otimes G_{\varphi_1}(\bar{Y}) \to (Z_0, Z_1)_{\varphi(\ell_{q_0}, \ell_{q_1}(2^{-n}))}$$

holds for any Banach couples $\bar{X}$ and $\bar{Y}$.

Proof. An immediate consequence of Theorem 2.2 is that the triple

$$(\ell_\infty(\varphi_0^*(1,2^{-n})), \ell_\infty(\varphi_1^*(1,2^{-n})); \bar{Z}_{\varphi(\ell_{q_0}, \ell_{q_1}(2^{-n}))})$$

is a bilinear interpolation with respect to

$$((c_0, c_0(2^{-n})), (c_0, c_0(2^{-n})); \bar{Z}).$$

Further, under our hypothesis on φ_0, the functor G_{φ_0} is approximable on any Banach couple (see [15] or [28, pp. 451–452] or [17]). By Theorems 2.2 and 3.1, this completes the proof. □

In what follows, for any $\varphi \in \Phi$ and $1 \leqslant p < \infty$ we let $H_{\varphi,p}$ be the maximal interpolation functor $H_A^{\bar{A}}$ with $\bar{A} := (\ell_p, \ell_p(2^n))$ and $A := \ell_p(\varphi^*(1, 2^n))$ defined on $\mathbb{Z}$.

If $p = 1$, then $H_{\varphi,1}$ is the upper Ovchinnikov functor φ_u (see [28]) which is denoted in short by H_φ.

Theorem 3.3. *Assume that the functions* φ_0, φ_1 *and* φ *in* Φ_0 *with* φ_0 *being non-degenerate and* $\varphi \in \Phi^{+-}$ *satisfy the condition* (SM). *Then the following bilinear interpolation theorem holds*:

$$G_{\varphi_0} \otimes G_{\varphi_1} \to H_{\varphi,2}.$$

Proof. It follows by Theorem 2.1 that for any $\varphi \in \Phi^{+-}$,

$$(\ell_2, \ell_2(2^n))_{\varphi,2,2} = \varphi(\ell_2, \ell_2(2^n)) = \ell_2(\varphi^*(1, 2^n)).$$

Since any Hilbert space has the periodic Fourier cotype 2, the argumentation used in the prove of the Theorem 3.2 completes the proof. □

In order to present more general applications, we will need first a preparatory theorem which is probably of independent interest.

Theorem 3.4. *Assume that* $\bar{X} = (X_0, X_1)$ *is a Banach couple and* $\varphi \in \Phi$. *Then the following statements are true*:

(i) $H_\varphi(\bar{X}) \hookrightarrow H_{\varphi,2}(\bar{X})$.
(ii) *If* $\varphi \in \Phi_0$ *and both* X_0^* *and* X_1^* *have cotype* 2, *then* $H_{\varphi,2}(\bar{X}) \hookrightarrow H_\varphi(\bar{X})$ *with the constant of embedding depending only on the cotype* 2*-constants of* X_0^* *and* X_1^*.

Proof. (i) Since $H_{\varphi,2}$ is a maximal interpolation functor and

$$H_\varphi(\ell_2, \ell_2(2^n)) = \varphi(\ell_2, \ell_2(2^n)) = \ell_2(\varphi^*(1, 2^n)),$$

the required continuous inclusion follows.

(ii) We use the abstract Grothendieck theorem of Pisier. It states (see [29, Theorem 4.1]) that if X, Y are Banach spaces, X^* and Y have cotype 2 and $T : X \to Y$ is approximable operator, then for a given $\varepsilon > 0$ there is a Hilbert space $\mathscr{H}$ and operators $V : X \to \mathscr{H}$, $U : \mathscr{H} \to Y$ such that $T = UV$, and

$$||U|| \, ||V|| \leqslant (1 + \varepsilon)(2C_2(X^*)C_2(Y))^{3/2}.$$

Fix $x \in H_{\varphi,2}(\bar{X})$. To show that $x \in H_\varphi(\bar{X})$ we need to prove that for any operator $T : \bar{X} \to (\ell_1, \ell_1(2^n))$, we have $Tx \in \ell_1(\varphi^*(1, 2^n))$. Since both spaces ℓ_1 and $\ell_1(2^n)$ with bases have cotype 2, $T : X_j \to \ell_1(2^{jn})$ is an approximable operator ($j = 0, 1$). Thus, combining the abstract Grothendieck theorem of Pisier with Lemma 11.1.1 in [28], we conclude that there exist a Hilbert couple $\bar{H} = (\mathscr{H}_0, \mathscr{H}_1)$ and operators

$$V : (X_0, X_1) \to (\mathscr{H}_0, \mathscr{H}_1) \quad \text{and} \quad U : (\mathscr{H}_0, \mathscr{H}_1) \to (\ell_1, \ell_1(2^n))$$

such that $T = UV$. Since $x \in H_{\varphi,2}(\bar{X})$, $K(t,x;\bar{X}) \asymp \varphi(1,t)$ for any $t>0$, $K(s,t,x;\bar{X}) \in \Phi_0$ by $\varphi \in \Phi_0$. In consequence $K(s,t,Vx;\bar{\mathscr{H}}) \in \Phi_0$. This yields that there exists $\xi \in \ell_2 + \ell_2(2^n)$ such that

$$K(t,\xi;\ell_2,\ell_2(2^n)) \asymp K(t,Vx;\bar{\mathscr{H}}).$$

By Sedaev's result [31], we know that interpolation between Hilbert couples is described by the K-method of interpolation. Thus, we can construct operators

$$S_1 : (\mathscr{H}_0,\mathscr{H}_1) \to (\ell_2,\ell_2(2^n)) \quad \text{and} \quad S_2 : (\ell_2,\ell_2(2^n)) \to (\mathscr{H}_0,\mathscr{H}_1)$$

such that $||S_j|| \leqslant \sqrt{2}$ for $j = 0, 1$ and

$$Vx = S_2\, S_1(Vx).$$

Since $S_1 V : (X_0, X_1) \to (\ell_2,\ell_2(2^n))$ and $x \in H_{\varphi,2}(\bar{X})$,

$$S_1(Vx) \in \ell_2(\varphi^*(1,2^n)) = H_\varphi(\ell_2,\ell_2(2^n)).$$

Further, $US_2 : (\ell_2,\ell_2(2^n)) \to (\ell_1,\ell_1(2^n))$ implies by interpolation property that

$$US_2 : H_\varphi(\ell_2,\ell_2(2^n)) \to H_\varphi(\ell_1,\ell_1(2^n)) = \ell_1(\varphi^*(1,2^n)).$$

Combining the above relations, we obtain

$$Tx = US_2(S_1 Vx) \in \ell_1(\varphi^*(1,2^n))$$

and in consequence $x \in H_\varphi(\bar{X})$. $\square$

We can now state main applications of Theorems 3.3 and 3.4.

Theorem 3.5. *Assume that the functions φ_0, φ_1 and φ in Φ_0 with φ_0 non-degenerate and $\varphi \in \Phi^{+-}$ satisfy condition* (SM). *Assume that $\bar{Z} = (Z_0, Z_1)$ is a Banach couple such that both Z_0^* and Z_1^* have cotype* 2. *Then the following bilinear interpolation theorem*:

$$G_{\varphi_0}(\bar{X}) \otimes G_{\varphi_1}(\bar{Y}) \to H_\varphi(\bar{Z})$$

holds for any Banach couples $\bar{X}$ and $\bar{Y}$.

Let us show an application of this result for Calderón–Lozanovsky spaces. We do not discuss here applications to concrete bilinear operators. The interested reader may consult the results on boundedness of a large class of bilinear operators on a product of L_p-spaces presented, e.g. in [8] or [13] in order to get via the bilinear interpolation Theorem 3.7 the results on boundedness of these operators on a product of Orlicz spaces.

Theorem 3.6. *Assume that the concave functions φ_0, φ_1 and φ in Φ_0 with φ_0 being non-degenerate and $\varphi \in \Phi^{+-}$ satisfy condition* (SM). *Assume that $\bar{Z} = (Z_0, Z_1)$ is a couple of maximal Banach lattices such that both Z^* and Z_1^* have cotype* 2. *Then the following bilinear interpolation theorem*:

$$\varphi_0(\bar{X}) \otimes \varphi_1(\bar{Y}) \to \varphi(\bar{Z})$$

holds for any Banach couples of lattices $\bar{X}$ and $\bar{Y}$.

Proof. It is well-known that if $\varphi \in \Phi_0$ is a concave function, then

$$\varphi(E_0, E_1) \hookrightarrow G_\varphi(E_0, E_1)$$

for any couple (E_0, E_1) of Banach lattices (see [27] or [28, p. 453]). Since $H_\varphi(\bar{Z}) = \varphi(\bar{Z})$ whenever Z_j is maximal $(j = 0, 1)$ (see [28, pp. 474–475]), Theorem 3.5 applies. □

We note that by the well-known description of Calderón–Lozanovsky construction for any weighted L_p spaces (see, e.g., [27]) the application of the above theorem yields a bilinear interpolation theorem for the products of Musielak–Orlicz spaces with respect to the products of weighted L_p spaces. For the sake of completeness we recall that if $1 \leqslant p_0 < p_1 \leqslant \infty$ and $(L_{p_0}(w_0), L_{p_1}(w_1))$ is a Banach couple of weighted L_p spaces on a measure space (Ω, μ), then for any concave $\varphi \in \Phi$, we have

$$\varphi(L_{p_0}(w_0), L_{p_1}(w_1)) = L_{\mathscr{M}},$$

where $L_{\mathscr{M}}$ is the Musielak–Orlicz space generated by the function

$$\mathscr{M}(u, t) := M((w_1(t)^{1/p_0} w_0(t)^{-1/p_1})^q u)(w_0(t)/w_1(t))^q$$

for $u \geqslant 0$ and $t \in \Omega$ with $1/q = 1/p_0 - 1/p_1$ and $M^{-1}(s) = \varphi(s^{1/p_0}, s^{1/p_1})$ for $s \geqslant 0$ (see [27]) equipped with the norm

$$||x|| := \inf\left\{\lambda > 0;\ \int_\Omega \mathscr{M}(|x(t)|/\lambda, t)\, d\mu \leqslant 1\right\}.$$

Combining the above remarks with the well-known fact that any L_p-space with $1 \leqslant p \leqslant 2$ has cotype 2 (see, e.g., [20]), we obtain the following bilinear theorem for Orlicz spaces.

Theorem 3.7. *Assume that the concave functions φ_0, φ_1 and φ in Φ_0 with φ_0 being non-degenerate and $\varphi \in \Phi^{+-}$ satisfy condition* (SM). *Let $1 \leqslant p_0 < p_1 \leqslant \infty$, $1 \leqslant q_0 < q_1 \leqslant \infty$ and $2 \leqslant r_0 < r_1 < \infty$ and let $\mathscr{M}_0$, $\mathscr{M}_1$ and $\mathscr{M}$ be Orlicz functions defined by $\mathscr{M}_0^{-1}(s) = \varphi_0(s^{1/p_0}, s^{1/p_1})$, $\mathscr{M}_1^{-1}(s) = \varphi_1(s^{1/q_0}, s^{1/q_1})$ and $\mathscr{M}^{-1}(s) = \varphi(s^{1/r_0}, s^{1/r_1})$ for any $s \geqslant 0$. Then the triple of spaces $(L_{\mathscr{M}_0}, L_{\mathscr{M}_0}; L_{\mathscr{M}})$ is bilinear interpolation with respect to the* *triple* $((L_{p_0}, L_{p_1}), (L_{q_0}, L_{q_1}); (L_{r_0}, L_{r_1}))$ *of couples of* L_p*-spaces defined on any measure spaces.*

4. Interpolation spaces of operators

In this section we present certain applications of the obtained results. We first present a result which shows that bilinear interpolation theorem yields an information on relations about spaces of operators between interpolation spaces and interpolation spaces with respect to couples of operator spaces. To be more precise, let $\bar{X} = (X_0, X_1)$, $\bar{Y} = (Y_0, Y_1)$ be Banach couples. It is clear that we have the following continuous inclusion:

$$\mathscr{L}(X_{j\Delta}, Y_j) \hookrightarrow \mathscr{L}(X_0 \cap X_1, Y_0 + Y_1)$$

for $j = 0, 1$, where for a Banach space X intermediate with respect to (X_0, X_1), X_Δ (resp., X°) denotes a normed space $(X_0 \cap X_1, \|\cdot\|_X)$ (resp., the closed hull of $X_0 \cap X_1$ in X). In consequence $(\mathscr{L}(X_{0\Delta}, Y_0), \mathscr{L}(X_{1\Delta}, Y_1))$ forms a Banach couple denoted for simplicity by $(\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1))$.

Theorem 4.1. *Let* F_0, F_1 *and* F_2 *be interpolation functors such that the bilinear interpolation theorem* $F_0 \otimes F_1 \to F_2$ *holds with a constant* $C > 0$. *Then for any Banach couples* $\bar{X}$ *and* $\bar{Y}$ *the following continuous inclusion holds*

$$F_1(\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1)) \hookrightarrow \mathscr{L}(F_0(\bar{X}^\circ)_\Delta, F_2(\bar{Y})).$$

Proof. Assume that a bilinear interpolation theorem $F_0 \otimes F_1 \to F_2$ holds with a constant $C > 0$. Let $P_j : X_j^\circ \times \mathscr{L}(X_j, Y_j) \to Y_j$ be a bilinear operator defined by

$$P_j(x, T_j) \coloneqq \bar{T}_j x \quad \text{for } (x, T_j) \in X_j^\circ \times \mathscr{L}(X_j, Y_j),$$

where $\bar{T}_j : X_j^\circ \to Y_j$ is a norm preserving the extension of $T_j : X_{j\Delta} \to Y_j$. Since

$$\|P_j(x, T_j)\|_{Y_j} = \|\bar{T}x\|_{Z_j} \leqslant \|\bar{T}\|_{X_j^\circ \to Y_j} \|x\|_{X_j^\circ} = \|x\|_{X_j^\circ} \|T\|_{X_{j\Delta} \to Y_j},$$

P_j is a bounded bilinear operator. Further, for any $x \in X_0^\circ \cap X_1^\circ = X_0 \cap X_1$ and $T \in \mathscr{L}(X_0, Y_0) \cap \mathscr{L}(X_1, Y_1)$, we have

$$P_0(x, T) = \bar{T}x = Tx = P_1(x, T).$$

Hence, (P_0, P_1) is a bounded bilinear operator from the product $(X_0^\circ, X_1^\circ) \times (\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1))$ into (Y_0, Y_1). Let

$$P : \Delta(\bar{X}) \times (\mathscr{L}(X_0, Y_0) + \mathscr{L}(X_1, Y_1)) \to Y_0 + Y_1$$

be a bounded bilinear extension of (P_0, P_1). Note that $P(x, T) = Tx$ for any $x \in \Delta(\bar{X})$ and $T \in \mathscr{L}(X_0, Y_0) + \mathscr{L}(X_1, Y_1)$. In fact, it follows by definition of P that

$$P(x, T) = P_0(x, T_0) + P_1(x, T_1)$$

for any $x \in \Delta(\bar{X})$, $T = T_0 + T_1 \in \mathscr{L}(X_0, Y_0) + \mathscr{L}(X_1, Y_1)$, where $T_j \in \mathscr{L}(X_j, Y_j)$ for $j = 0, 1$. Hence

$$P(x, T) = \bar{T}_0 x + \bar{T}_1 x = T_0 x + T_1 x = Tx.$$

By our hypothesis the bilinear interpolation theorem $F_0 \otimes F_1 \to F_2$ holds with a constant $C > 0$. Thus

$$P : F_0(\bar{X}^\circ)_\Delta \times F_1(\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1)) \to F_2(\bar{Y})$$

is a bounded bilinear operator with a norm less or equal to C. This implies that for any $x \in \Delta(\bar{X})$ and $T \in F_1(\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1))$, we have

$$||Tx||_{F_2(\bar{Y})} = ||P(x, T)||_{F_2(\bar{Y})} \leqslant C||x||_{F_0(\bar{X}^\circ)}||T||_{F_1(\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1))}.$$

In consequence

$$\begin{aligned} ||T||_{F_0(\bar{X}^\circ)_\Delta \to F_2(\bar{Y})} &= \sup\{||Tx||_{F_2(\bar{Y})};\ ||x||_{F_0(\bar{X}^\circ)} \leqslant 1,\ ||x||_{\Delta(\bar{X})} \leqslant 1\} \\ &\leqslant C||T||_{F_1(\mathscr{L}(X_0, Y_0), \mathscr{L}(X_1, Y_1))}, \end{aligned}$$

which completes the proof. □

We conclude the paper with a result concerning the Banach operator ideal of (E, p)-summing operators. We recall, following [9], that if $1 \leqslant p < \infty$ and E is a Banach sequence space on $\mathbb{N}$ such that $\ell_p \hookrightarrow E$, then an operator $T : X \to Y$ between Banach spaces is said to be (E, p)-summing if there is a constant $C > 0$ such that for all finite sequences $\{x_1, \ldots, x_n\}$ in X, we have

$$\left|\left|\sum_{k=1}^{n} ||T(x_k)||_Y\, e_k\right|\right|_E \leqslant C \sup\left\{\left(\sum_{k=1}^{n} |x^*(x_k)|^p\right)^{1/p};\ ||x^*||_{X^*} \leqslant 1\right\}.$$

We denote by $\pi_{E,p}(T)$ the smallest constant C satisfying this property, and by $\Pi_{E,p}(X, Y)$ the space of (E, p)-summing operators from X into Y. Then $(\Pi_{E,p}, \pi_{E,p})$ is a Banach operator ideal in the sense of Pietsch provided that $||e_n||_E = 1$. In particular, if $E = \ell_p$, we obtain the Banach ideal of p-summing operators (see [29]).

In what follows, $\mathscr{M}_{n,m} := \mathscr{M}_{n,m}(\mathbb{K})$ denotes the space of $n \times m$ matrices on $\mathbb{K}$. A matrix $T \in \mathscr{M}_{n,m}$ will be identified with a linear operator from $\mathbb{K}^m$ into $\mathbb{K}^n$ built by using the canonical bases.

Theorem 4.2. *Let* (X_0, X_1) *and* (Y_0, Y_1) *be two couples of finite-dimensional Banach spaces with* $\dim X_0 = \dim X_1 = m$, $\dim Y_0 = \dim Y_1 = n$ *and assume that* X_0, X_1 *are* 2-*concave lattices and* Y_0^*, Y_1^* *have cotype* 2. *Then, for any super-multiplicative function* $\varphi \in \Phi^{+-}$, $1 \leqslant p \leqslant 2$ *and* 2-*convex maximal separable Banach sequence space* E *such that* $\ell_p \hookrightarrow E$ *the following holds*:

$$G_\varphi(\Pi_{E,p}(X_0, Y_0), \Pi_{E,p}(X_1, Y_1)) \hookrightarrow \Pi_{E,p}(G_\varphi(X_0, X_1), H_\varphi(Y_0, Y_1))$$

with a norm of the continuous inclusion not depending on m *and* n.

Proof. We observe that an operator $T : X \to Y$ is (E, p)-summing if and only if

$$\sup_{k \geqslant 1} \|\hat{T}_k\|_{\mathscr{L}(\ell_{p'}^k, X) \to E(Y)} < \infty,$$

where $\hat{T}_k : \mathscr{L}(\ell_{p'}^k, X) \to E(Y)$ is defined by

$$\hat{T}_k(S) = \sum_{i=1}^{k} (TSe_i)e_i \quad \text{for } S \in \mathscr{L}(\ell_{p'}^k, X).$$

Now we follow the method of Pisier [30] for the complex method of interpolation. Fix a positive integer k and consider the bilinear operator

$$U_j : \mathscr{L}(\ell_{p'}^k, X_j) \times \Pi_{E,p}(X_j, Y_j) \to E(Y_j),$$

defined by $U_j(S, T) \coloneqq \sum_{i=1}^k (TSe_i)e_i$ for $j = 0, 1$. Then we have

$$\|U_j\|_{\mathscr{L}(\ell_{p'}^k, X_j) \times \Pi_{E,p}(X_j, Y_j) \to E(Y_j)} \leqslant 1.$$

It is easy to check that if F is a 2-concave Banach sequence space and a Banach space X has cotype 2, then $F(X)$ has cotype 2. Since E is separable, $E(Y_j)^*$ is isometrically isomorphic to $E'(Y_j^*)$. Thus, our hypothesis implies that $E(Y_j)^*$ has cotype 2 for $j = 0, 1$ (by 2-convexity of E it follows that E' has cotype 2, see [20]). Now, the bilinear interpolation Theorem 3.5 gives that there is a constant $C > 0$ such that

$$\begin{aligned} &\left\| \sum_{i=1}^{k} (TSe_i)e_i \right\|_{H_\varphi(E(Y_0), E(Y_1))} \\ &\quad \leqslant C \|S\|_{G_\varphi(\mathscr{L}(\ell_{p'}^k, X_0), \mathscr{L}(\ell_{p'}^k, X_1))} \|T\|_{G_\varphi(\Pi_{E,p}(X_0, Y_0), \Pi_{E,p}(X_1, Y_1))} \end{aligned}$$

for all $T \in \mathscr{M}_{m,k}$ and $S \in \mathscr{M}_{n,m}$. Since E is a maximal Banach sequence space, the following continuous inclusion holds (see [23])

$$H_\varphi(E(Y_0), E(Y_1)) \hookrightarrow E(H_\varphi(Y_0, Y_1)).$$

To complete the proof, we use the following result (which follows from the proof of Proposition 4 in [10]): if $\varphi \in \Phi$ is a non-degenerate function, then for any finite-dimensional Banach space N of cotype 2 and any couple (M_0, M_1) of finite-dimensional 2-concave Banach lattices with $\dim(M_0) = \dim(M_1)$,

$$\mathscr{L}(N^*, G_\varphi(M_0, M_1)) \hookrightarrow G_\varphi(\mathscr{L}(N^*, M_0), \mathscr{L}(N^*, M_1))$$

with a norm of the continuous inclusion depending only on the 2-concavity constants of M_0 and M_1 and a cotype 2 constant of N.

Combining the above with the fact that ℓ_p^k has cotype 2 for any $1 \leqslant p \leqslant 2$, we obtain that

$$\left\| \sum_{i=1}^{k} \|(TSe_i)\|_{H_\varphi(\bar{Y})} e_i \right\|_E \leqslant C \|S\|_{\mathscr{L}(\ell_{p'}^k, G_\varphi(X_0, X_1))} \|T\|_{G_\varphi(\Pi_{E,p}(X_0, Y_0), \Pi_{E,p}(X_1, Y_1))}$$

for all $T \in \mathscr{M}_{m,k}$ and $S \in \mathscr{M}_{n,m}$. This shows that the sequence $\{\hat{T}_k\}_{k=1}^{\infty}$ of operators with

$$\hat{T}_k : \mathscr{L}(\ell_{p'}^k, G_\varphi(X_0, X_1)) \to E(H_\varphi(Y_0, Y_1))$$

is uniformly bounded, which yields the desired continuous inclusion. □

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