P. MANCOSU, K.F. JØRGENSEN AND S.A. PEDERSEN

INTRODUCTION

In the 20th century philosophy of mathematics has to a great extent been dominated by views developed during the so-called foundational crisis in the beginning of that century. These views have primarily focused on questions pertaining to the logical structure of mathematics and questions regarding the justification and consistency of mathematics. Paradigmatic in this respect is Hilbert's program which inherits from Frege and Russell the project to formalize all areas of ordinary mathematics and then adds the requirement of a proof, by epistemically privileged means (finitistic reasoning), of the consistency of such formalized theories. While interest in modified versions of the original foundational programs is still thriving, in the second part of the twentieth century several philosophers and historians of mathematics have questioned whether such foundational programs could exhaust the realm of important philosophical problems to be raised about the nature of mathematics. Some have done so in open confrontation (and hostility) to the logically based analysis of mathematics which characterized the classical foundational programs, while others (and many of the contributors to this book belong to this tradition) have only called for an extension of the range of questions and problems that should be raised in connection with an understanding of mathematics. The focus has turned thus to a consideration of what mathematicians are actually doing when they produce mathematics. Questions concerning concept-formation, understanding, heuristics, changes in style of reasoning, the role of analogies and diagrams etc. have become the subject of intense interest. These historians and philosophers agree that there is more to understanding mathematics than a study of its logical structure and put much emphasis on mathematical activity as a human activity. How are mathematical objects and concepts generated? How does the process tie up with justification? What role do visual images and diagrams play in mathematical activity? In addition to these cognitive issues one might also investigate how mathematics interacts with the natural sciences, and how mathematical thinking might depend on the culture it is embedded in.

This book is based on the meeting "Mathematics as Rational Activity" held at Roskilde University, Denmark, from November 1 to November 3, 2001. The meeting focused on recent work in the study of mathematical activity understood according to the outline given above. The lectures, by some of the most outstanding scholars in this area, addressed a variety of issues related to mathematical reasoning. Despite the variety of the contributions there were strong unifying themes which recur in these lectures thereby providing a strong sense of unity and purpose to the present book.

The title of the book “Visualization, Explanation, and Reasoning Styles in Mathematics” is indeed an accurate description of these recurring themes.

The volume is divided into two parts. The first part is called *Mathematical Reasoning and Visualization*.

One question which arises pertaining to mathematical reasoning is to what extent, if any, diagrams and visual imagery can provide us with mathematical knowledge. Most of the contributions in the book touch upon this question but the first part of the book is fully devoted to it. In “Visualization in Logic and Mathematics”, Paolo Mancosu provides a broad introductory discussion of visualization and diagrammatic reasoning and their relevance for recent discussions in the philosophy of mathematics. Mancosu begins by outlining how visual intuition and diagrammatic reasoning were discredited in late nineteenth century and twentieth century analysis and geometry. While diagrams and visual imagery were considered heuristically fruitful their role for justificatory purposes was considered to be unreliable and thus to be avoided. However, recent developments in mathematics and logic have brought back to the forefront the importance of visual imagery and diagrammatic reasoning. Mancosu describes how many mathematicians are calling for more visual approaches to mathematics and the recent developments in logic related to diagrammatic reasoning. In the final part of the paper he discusses how these recent developments affect the traditional foundational debates and describes some recent philosophical attempts to grant to visualization (Giaquinto) and diagrammatic reasoning (Barwise and Etchemendy) an epistemic status which goes beyond the mere heuristic role attributed to them in the past.

With this background the reader can then move on to Marcus Giaquinto’s “From Symmetry Perception to Basic Geometry”. In Frege’s approach to the foundations of mathematics, Frege explicitly excluded that psychological investigations might be relevant to the foundational goal. This was basically motivated by the idea that experience, whether physical or psychological, could not warrant the generalizations drawn from it and thus in this way one could not account for the objectivity of mathematics. The Fregean approach had no account of how our psychological processes relate to our grasping of mathematical truths. Frege’s position rests on the assumption that the only role a perception or an experience of visual imaging can play is that of evidence for a further generalization. By contrast, Giaquinto proposes that experiences of seeing or visual imaging might play the role of “triggers” for belief-forming dispositions which in turn give us geometrical knowledge. He gives an account, meant to be empirically testable, of how we could come to the knowledge of a simple geometrical truth, such as that in a perfect square the two parts either side of the diagonal are congruent. The account proceeds by stages. First, a description of how we perceive squareness (and the role that visual detection of symmetries plays in the process) and how this results in a category representation (in the form of a description set) for the perceptual concept of square. Second, how a modification of the perceptual concept of square gives rise to the concept of a perfect square. Third, it is argued that the possession of these concepts is tantamount to having certain belief forming dispositions which can be triggered by experiences of visual seeing or imaging. However, the role of the visual or imaging experience is not that of evidence but rather of "trigger" for the belief-forming dispositions. Finally, if the mode of acquisition of such triggered beliefs is reliable and there is no violation of epistemic rationality, Giaquinto claims that the beliefs thus obtained constitute knowledge. What type of knowledge? Being non-logical and non-empirical (since the role of experience is not that of providing evidence) the beliefs thus obtained are synthetic a priori. Whether any individual or we as a community of mathematical learners come to have the beliefs in question through the process described ought, according to Giaquinto, to be subject of empirical investigation. If he is right the gap between experience and mathematical knowledge would finally be filled by an account that does justice both to the role of psychological processes and to the objectivity of mathematics.

In a more traditional philosophical context, namely the perennial dispute between platonism and naturalism, James Robert Brown also addresses the role of "seeing" and "intuition" in mathematics and the relevance of diagrams in this context. While providing a defense of Platonism, Brown agrees that with respect to epistemological questions traditional Platonism has always been problematic: How are we to have access to the mathematical entities which exist in an abstract non-causal world? Modern Platonists typically claim that we can "see" or "intuit" the mathematical entities with a special non-sensible intuition. K. Gödel and G. H. Hardy are two of the most well known mathematicians holding this view. Thus, Hardy for instance, sees all mathematical evidence as some sort of perception. But in this respect, according to Brown, he goes too far. We only need, Brown says, to commit ourselves to the perception of some basic mathematical objects and facts. These can then serve as grounds for more advanced mathematics which we cannot directly "see" and this is quite close to Gödel's position. By stressing the fact that much of what is called "seeing" in natural science is quite remote from visual perception, Brown goes on to describe how we can have "seeing" in mathematics through diagrams. Using a simple example of a picture-proof, Brown claims that a diagram can function as a "telescope" that allows us to "see" into the Platonic realm. The diagram displays only a specific case but through the diagram we are able to intuit a general truth and this intuition cannot be confused with the sensory information given by the diagram. Brown's position on intuition is then elaborated further through an analysis of how Freiling's well known informal disproof of the Continuum Hypothesis (by throwing darts at the real line) affects Maddy's naturalism in philosophy of mathematics. The resulting claim is that, according to Brown, there is "some sort of mathematical perception which cannot be reduced to either physical perception or to disguised logical inference".

In his second contribution, Marcus Giaquinto addresses the varieties of mathematical activities which are encountered in mathematical practice. These include, to name only some paradigmatic examples, discovery, explanation, formulation, application, justification and representation. All of these activities provide rich material for a philosophical analysis of mathematics. Unfortunately, until recently philosophers of mathematics have mainly paid attention to only a few of these and, moreover, the of attention has often been too narrowly focused. The extension proposed by Giaquinto concerns not only the proposal to take into account the above mentioned activities but also the various aspects in which mathematics is done and communicated (making, presenting, taking in).

Three important ingredients of mathematical activity are discovery, explanation and justification. The discussion of discovery through visual imaging nicely ties up with the previous material on visualization and again Giaquinto points out that although we might reach knowledge by such means this need not be a proof. Explanation is also a theme touched upon by many contributors in the book. Giaquinto points out that there are proofs which are not explanations and explanations which are not proofs. Of course, there are also examples of proofs which are explanations, and Giaquinto refers to Chemla's paper for an important historical example. Moreover, explanations might play a role in motivating definitions, as illustrated by the moving particle argument which gives a satisfactory account of the use of Euler's formula

$$e^{i\pi} = \cos\pi + i\sin\pi.$$

as a definition in extending the exponential function to complex numbers. Motivating a definition through an explanation is thus an important type of mathematical activity and it can be seen as a form as justification which is distinct from proving a theorem. Another such activity is motivating or 'justifying' the axioms. Giaquinto concludes that extending the philosophical analysis of mathematics to all these aspects, and the many more discussed in his paper, 'would restore to philosophy of mathematics its ancient depth and succulence'.

Part II of the book is entitled *Mathematical Explanation and Proof Styles*.

Jens Høyrup in "On Reasoning Styles in Early Mathematics" discusses aspects of reasoning in Babylonian and Greek mathematics. Høyrup's essay takes its start from a criticism of those historians of mathematics who formulate a distinction between Babylonian and Greek mathematics claiming that the former is basically a collection of ad hoc rules whereas the latter is a reasoned discipline. In addition to show that this is an incorrect characterization of the situation, Høyrup also wants to characterize the reasoning involved in Old Babylonian mathematics in constant comparison with Greek mathematics. What he points out is that there are certainly mathematical tablets in which solutions to problems are given where "no attempt is made to discuss why or under which conditions the operations performed are legitimate and lead to correct results". The situation is made worse by the fact that most of these clay tablets contain no diagrams. But obviously there must have been more that accompanied the process of instruction and learning. Høyrup argues that a few remaining texts from Susa allows us to see the kind of explanations that would have been given orally in a learning context. Moreover, these explanations are 'critical', i.e. provide reasons for the extent of the validity of the procedure under discussion and for why the procedure works. Thus, Old Babylonian mathematics displays its own characteristic style of thought. The real difference with Greek mathematics is that in the Old Babylonian school "the role of critique had been peripheral and accidental; in Greek theoretical mathematics it was, if not the very centre then at least an essential gauge". Høyrup concludes that we cannot count as mathematics any activity that is devoid of understanding and that when the historian works on a mathematical culture for which the sources do not reveal an appeal to reasoning then either we are not understanding the sources or the sources are not an accurate mirror of the mathematical practice.

Another area in history of mathematics which has traditionally been judged against the yardstick of Greek mathematics is Chinese mathematics. Karine Chemla's "The Interplay between Proof and Algorithm in 3-rd Century China" dovetails well with Høyrup's contribution by showing that Chinese mathematics also presents reasoning styles which differ from Greek mathematics but should nonetheless be seen as part of the history of proof. A key case study in this connection is Liu Hui's commentary on *The nine chapters on mathematical procedures*. In particular, Chemla focuses on Liu Hui's commentary on the measurement of the circle. The commentary, made up of two parts, reveals Liu Hui's concerns for explaining why a certain algorithm is correct and it functions as an explanation of the algorithm. The work bears witness to a sophisticated practice of proving mathematical results in ancient China which differs from proving practices in Greek mathematics. Liu Hui was a commentator and this is significant as proofs seem to emerge in Chinese mathematics as the result of such activity. Chemla claims that whereas in Greek mathematics proofs seem mainly aimed at establishing the truth of propositions in the case of Liu Hui what is at stake is the establishment of the correctness of a certain algorithm or possibly showing why the algorithm is correct. Without entering into the details of the analysis let us point out that we have here another interesting case study of mathematical practice which highlights two important facts. First, it shows that the role of proof might be explanatory, in addition to that of certifying a result. Moreover, the reasoning style displayed in these texts represent a characterizing feature of Chinese mathematics and thus it reminds us about the importance of the mathematical culture in which different proof practices are embedded.

Thus, Høyrup's and Chemla's case studies, in addition to their intrinsic importance for the historiographical characterization of Old Babylonian mathematics and Chinese mathematics vs. Greek mathematics, raise important issues concerning mathematical understanding and mathematical explanation and show that these notions are also context-dependent.

Jamie Tappenden's article "Proof style and Understanding in Mathematics I" touches on several topics central to the book such as visualization, explanation, justification, and concept formation. The article focuses on the different styles within complex analysis represented by Weierstrass and Riemann. Weierstrass's methodology was computationally motivated: it aimed at finding explicit representations of functions and algorithms to compute their values. Riemann, by contrast, was more abstract in his approach, more "conceptual". With the introduction of the concept of a Riemann surface, Riemann not only reorganized the subject matter of complex analysis but introduced a whole new style in the area. This new approached yielded new discoveries, new proofs and it deepened our understanding of the subject in unexpected ways. Tappenden explores here the important role that the visualization allowed by Riemann's approach played in this reconfiguration of the subject. The unification yielded by Riemann's approach is also analyzed by Tappenden with reference to contemporary debates on the nature of unification, understanding, and explanation (Friedman, Toulmin, Kitcher). The topic of unification is intimately tied to the discussion of 'fruitful' concepts. Fruitful concepts have unifying and explanatory roles but it is often difficult to say what makes a concept fruitful in mathematics. Tappenden mentions the unification of the theory of algebraic functions of one variable and the theory of algebraic numbers given by Dedekind and Weber. In this case, notions like "ideals" and "fields" turned out to be extremely fruitful concepts: they help us understand "what is going on" and as such discharge an explanatory role. In connecting the topics of visualization, explanation, and unification, Tappenden notes that often, as in the case of Riemann's approach to function theory and Artin's *Geometric Algebra*, a contributor to the "fruitfulness" or "naturalness" of the approach is that the arguments and categories characterizing the approach can be visualized. These aspects of mathematical practice (visualization, explanation, fruitfulness) are often relegated to 'subjective' matters of taste but Tappenden makes a strong case that they can be the topic of fruitful philosophical and methodological analysis. However, he concludes, one should not hope to provide an a priori account of such notions; rather, only detailed case studies of mathematical practice will be able to enlighten us on these complex issues.

The notion of explanation in mathematics, which has appeared in many of the articles discussed above, is the focus of Johannes Hafner and Paolo Mancosu's "The Varieties of Mathematical Explanations". While Hafner and Mancosu emphasize that explanations in mathematics need not be proofs (for instance, theories as a whole might be explanatory), in this paper they restrict attention to proofs. They begin by providing evidence for the claim that mathematicians seek explanations in their ordinary practice and cherish different types of explanations (for instance, many mathematicians are often critical of proofs that only show *that* something is true but do not give an hint of *why* it is true). They go on to suggest that a fruitful approach to the topic of mathematical explanation would consist in providing a taxonomy of recurrent types of mathematical explanation and then trying to see whether these patterns are heterogeneous or can be subsumed under a general account. In the literature on explanation in mathematics there are basically only two philosophical theories on offer. One proposed by Steiner (1978) and an account based on unification due to Kitcher (discussed in the previous article by Tappenden). Mancosu and Hafner provide a case study of how to use mathematical explanations as found in mathematical practice to test theories of mathematical explanation. The case study focuses on Steiner's theory of mathematical explanation. This theory singles out two criteria for a proof to count as explanatory: dependence on a characterizing property and generalizability through varying of that property. The authors argue that Pringsheim's explanatory proof of Kummer's convergence criterion in the theory of infinite series defies both criteria and thus cannot be accounted for by Steiner's model of explanation. This can be seen, as it were, as a case study of how to show that the variety of mathematical explanations cannot be easily reduced to a single model.

The last article of the book is devoted to a very neglected part of mathematical activity: the role of aesthetical factors in mathematics. Obviously, mathematics displays aesthetic features. Mathematicians often talk about the elegance of certain constructions or the beauty of various geometrical figures. But is the 'aesthetic dimension' a rational feature of mathematical activity or a completely subjective and non-analyzable aspect of the mathematical experience? Reviel Netz's "Towards an aesthetic of mathematics" develops several analytical tools required for a productive discussion of this difficult topic. Netz argues from the outset that every type of human expression possesses an aesthetic dimension. Moreover, the aesthetic dimension is an objective fact, although a difficult one to analyze. Whereas most driving factors in mathematical activity are epistemic in the case of beauty we have a clear case of a non-epistemic value which is intrinsic to mathematical research. Netz thus expands the range of topics addressed in the other contributions of the book: "The thrust of the articles collected in this volume is, I believe, to widen our picture of the field of mathematical practice as a rational activity: one that appeals to the visual and not merely to the symbolic, that aims at explanation and not merely at proof. It also appeals, I suggest, to the aesthetic. Among other things – and still as rational practitioners – mathematicians aim at beauty." Netz's paper proceeds by giving a typology of sources of mathematical beauty. Mathematical beauty can be predicated of states of minds, of the products of mathematical activity (say theorems as embodied in texts), and of the objects studied in the previous categories. Netz's analysis focuses on mathematical texts and he proposes to bring to bear for the task a body of theory already developed in poetics. In order to limit the scope he discusses Greek mathematical texts and explores the sense in which techniques of "narrative" and "prosody" can be fruitfully exploited for an analysis of the aesthetic dimension of these texts. In this approach "narrative" will account for the content and "prosody" for the form of the mathematical text. Netz claims that just as in literature one source of beauty in mathematics is the interaction (he calls it "correspondence") between form and content.

Given the emphasis on the heterogeneity of mathematical practice displayed in most of the articles in the present collection, the outcome of the work is not that of claiming that some unique model or theory will account for the great wealth of mathematical activities. Even if such a theory were to be found in the future, it would be premature to suggest anything of the sort at this stage. Rather, through their mathematical, historical, and philosophical richness, these contributions show that there is a wide virgin territory open to investigation. Our hope is that others will also embark in its exploration.

Department of Philosophy
U.C. Berkeley
USA

Section for Philosophy and Science Studies
Roskilde University
Denmark