Igusa Local Zeta Function of the Cubic Polynomial $f(x) = x_1^3 + \cdots + x_n^3$

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Abstract. The purpose of this paper is to give the Igusa Local Zeta Function of the cubic polynomial $f(x) = x_1^3 + \cdots + x_n^3$. We solve this problem in general for primes congruent to 2 modulo 3 and in a special case for primes congruent to 1 modulo 3. This work was completed as part of the Mount Holyoke Summer Mathematics Institute, an NSF funded REU Program.[1]

1 Introduction

Throughout this paper, let p be a rational prime number, $\mathbb{Q}_p$ denote the field of p-adic numbers, $\mathbb{Z}_p$ be the ring of p-adic integers, and $\mathbb{Z}_p^\times$ represent the units of $\mathbb{Z}_p$. Further, let χ be a multiplicative character from $\mathbb{Q}_p$ to the complex unit circle. For a polynomial $f(x) = f(x_1, ..., x_n)$, we define the integral

$$Z_\chi(t) = \int_{\mathbb{Z}_p^n} \chi(\mathrm{ac}(f(x))) \, |f(x)|_p^s dx$$

to be the Igusa Local Zeta Function associated to $f(x)$ and χ. In this function, $t = p^{-s}$ and $dx = dx_1...dx_n$ is a product Haar measure on $\mathbb{Z}_p^n$, normalized so that

$$\int_{\mathbb{Z}_p^n} dx = 1.$$

Further, $\mathrm{ac}(f(x))$ denotes the angular component of $f(x)$. The purpose of this paper is to give the Igusa Local Zeta Function associated to the cubic polynomial $f(x) = x_1^3 + \cdots + x_n^3$.

The method we use to compute $Z\chi(t)$ has three steps. First, we define a function $F^*(i^*)$ associated to $f(x)$ by

$$F^*(i^*) = \int_{\mathbb{Z}_p^n} \Psi(i^* f(x)) \, dx,$$

where Ψ is an additive character on $\mathbb{Q}_p$ that is trivial on $\mathbb{Z}_p$. This is called the Generalized Exponential Sum, first formulated by Weil in 1965 [3]. So we must first compute $F^*(i^*)$ for $f(x) = x_1^3 + \cdots + x_n^3$. Next, we take the inverse Fourier Transform of $F^*(i^*)$ to obtain a new function,

$$F(i) = \int_{\mathbb{Q}_p} F^*(i^*) \, \Psi(-ii^*) \, di^*.$$

This function, $F(i)$, is called the Local Singular Series, and it gives information about the density of solutions of $f(x)$. The final step is to take the Mellin Transform of $F(i)$ to obtain the Igusa Local Zeta Function:

$$Z_\chi(t) = \int_{\mathbb{Z}_p} \chi(\mathrm{ac}(i)) \, F(i) \, |i|_p^s di.$$

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Throughout this paper, we will let $i^* = p^{-e}v$, where $v = \mathrm{ac}(i)$, and $i = p^k u$, where $u = \mathrm{ac}(i)$.

Also used in this paper are the orthogonality relations given by Igusa in [2]. Let χ be a nontrivial character with conductor equal to 1, the conductor being the smallest positive integer, c, such that χ is trivial on the ball $1 + p^c\mathbb{Z}_p$. Then we have

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-m}y\right) dy = \begin{cases} 1-p^{-1} & \text{if} \quad m \leq 0 \\ -p^{-1} & \text{if} \quad m = 1 \\ 0 & \text{if} \quad m > 1 \end{cases}$$

and

$$\int_{\mathbb{Z}_p^\times} \chi(y)\, \Psi\left(p^{-m}y\right) dy = \begin{cases} g_\chi & \text{if} \quad m = 1 \\ 0 & \text{if} \quad m \neq 1 \end{cases}.$$

In the second relation, g_χ is a Generalized Gaussian Sum:

$$g_\chi = \int_{\mathbb{Z}_p^\times} \chi(u)\, \Psi\left(p^{-1}u\right) du.$$

2 Computation of $F^*(i^*)$

First we compute $F^*(i^*)$ for $\tilde{f}(x) = x^3$. If we break up the intregal into an infinite sum of integrals over areas in which the absolute value is constant, we have:

$$\begin{aligned} F^*(i^*) &= \int_{\mathbb{Z}_p} \Psi\left(i^* x^3\right) dx \\ &= \sum_{j=0}^{\infty} \int_{p^j\left(\mathbb{Z}_p^\times\right)} \Psi\left(p^{-e}vx^3\right) dx. \end{aligned}$$

We will make the following change of variables:

$$\begin{aligned} x &= p^j y \\ dx &= p^{-j} dy, \end{aligned}$$

which leads us to

$$F^*(i^*) = \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}vy^3\right) dy.$$

Now we must break this up into two cases, based on whether $p \equiv 2 \pmod 3$ or $p \equiv 1 \pmod 3$.

2.1 $p \equiv 2 \pmod 3$

Since every $y \in \mathbb{Z}_p^\times$ is a cube, y^3 ranges through the same values as y; hence we can write

$$\begin{aligned} F^*(i^*) &= \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}vy^3\right) dy \\ &= \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}vy\right) dy. \end{aligned}$$

Similarly, vy ranges through the same values as y, so we have

$$F^*(i^*) = \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}y\right) dy.$$

Recall the orthogonality relations given in the Introduction. They imply that

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}y\right) dy = \begin{cases} 1-p^{-1} & \text{if} \quad j \geq \frac{e}{3} \\ -p^{-1} & \text{if} \quad j = \left(\frac{e-1}{3}\right) \\ 0 & \text{if} \quad j < \left(\frac{e-1}{3}\right) \end{cases},$$

and so we have

$$F^*\left(i^*\right) = \begin{cases} p^{-\frac{e}{3}} & \text{if} \quad e \equiv 0 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{e+1}{3}} & \text{if} \quad e \equiv 2 \pmod 3 \text{ and } e > 0 \\ 0 & \text{if} \quad e \equiv 1 \pmod 3 \text{ and } e > 0 \\ 1 & \text{if} \quad e \leq 0 \end{cases}.$$

2.2 $p \equiv 1 \pmod 3$

Note that for any continuous function φ on $\mathbb{Z}_p^\times$,

$$\int_{\mathbb{Z}_p^\times} \varphi\left(y^3\right) dy = \int_{\mathbb{Z}_p^\times} \varphi\left(y\right)\left(1+\chi_3(y)+\chi_3^{-1}(y)\right)dy,$$

since if y is a cube, then $(1+\chi_3(y)+\chi_3^{-1}(y)) = 3$, and if y is not a cube, then $(1+\chi_3(y)+\chi_3^{-1}(y)) = 0$. Thus we have

$$\begin{aligned} F^*\left(i^*\right) &= \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}vy^3\right) dy \\ &= \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}vy\right) dy + \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \chi_3(y)\Psi\left(p^{-(e-3j)}vy\right) dy \\ &\quad + \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(y)\Psi\left(p^{-(e-3j)}vy\right) dy. \end{aligned}$$

Now we make the following change of variables:

$$\begin{aligned} vy &= y' \\ dy &= dy'. \end{aligned}$$

Noting that $y = y'v^{-1}$ leads to

$$\begin{aligned} F^*\left(i^*\right) &= \sum_{j=0}^{\infty} p^{-j} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}y'\right) dy' + \sum_{j=0}^{\infty} p^{-j}\chi_3^{-1}(v) \int_{\mathbb{Z}_p^\times} \chi_3(y')\Psi\left(p^{-(e-3j)}y'\right) dy' \\ &\quad + \sum_{j=0}^{\infty} p^{-j}\chi_3(v) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(y')\Psi\left(p^{-(e-3j)}y'\right) dy'. \end{aligned}$$

Again, recall the orthogonality relations given in the Introduction. They imply that

$$\begin{aligned} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(e-3j)}y'\right) dy' &= \begin{cases} 1-p^{-1} & \text{if} \quad j \geq \frac{e}{3} \\ -p^{-1} & \text{if} \quad j = \left(\frac{e-1}{3}\right) \\ 0 & \text{if} \quad j < \left(\frac{e-1}{3}\right) \end{cases}; \\ \int_{\mathbb{Z}_p^\times} \chi_3(y')\Psi\left(p^{-(e-3j)}y'\right) dy' &= \begin{cases} g_{\chi_3} & \text{if} \quad j = \left(\frac{e-1}{3}\right) \\ 0 & \text{if} \quad j \neq \left(\frac{e-1}{3}\right) \end{cases}; \\ \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(y')\Psi\left(p^{-(e-3j)}y'\right) dy' &= \begin{cases} g_{\chi_3^{-1}} = \bar{g}_{\chi_3} & \text{if} \quad j = \left(\frac{e-1}{3}\right) \\ 0 & \text{if} \quad j \neq \left(\frac{e-1}{3}\right) \end{cases}. \end{aligned}$$

Utilizing these relations, we get

$$F^*(i^*) = \begin{cases} p^{-\frac{e}{3}} & \text{if} \quad e \equiv 0 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{e+1}{3}} & \text{if} \quad e \equiv 2 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{e-1}{3}}(\chi_3^{-1}(v)g_{\chi_3} + \chi_3(v)\bar{g}_{\chi_3}) & \text{if} \quad e \equiv 1 \pmod 3 \text{ and } e > 0 \\ 1 & \text{if} \quad e \leq 0 \end{cases}.$$

2.3 Computation of $F^*(i^*)$ for $f(x) = x_1^3 + \cdots + x_n^3$

Because of the additive nature of the function Ψ, we see that

$$\int_{\mathbb{Z}_p} \Psi(i^*(y_1 + y_2))\, dy_1 dy_2 = \int_{\mathbb{Z}_p} \Psi(i^* y_1)\, dy_1 \cdot \int_{\mathbb{Z}_p} \Psi(i^* y_2)\, dy_2.$$

Thus, to obtain the formula for $F^*(i^*)$ for our polynomial $f(x) = x_1^3 + \cdots + x_n^3$, we simply raise the formula for $F^*(i^*)$ for the polynomial $\tilde{f}(x) = x^3$ to the n^{th} power. Therefore, for $f(x) = x_1^3 + \cdots + x_n^3$ we have the following:

$$F^*(i^*) = \begin{cases} p^{-\frac{ne}{3}} & \text{if} \quad e \equiv 0 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{n(e+1)}{3}} & \text{if} \quad e \equiv 2 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{n(e-1)}{3}}(\chi_3^{-1}(v)g_{\chi_3} + \chi_3(v)\bar{g}_{\chi_3})^n & \text{if} \quad e \equiv 1 \pmod 3 \text{ and } e > 0 \\ 1 & \text{if} \quad e \leq 0 \end{cases}.$$

3 Computation of the Zeta Function for $p \equiv 2 \pmod 3$

3.1 Zeta Function for $\chi \neq \chi_0,\ \chi_3,\ \chi_3^{-1}$

Hosokawa gives the following lemma in [1]:

Lemma 1 *Let $f(x) = f(x_1, ..., x_n)$ be a polynomial with coefficients in $\mathbb{Q}_p$ and $Z_\chi(t)$ the Igusa Local Zeta Function for the multiplicative character χ associated to $f(x)$. If the group $\mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p^n$ and this action gives a measure-preserving analytic homeomorphism from $\mathbb{Z}_p^n$ onto itself, satisfying*

$$f(u \cdot x) = u^m f(x) \qquad (u \in \mathbb{Z}_p^\times)$$

for some positive integer m, then for any χ satisfying $\chi^m \neq \chi_0$, we have

$$Z_\chi(t) = 0.$$

Consider the usual action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p^n$:

$$u \cdot (x_1, ..., x_n) = (ux_1, ..., ux_n) \qquad \left(u \in \mathbb{Z}_p^\times; (x_1, ..., x_n) \in \mathbb{Z}_p^n\right).$$

This action is a measure-preserving homeomorphism. Furthermore, if $f(x) = x_1^3 + \cdots + x_n^3$, we see that

$$f(u \cdot x) = u^3 f(x).$$

Therefore, it follows from the above lemma that the Igusa Local Zeta Function for $\chi \neq \chi_0,\ \chi_3,\ \chi_3^{-1}$ associated to the cubic diagonal $x_1^3 + \cdots + x_n^3$ is equal to 0.

3.2 Zeta Function for $\chi = \chi_0$

Beginning this computation, we start with the fact that

$$
\begin{aligned}
F(i) &= \int_{\mathbb{Q}_p} F^*(i^*)\,\Psi(-ii^*)\,di^* \\
&= 1 + \sum_{e=1}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^\times\right)} F^*(i^*)\,\Psi(-ii^*)\,di^* \\
&= 1 + \sum_{\substack{e=3\\ e\equiv 0(3)}}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^\times\right)} p^{\frac{-ne}{3}}\,\Psi(-ii^*)\,di^* + \sum_{\substack{e=2\\ e\equiv 2(3)}}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^\times\right)} p^{\frac{-n(e+1)}{3}}\,\Psi(-ii^*)\,di^*.
\end{aligned}
$$

We next make the following change of variables:

$$
\begin{array}{cc}
\text{when } e \equiv 0 \pmod 3, & \text{when } e \equiv 2 \pmod 3, \\
i^* = p^{-3a} v & i^* = p^{-(3a+2)} v \\
di^* = p^{3a} dv & di^* = p^{3a+2} dv
\end{array}
$$

which leads us to

$$
F(i) = 1 + \sum_{a=1}^{\infty} p^{3a} \int_{\mathbb{Z}_p^\times} p^{-na}\,\Psi\left(-p^k u p^{-3a} v\right) dv + \sum_{a=0}^{\infty} p^{3a+2} \int_{\mathbb{Z}_p^\times} p^{-n(a+1)}\,\Psi\left(-p^k u p^{-(3a+2)} v\right) dv.
$$

We then make another change of variables,

$$
\begin{aligned}
-uv &= u' \\
dv &= du',
\end{aligned}
$$

to get

$$
F(i) = 1 + \sum_{a=1}^{\infty} p^{(3-n)a} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a-k)} u'\right) du' + \sum_{a=1}^{\infty} p^{(3-n)a+2-n} \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a+2-k)} u'\right) du'.
$$

We then use the orthogonality relations to get the following:

$$
\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a-k)} u'\right) du' = \begin{cases} 1 - p^{-1} & \text{if} \quad a \le \frac{k}{3} \\ -p^{-1} & \text{if} \quad a = \frac{k+1}{3} \\ 0 & \text{if} \quad a > \frac{k+1}{3} \end{cases};
$$

$$
\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a+2-k)} u'\right) du' = \begin{cases} 1 - p^{-1} & \text{if} \quad a \le \frac{k-2}{3} \\ -p^{-1} & \text{if} \quad a = \frac{k-1}{3} \\ 0 & \text{if} \quad a > \frac{k-1}{3} \end{cases}.
$$

First we find $F(i)$ in the case that $k \equiv 0\,(3)$ and $k \ge 0$. We have

$$
F(i) = 1 + \sum_{a=1}^{\frac{k}{3}} p^{(3-n)a}\left(1 - p^{-1}\right) + \sum_{a=0}^{\frac{k}{3}-1} p^{(3-n)a} p^{2-n}\left(1 - p^{-1}\right).
$$

Next, recall that

$$
\sum_{m=1}^{c} \alpha^m = \alpha \frac{(\alpha^c - 1)}{\alpha - 1} \text{ and } \sum_{m=0}^{c} \alpha^m = \frac{\alpha^{c+1} - 1}{\alpha - 1}, \text{where } c < \infty.
$$

We use these formulas to find that

$$
F(i) = 1 + \left(1 - p^{-1}\right)\left(p^{3-n}\right)\left(\frac{p^{(3-n)\frac{k}{3}} - 1}{p^{3-n} - 1}\right) + \left(1 - p^{-1}\right)\left(p^{2-n}\right)\left(\frac{p^{(3-n)\left(\frac{k}{3}\right)} - 1}{p^{3-n} - 1}\right).
$$

We next find $F(i)$ for $k \equiv 2 \pmod 3$ and $k \geq 0$. Again, we use the orthogonality relations and finite geometric sum formulas to find that

$$\begin{aligned} F(i) &= 1 + \sum_{a=1}^{\frac{k-2}{3}} p^{(3-n)a}\left(1-p^{-1}\right) + p^{(3-n)\left(\frac{k+1}{3}\right)}\left(-p^{-1}\right) + \sum_{a=0}^{\frac{k-2}{3}} p^{(3-n)a} p^{2-n}\left(1-p^{-1}\right) \\ &= 1 + \left(1-p^{-1}\right)\left(p^{3-n}\right)\left(\frac{p^{(3-n)\left(\frac{k-2}{3}\right)}-1}{p^{3-n}-1}\right) + \left(1-p^{-1}\right)\left(p^{2-n}\right)\left(\frac{p^{(3-n)\left(\frac{k+1}{3}\right)}-1}{p^{3-n}-1}\right) - p^{\frac{(3-n)(k+1)}{3}-1}. \end{aligned}$$

Finally, we find $F(i)$ for $k \equiv 1 \pmod 3$ and $k \geq 0$ by using the orthogonality relations and geometric sum formulas:

$$\begin{aligned} F(i) &= 1 + \sum_{a=1}^{\frac{k-1}{3}} p^{(3-n)a}(1-p^{-1}) + \sum_{a=0}^{\frac{k-1}{3}-1} p^{(3-n)a} p^{2-n}(1-p^{-1}) + p^{(3-n)\left(\frac{k-1}{3}\right)} p^{2-n}(-p^{-1}) \\ &= 1 + (1-p^{-1})(p^{3-n})\left(\frac{p^{(3-n)\left(\frac{k-1}{3}\right)}-1}{p^{3-n}-1}\right) + (1-p^{-1})(p^{2-n})\left(\frac{p^{(3-n)\left(\frac{k-1}{3}\right)}-1}{p^{3-n}-1}\right) - p^{\frac{(3-n)(k-1)}{3}+1-n}. \end{aligned}$$

By letting $A_n = \frac{(1-p^{-1})(p^{3-n})}{p^{3-n}-1}$ and $B_n = \frac{(1-p^{-1})(p^{2-n})}{p^{3-n}-1}$, we have for $k \geq 0$:

$$F(i) = \begin{cases} 1 + A_n(p^{(3-n)(\frac{k}{3})}-1) + B_n(p^{(3-n)(\frac{k}{3})}-1) & \text{if} \quad k \equiv 0 \pmod 3 \\ 1 + A_n(p^{(3-n)(\frac{k-2}{3})}-1) + B_n(p^{(3-n)(\frac{k+1}{3})}-1) - p^{\frac{(3-n)(k+1)}{3}-1} & \text{if} \quad k \equiv 2 \pmod 3 \\ 1 + A_n(p^{(3-n)(\frac{k-1}{3})}-1) + B_n(p^{(3-n)(\frac{k-1}{3})}-1) - p^{\frac{(3-n)(k-1)}{3}+1-n} & \text{if} \quad k \equiv 1 \pmod 3 \end{cases}$$

We now move on to computing the Igusa Local Zeta Function from the function $F(i)$. As before, we write the integral as an intinite sum of integrals over areas in which the absolute value is constant:

$$\begin{aligned} Z_{\chi_0}(t) &= \int_{\mathbb{Z}_p} \chi_0(u) F(i) \left|i\right|_p^s di \\ &= \sum_{k=0}^{\infty} \int_{p^k\left(\mathbb{Z}_p^\times\right)} F(i) \left|i\right|_p^s di \\ &= \sum_{\substack{k=0 \\ k\equiv 0(3)}}^{\infty} \int_{p^k\left(\mathbb{Z}_p^\times\right)} \left(1 + A_n\left(p^{(3-n)\left(\frac{k}{3}\right)}-1\right) + B_n\left(p^{(3-n)\left(\frac{k}{3}\right)}-1\right)\right) t^k di \\ &\quad + \sum_{\substack{k=2 \\ k\equiv 2(3)}}^{\infty} \int_{p^k\left(\mathbb{Z}_p^\times\right)} \left(1 + A_n\left(p^{(3-n)\left(\frac{k-2}{3}\right)}-1\right) + B_n\left(p^{(3-n)\left(\frac{k+1}{3}\right)}-1\right) - p^{\frac{(3-n)(k+1)}{3}-1}\right) t^k di \\ &\quad + \sum_{\substack{k=1 \\ k\equiv 1(3)}}^{\infty} \int_{p^k\left(\mathbb{Z}_p^\times\right)} \left(1 + A_n\left(p^{(3-n)\left(\frac{k-1}{3}\right)}-1\right) + B_n\left(p^{(3-n)\left(\frac{k-1}{3}\right)}-1\right) - p^{\frac{(3-n)(k-1)}{3}+1-n}\right) t^k di. \end{aligned}$$

We then make the following change of variables:

when $k \equiv 0 \pmod 3$	when $k \equiv 2 \pmod 3$	when $k \equiv 1 \pmod 3$
$i = p^{3a}u$	$i = p^{3a+2}u$	$i = p^{3a+1}u$
$di = p^{-3a}du$	$di = p^{-(3a+2)}du$	$di = p^{-(3a+1)}du$

to get

$$\begin{aligned}
Z_{\chi_0}(t) &= \sum_{a=0}^{\infty} p^{-3a} t^{3a} \left(1 + A_n \left(p^{(3-n)a} - 1\right) + B_n \left(p^{(3-n)a} - 1\right)\right) \int_{\mathbb{Z}_p^\times} du \\
&\quad + \sum_{a=0}^{\infty} p^{-(3a+2)} t^{3a+2} \left(1 + A_n \left(p^{(3-n)a} - 1\right) + B_n \left(p^{(3-n)(a+1)} - 1\right) - p^{(3-n)a+2-n}\right) \int_{\mathbb{Z}_p^\times} du \\
&\quad + \sum_{a=0}^{\infty} p^{-(3a+1)} t^{3a+1} \left(1 + A_n \left(p^{(3-n)a} - 1\right) + B_n \left(p^{(3-n)a} - 1\right) - p^{(3-n)a+1-n}\right) \int_{\mathbb{Z}_p^\times} du \\
&= \left(1 - p^{-1}\right) \sum_{a=0}^{\infty} \left(p^{-3a} t^{3a} + p^{-3a} t^{3a} A_n p^{(3-n)a} - p^{-3a} t^{3a} A_n + p^{-3a} t^{3a} B_n p^{(3-n)a} - p^{-3a} t^{3a} B_n\right) \\
&\quad + \left(1 - p^{-1}\right) \sum_{a=0}^{\infty} (p^{-(3a+2)} t^{3a+2} + p^{-(3a+2)} t^{3a+2} A_n p^{(3-n)a} - p^{-(3a+2)} t^{3a+2} A_n \\
&\qquad + p^{-(3a+2)} t^{3a+2} B_n p^{(3-n)(a+1)} - p^{-(3a+2)} t^{3a+2} B_n - p^{-(3a+2)} t^{3a+2} p^{(3-n)a+2-n}) \\
&\quad + \left(1 - p^{-1}\right) \sum_{a=0}^{\infty} (p^{-(3a+1)} t^{3a+1} + p^{-(3a+1)} t^{3a+1} A_n p^{(3-n)a} - p^{-(3a+1)} t^{3a+1} A_n \\
&\qquad + p^{-(3a+1)} t^{3a+1} B_n p^{(3-n)a} - p^{-(3a+1)} t^{3a+1} B_n - p^{-(3a+1)} t^{3a+1} p^{(3-n)a+1-n}).
\end{aligned}$$

We will then make the following substitution to simplify matters:

$$\begin{aligned}
C_n &= \sum_{a=0}^{\infty} p^{-3a} t^{3a} = \frac{1}{1 - p^{-3} t^3} \\
D_n &= \sum_{a=0}^{\infty} p^{-3a} t^{3a} p^{(3-n)a} = \frac{1}{1 - p^{-n} t^3}.
\end{aligned}$$

We then get

$$\begin{aligned}
Z_{\chi_0}(t) &= \left(1 - p^{-1}\right) (C_n + A_n D_n - A_n C_n + B_n D_n - B_n C_n + p^{-2} t^2 C_n + p^{-2} t^2 A_n D_n - p^{-2} t^2 A_n C_n \\
&\quad + p^{1-n} t^2 B_n D_n - p^{-2} t^2 B_n C_n - p^{-n} t^2 D_n + p^{-1} t C_n + p^{-1} t A_n D_n - p^{-1} t A_n C_n \\
&\quad + p^{-1} t B_n D_n - p^{-1} t B_n C_n - p^{-n} t D_n).
\end{aligned}$$

Amazingly enough, this rather ugly expression can be simplified to the following:

$$Z_{\chi_0}(t) = \frac{\left(1 - p^{-1}\right)\left(1 - p^{-n} t\right)}{\left(1 - p^{-1} t\right)\left(1 - p^{-n} t^3\right)}.$$

3.3 Zeta Function for $\chi = \chi_3, \chi_3^{-1}$

The first step in the computation of the Igusa Local Zeta Function for χ_3 and χ_3^{-1} is to find $F(i)$; this was done in 3.2. The Zeta Function for χ_3 is

$$Z_{\chi_3}(t) = \int_{\mathbb{Z}_p} \chi_3(u) F(i) \, |i|_p^s \, di,$$

and the Zeta Function for χ_3^{-1} is

$$Z_{\chi_0^{-1}}(t) = \int_{\mathbb{Z}_p} \chi_3^{-1}(u) F(i) \, |i|_p^s \, di.$$

However, since $p \equiv 2 \pmod 3$, every element of $\mathbb{Z}_p$ is a cube. Thus, χ_3 and χ_3^{-1}act exactly the same as the trivial character. Therefore, the formula for the Igusa Local Zeta Function for both χ_3 and χ_3^{-1} will be identical to the Zeta Function for the trivial character, found in 3.2.

4 Computation of the Zeta Function for $p \equiv 1 \pmod 3$, $n = 2$

4.1 Zeta Function for $\chi \neq \chi_0$, χ_3, χ_3^{-1}

By the exact same reasoning given in 3.1, the Zeta Function for $\chi \neq \chi_0$, χ_3, χ_3^{-1} is equal to 0.

4.2 Zeta Function for $\chi = \chi_0$, χ_3, χ_3^{-1}

Although we apply the same method used to calculate the Zeta Function for primes congruent to 2 modulo 3, the computation becomes much more difficult in the case that $p \equiv 1 \pmod 3$. The computation would begin with the following:

$$\begin{aligned} F(i) &= \int_{\mathbb{Q}_p} F^*(i^*)\,\Psi(-ii^*)\,di^* \\ &= 1 + \sum_{e=1}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^{\times}\right)} F^*(i^*)\,\Psi(-ii^*)\,di^* \\ &= 1 + \sum_{\substack{e=3 \\ e\equiv 0(3)}}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^{\times}\right)} p^{-\frac{ne}{3}}\,\Psi(-ii^*)\,di^* + \sum_{\substack{e=2 \\ e\equiv 2(3)}}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^{\times}\right)} p^{-\frac{n(e+1)}{3}}\,\Psi(-ii^*)\,di^* \\ &\quad + \sum_{\substack{e=1 \\ e\equiv 1(3)}}^{\infty} \int_{p^{-e}\left(\mathbb{Z}_p^{\times}\right)} p^{-\frac{n(e-1)}{3}} (\chi_3^{-1}(v) g_{\chi_3} + \chi_3(v)\bar{g}_{\chi_3})^n\,\Psi(-ii^*)\,di^*. \end{aligned}$$

The difficulty in evaluating this expression lies in the difficulty in evaluating

$$(\chi_3^{-1}(v) g_{\chi_3} + \chi_3(v)\bar{g}_{\chi_3})^n.$$

In fact, this turns out to be such a difficult problem that we do not solve it in general here. We do, however, give a solution in the case where $n = 2$. That is, we find the Igusa Local Zeta Function assocaited to the polynomial $x_1^3 + x_2^3$.

4.3 Zeta Function for $\chi = \chi_0$ Associated to the Polynomial $x_1^3 + x_2^3$

We begin by noting that, in this case, we have

$$F^*(i^*) = \begin{cases} p^{-\frac{2e}{3}} & \text{if} \quad e \equiv 0 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{2e+2}{3}} & \text{if} \quad e \equiv 2 \pmod 3 \text{ and } e > 0 \\ p^{-\frac{2e-2}{3}} (\chi_3(v) g_{\chi_3}^2 + \chi_3^{-1}(v)\bar{g}_{\chi_3}^2 + 2p^{-1}) & \text{if} \quad e \equiv 1 \pmod 3 \text{ and } e > 0 \\ 1 & \text{if} \quad e \leq 0 \end{cases}.$$

The expression for $e \equiv 1 \pmod 3$ and $e > 0$ comes about in the following way:

$$\begin{aligned} (\chi_3^{-1}(v) g_{\chi_3} + \chi_3(v)\bar{g}_{\chi_3})^2 &= (\chi_3^{-1}(v) g_{\chi_3})^2 + (\chi_3(v)\bar{g}_{\chi_3})^2 + 2\chi_3^{-1}(v) g_{\chi_3} \chi_3(v)\bar{g}_{\chi_3} \\ &= \chi_3(v) g_{\chi_3}^2 + \chi_3^{-1}(v)\bar{g}_{\chi_3}^2 + 2p^{-1}, \end{aligned}$$

since $(\chi_3^{-1})^2 = \chi_3$, $(\chi_3)^2 = \chi_3^{-1}$, and $g_{\chi_3}\bar{g}_{\chi_3} = p^{-1}$ [2]. We now compute $F(i)$:

$$\begin{aligned}
F(i) &= \int_{\mathbb{Q}_p} F^*(i^*)\,\Psi(-ii^*)\,di^* \\
&= 1 + \sum_{e=1}^{\infty} \int_{p^{-e}(\mathbb{Z}_p^\times)} F^*(i^*)\,\Psi(-ii^*)\,di^* \\
&= 1 + \sum_{\substack{e=3 \\ e\equiv 0(3)}}^{\infty} \int_{p^{-e}(\mathbb{Z}_p^\times)} p^{-\frac{2e}{3}}\,\Psi(-ii^*)\,di^* + \sum_{\substack{e=2 \\ e\equiv 2(3)}}^{\infty} \int_{p^{-e}(\mathbb{Z}_p^\times)} p^{-\frac{2e+2}{3}}\,\Psi(-ii^*)\,di^* \\
&\quad + \sum_{\substack{e=1 \\ e\equiv 1(3)}}^{\infty} \int_{p^{-e}(\mathbb{Z}_p^\times)} p^{-\frac{2e-2}{3}}(\chi_3(v)g_{\chi_3}^2 + \chi_3^{-1}(v)\bar{g}_{\chi_3}^2 + 2p^{-1})\Psi(-ii^*)\,di^*.
\end{aligned}$$

We next make the follwoing change of variables:

$$\begin{array}{ccc}
\text{when } e \equiv 0 \pmod 3 & \text{when } e \equiv 2 \pmod 3 & \text{when } e \equiv 2 \pmod 3 \\
i^* = p^{-3a}v & i^* = p^{-(3a+2)}v & i^* = p^{-(3a+1)}v \\
di^* = p^{3a}dv & di^* = p^{3a+2}dv & di^* = p^{3a+1}dv
\end{array},$$

and we get

$$\begin{aligned}
F(i) &= 1 + \sum_{a=1}^{\infty} p^{3a}p^{-2a} \int_{\mathbb{Z}_p^\times} \Psi\left(-p^k u p^{-3a} v\right) dv + \sum_{a=0}^{\infty} p^{3a+2}p^{-2a-2} \int_{\mathbb{Z}_p^\times} \Psi\left(-p^k u p^{-(3a+2)} v\right) dv \\
&\quad + \sum_{a=0}^{\infty} p^{3a+1}p^{-2a} g_{\chi_3}^2 \int_{\mathbb{Z}_p^\times} \chi_3(v)\,\Psi\left(-p^k u p^{-(3a+1)} v\right) dv \\
&\quad + \sum_{a=0}^{\infty} p^{3a+1}p^{-2a} \bar{g}_{\chi_3}^2 \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(v)\,\Psi\left(-p^k u p^{-(3a+1)} v\right) dv \\
&\quad + \sum_{a=0}^{\infty} p^{3a+1}p^{-2a} \left(2p^{-1}\right) \int_{\mathbb{Z}_p^\times} \Psi\left(-p^k u p^{-(3a+1)} v\right) dv.
\end{aligned}$$

We now make another change of variables:

$$\begin{aligned}
-uv &= u' \\
dv &= du' \\
v &= -u'u^{-1}
\end{aligned}$$

to get

$$\begin{aligned}
F(i) &= 1 + \sum_{a=1}^{\infty} p^a \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a-k)}u'\right) du' + \sum_{a=0}^{\infty} p^a \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a+2-k)}u'\right) du' \\
&\quad + p g_{\chi_3}^2 \chi_3^{-1}(u) \sum_{a=0}^{\infty} p^a \int_{\mathbb{Z}_p^\times} \chi_3(u')\,\Psi\left(p^{-(3a+1-k)}u'\right) du' \\
&\quad + p \bar{g}_{\chi_3}^2 \chi_3(u) \sum_{a=0}^{\infty} p^a \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u')\,\Psi\left(p^{-(3a+1-k)}u'\right) du' \\
&\quad + 2\sum_{a=0}^{\infty} p^a \int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a+1-k)}u'\right) du'.
\end{aligned}$$

We next get the follwoing orthogonality relations:

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a-k)}u'\right)du' = \begin{cases} 1-p^{-1} & \text{if} \quad a \le \frac{k}{3} \\ -p^{-1} & \text{if} \quad a = \frac{k+1}{3} \\ 0 & \text{if} \quad a > \frac{k+1}{3} \end{cases} ;$$

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a+2-k)}u'\right)du' = \begin{cases} 1-p^{-1} & \text{if} \quad a \le \frac{k-2}{3} \\ -p^{-1} & \text{if} \quad a = \frac{k-1}{3} \\ 0 & \text{if} \quad a > \frac{k-1}{3} \end{cases} ;$$

$$\int_{\mathbb{Z}_p^\times} \chi_3\left(u'\right)\Psi\left(p^{-(3a+1-k)}u'\right)du' = \begin{cases} g_{\chi_3} & \text{if} \quad a = \frac{k}{3} \\ 0 & \text{if} \quad a \ne \frac{k}{3} \end{cases} ;$$

$$\int_{\mathbb{Z}_p^\times} \chi_3^{-1}\left(u'\right)\Psi\left(p^{-(3a+1-k)}u'\right)du' = \begin{cases} g_{\chi_3^{-1}} = \bar{g}_{\chi_3} & \text{if} \quad a = \frac{k}{3} \\ 0 & \text{if} \quad a \ne \frac{k}{3} \end{cases} ;$$

$$\int_{\mathbb{Z}_p^\times} \Psi\left(p^{-(3a+1-k)}u'\right)du' = \begin{cases} 1-p^{-1} & \text{if} \quad a \le \frac{k-1}{3} \\ -p^{-1} & \text{if} \quad a = \frac{k}{3} \\ 0 & \text{if} \quad a > \frac{k}{3} \end{cases} .$$

Hence, when $k \equiv 0 \pmod 3$,

$$\begin{aligned} F(i) &= 1+\sum_{a=1}^{\frac{k}{3}} p^a\left(1-p^{-1}\right)+\sum_{a=0}^{\frac{k}{3}-1} p^a\left(1-p^{-1}\right)+pg_{\chi_3}^2\chi_3^{-1}(u)\,p^{\frac{k}{3}}g_{\chi_3}+p\bar{g}_{\chi_3}^2\chi_3(u)\,p^{\frac{k}{3}}\bar{g}_{\chi_3} \\ &\quad +2\sum_{a=0}^{\frac{k}{3}-1} p^a\left(1-p^{-1}\right)+2p^{\frac{k}{3}}\left(-p^{-1}\right) \\ &= p^{\frac{k}{3}}+p^{\frac{k}{3}-1}-p^{-1}-2p^{-1}+g_{\chi_3}^3\chi_3^{-1}(u)\,p^{\frac{k}{3}+1}+\bar{g}_{\chi_3}^3\chi_3(u)\,p^{\frac{k}{3}+1}. \end{aligned}$$

Furthermore, when $k \equiv 2 \pmod 3$,

$$\begin{aligned} F(i) &= 1+\sum_{a=1}^{\frac{k-2}{3}} p^a\left(1-p^{-1}\right)+\sum_{a=0}^{\frac{k-2}{3}} p^a\left(1-p^{-1}\right)+p^{\frac{k+1}{3}}\left(-p^{-1}\right)+2\sum_{a=0}^{\frac{k-2}{3}} p^a\left(1-p^{-1}\right) \\ &= 3p^{\frac{k-2}{3}}-p^{-1}-2p^{-1}. \end{aligned}$$

Lastly, when $k \equiv 1 \pmod 3$,

$$\begin{aligned} F(i) &= 1+\sum_{a=1}^{\frac{k-1}{3}} p^a\left(1-p^{-1}\right)+\sum_{a=0}^{\frac{k-1}{3}-1} p^a\left(1-p^{-1}\right)+p^{\frac{k-1}{3}}\left(-p^{-1}\right)+2\sum_{a=0}^{\frac{k-1}{3}} p^a\left(1-p^{-1}\right) \\ &= 3p^{\frac{k-1}{3}}-p^{-1}-2p^{-1}. \end{aligned}$$

Now we can compute the Igusa Local Zeta Function. We have

$$\begin{aligned} Z_{\chi_0}(t) &= \int_{\mathbb{Z}_p} \chi_0(u)F(i)\,|i|_p^s\,di \\ &= \sum_{k=0}^{\infty}\int_{p^k\left(\mathbb{Z}_p^\times\right)} F(i)\,|i|_p^s\,di \\ &= \sum_{\substack{k=0 \\ k\equiv 0(3)}}^{\infty}\int_{p^k\left(\mathbb{Z}_p^\times\right)} (p^{\frac{k}{3}}+p^{\frac{k}{3}-1}-p^{-1}-2p^{-1}+g_{\chi_3}^3\chi_3^{-1}(u)\,p^{\frac{k}{3}+1}+\bar{g}_{\chi_3}^3\chi_3(u)\,p^{\frac{k}{3}+1})t^k di \\ &\quad +\sum_{\substack{k=2 \\ k\equiv 2(3)}}^{\infty}\int_{p^k\left(\mathbb{Z}_p^\times\right)} (3p^{\frac{k-2}{3}}-p^{-1}-2p^{-1})t^k di+\sum_{\substack{k=1 \\ k\equiv 1(3)}}^{\infty}\int_{p^k\left(\mathbb{Z}_p^\times\right)} (3p^{\frac{k-1}{3}}-p^{-1}-2p^{-1})t^k di. \end{aligned}$$

Next we make the following change of variables:

$$\begin{array}{ccc} \text{when } k \equiv 0 \pmod 3 & \text{when } k \equiv 2 \pmod 3 & \text{when } k \equiv 1 \pmod 3 \\ i = p^{3a}u & i = p^{3a+2}u & i = p^{3a+1}u \\ di = p^{-3a}du & di = p^{-(3a+2)}du & di = p^{-(3a+1)}du \end{array}$$

This gives

$$\begin{aligned} Z_{\chi_0}(t) &= \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^a + p^{a-1} - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} du + \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}g_{\chi_3}^3) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)du \\ &\quad + \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}\bar{g}_{\chi_3}^3) \int_{\mathbb{Z}_p^\times} \chi_3(u)du + \sum_{a=0}^{\infty} p^{-(3a+2)}t^{(3a+2)}(3p^a - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} du \\ &\quad + \sum_{a=0}^{\infty} p^{-(3a+1)}t^{(3a+1)}(3p^a - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} du. \end{aligned}$$

We note that $\int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)du = \int_{\mathbb{Z}_p^\times} \chi_3(u)du = 0$ and that $\int_{\mathbb{Z}_p^\times} du = 1 - p^{-1}$. If we put

$$A = \sum_{a=0}^{\infty} p^{-2a}t^{3a} = \frac{1}{1-p^{-2}t^3} \quad \text{and} \quad B = \sum_{a=0}^{\infty} p^{-3a}t^{3a} = \frac{1}{1-p^{-3}t^3},$$

then we have

$$\begin{aligned} Z_{\chi_0}(t) &= (1-p^{-1})(A + p^{-1}A - p^{-1}B - 2p^{-1}B + 3p^{-2}t^2A - p^{-3}t^2B - 2p^{-3}t^2B \\ &\qquad +3p^{-1}tA - p^{-2}tB - 2p^{-2}tB) \\ &= \frac{(1-p^{-1})(1-2p^{-1}+2p^{-1}t-p^{-2}t)}{(1-p^{-1}t)(1-p^{-2}t^3)}. \end{aligned}$$

4.4 Zeta Function for $\chi = \chi_3$ Associated to the Polynomial $x_1^3 + x_2^3$

The calculation of $F(i)$ was done in the previous section, so we begin with

$$\begin{aligned} Z_{\chi_3}(t) &= \int_{\mathbb{Z}_p} \chi_3(u)F(i)\,|i|_p^s\,di \\ &= \sum_{k=0}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} \chi_3(u)F(i)\,|i|_p^s\,di \\ &= \sum_{\substack{k=0 \\ k\equiv 0(3)}}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} \chi_3(u)(p^{\frac{k}{3}} + p^{\frac{k}{3}-1} - p^{-1} - 2p^{-1} + g_{\chi_3}^3\chi_3^{-1}(u)\,p^{\frac{k}{3}+1} + \bar{g}_{\chi_3}^3\chi_3(u)\,p^{\frac{k}{3}+1})t^k di \\ &\quad + \sum_{\substack{k=2 \\ k\equiv 2(3)}}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} \chi_3(u)(3p^{\frac{k-2}{3}} - p^{-1} - 2p^{-1})t^k di \\ &\quad + \sum_{\substack{k=1 \\ k\equiv 1(3)}}^{\infty} \int_{p^k(\mathbb{Z}_p^\times)} \chi_3(u)(3p^{\frac{k-1}{3}} - p^{-1} - 2p^{-1})t^k di. \end{aligned}$$

As before, we make the change of variables

$$\begin{array}{ccc} \text{when } k \equiv 0 \pmod 3 & \text{when } k \equiv 2 \pmod 3 & \text{when } k \equiv 1 \pmod 3 \\ i = p^{3a}u & i = p^{3a+2}u & i = p^{3a+1}u \\ di = p^{-3a}du & di = p^{-(3a+2)}du & di = p^{-(3a+1)}du \end{array}$$

to get

$$\begin{aligned}
Z_{\chi_3}(t) &= \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^a + p^{a-1} - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} \chi_3(u)du + \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}g_{\chi_3}^3) \int_{\mathbb{Z}_p^\times} \chi_3(u)\chi_3^{-1}(u)du \\
&+ \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}\bar{g}_{\chi_3}^3) \int_{\mathbb{Z}_p^\times} \chi_3(u)\chi_3(u)du + \sum_{a=0}^{\infty} p^{-(3a+2)}t^{(3a+2)}(3p^a - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} \chi_3(u)du \\
&+ \sum_{a=0}^{\infty} p^{-(3a+1)}t^{(3a+1)}(3p^a - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} \chi_3(u)du.
\end{aligned}$$

We now note that $\int_{\mathbb{Z}_p^\times} \chi_3(u)du = 0$, that $\int_{\mathbb{Z}_p^\times} \chi_3(u)\chi_3(u)du = \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)du = 0$, and that $\int_{\mathbb{Z}_p^\times} \chi_3(u)\chi_3^{-1}(u)du = \int_{\mathbb{Z}_p^\times} du = 1 - p^{-1}$. Therefore, we have

$$\begin{aligned}
Z_{\chi_3}(t) &= \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}g_{\chi_3}^3)(1-p^{-1}) \\
&= g_{\chi_3}^3 \frac{p-1}{1-p^{-2}t^3}.
\end{aligned}$$

4.5 Zeta Function for $\chi = \chi_3^{-1}$ Associated to the Polynomial $x_1^3 + x_2^3$

The computation for $\chi = \chi_3^{-1}$ is similar to the computation in the previous section, substituting χ_3^{-1} for χ_3. Noting further that $\int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)\chi_3^{-1}(u)du = \int_{\mathbb{Z}_p^\times} \chi_3(u)du = 0$, we have

$$\begin{aligned}
Z_{\chi_3^{-1}}(t) &= \int_{\mathbb{Z}_p} \chi_3^{-1}(u)F(i)\,|i|_p^s\,di \\
&= \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^a + p^{a-1} - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)du \\
&+ \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}g_{\chi_3}^3) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)\chi_3^{-1}(u)du \\
&+ \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}\bar{g}_{\chi_3}^3) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)\chi_3(u)du \\
&+ \sum_{a=0}^{\infty} p^{-(3a+2)}t^{(3a+2)}(3p^a - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)du \\
&+ \sum_{a=0}^{\infty} p^{-(3a+1)}t^{(3a+1)}(3p^a - p^{-1} - 2p^{-1}) \int_{\mathbb{Z}_p^\times} \chi_3^{-1}(u)du \\
&= \sum_{a=0}^{\infty} p^{-3a}t^{3a}(p^{a+1}\bar{g}_{\chi_3}^3)(1-p^{-1}) \\
&= \bar{g}_{\chi_3}^3 \frac{p-1}{1-p^{-2}t^3}.
\end{aligned}$$

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