Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER Applied Mathematics and Computation 166 (2005) 633–637
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

# On a class of quadratically convergent iteration formulae

Mamta, V. Kanwar *, V.K. Kukreja, Sukhjit Singh

*Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Sangrur 148 106, Longowal, Punjab, India*

**Abstract**

In this paper a new class of iterative formulae having quadratic convergence is presented. Furthermore, these algorithms are comparable to the well known method of Newton and the computed results support this theory.


*Keywords:* Newton's method; Iterative formulae; Order of convergence

## 1. Introduction

Almost all the iterative techniques of solution to an equation require the prior knowledge of one or more initial guesses for the desired root. Once an interval is known to contain a root, several classical procedures are available to refine it further. Some of them are modified Regula-falsi [1], Newton's method, Wu and Wu [2], Wu and Fu [3] etc. Ostrowski's [4] well-known book on the solution of equations contains the detailed description of an iterative method for finding

* Corresponding author.
*E-mail address:* vmithil@yahoo.co.in (V. Kanwar).

0096-3003/$ - see front matter 
doi:10.1016/j.amc.2004.07.008

roots. Newton's method is the best known and the most widely used algorithm for a wide variety of problems and used effectively in finding solutions especially of scalar problems. But Newton's method may fail to converge in case the initial guess is far from zero or the derivative is small in the vicinity of the required root. The purpose of this work is to present a new class of iterative techniques having quadratic convergence and can be used as an alternative to Newton's technique or in cases where Newton's technique is not successful.

## 2. Preliminary results

The following two theorems have been proved by Wu and Wu [2].

**Theorem 1.** *Suppose that* $f(x) \in C^1[a,b]$ *and* $pf(x) + f'(x) \neq 0$, *then equation* $f(x) = 0$ *has at most a root.*

**Theorem 2.** *If* $f(x) \in C^1[a,b]$, $f(a)f(b) < 0$ *and* $pf(x) + f'(x) \neq 0$, *then the equation* $f(x) = 0$ *has a unique root in* $(a,b)$.

## 3. A family of new algorithms

We shall present here two different classes of iteration techniques.

Consider the equation

$$f(x) = 0, \tag{3.1}$$

whose roots are to be found.

Let $r$ be the exact root and $x_0$ be the initial guess known for the required root.

Assume

$$x_1 = x_0 + h, \quad |h| \ll 1, \tag{3.2}$$

be the first approximation to the required root.

(a) Consider the following auxiliary equation with a parameter $p$

$$g(x) = p^2(x - x_0)^2 f^2(x) - f^2(x) = 0, \tag{3.3}$$

where $p \in R$ and $|p| < \infty$.

It is evident that the root of Eq. (3.1) is also the root of Eq. (3.3) and vice versa. If $x_1 = x_0 + h$ be the better approximation for the required root, Eq. (3.3) gives

$$p^2 h^2 f^2(x_0 + h) - f^2(x_0 + h) = 0. \tag{3.4}$$

Expanding by Taylor's theorem and simplifying, we get

$$h = \frac{2f(x_0)f'(x_0) \pm \sqrt{4f^2(x_0)f'^2(x_0) + 4f^2(x_0)[p^2f^2(x_0) - f'^2(x_0)]}}{2[p^2f^2(x_0) - f'^2(x_0)]}. \quad (3.5)$$

(Retaining the terms upto $o(h^2)$ excluding the term containing second derivative).

Because of the loss of significant errors implicit in this formula, we rationalize the numerator to obtain the new formula

$$h = \frac{-f(x_0)}{[f'(x_0) \pm pf(x_0)]}, \quad (3.6)$$

in which sign should be so chosen to make the denominator largest in magnitude.

Using (3.6) in (3.2), we get the first approximation to the required root as

$$x_1 = x_0 - \frac{f(x_0)}{[f'(x_0) \pm pf(x_0)]}. \quad (3.7)$$

Therefore, the general formula for successive approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{[f'(x_n) \pm pf(x_n)]}, \quad n = 0, 1 \ldots. \quad (3.8)$$

The parameter $p$ is chosen such that the corresponding function $pf(x_n)$ and $f'(x_n)$ have the same signs. If we let $p \to 0$ in (3.8), Newton's formula is obtained.

The sequence $\{x_n\}$ generated by iteration formula (3.8) with a parameter $p$ is at least quadratically convergent.

(b) If auxiliary equation of the following form is assumed

$$g(x) = p^2(x - x_0)^2 f(x) - f(x) = 0, \quad (3.9)$$

where $p \in R$, then again the root of (3.1) is also the root of (3.9).

Putting $x_1 = x_0 + h$ in (3.9) and using Taylor's expansion (retaining the terms upto $o(h^2)$ excluding the term containing second derivative), after some simplifications we have another general formula for successive approximation as

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) + 4p^2f^2(x_n)}}, \quad n = 0, 1 \ldots, \quad (3.10)$$

in which sign should be so chosen so as to make the denominator largest in magnitude.

If we let $p \to 0$ in (3.10), again Newton's formula is obtained.

## 4. Convergence analysis

Now we are presenting here the mathematical proof for the order of convergence of iterative technique (3.10) and the order of convergence for the remaining can be proved similarly.

**Theorem.** *Let f: $D \to R$, for an open interval D. Assume that f has first and second derivatives in D. If f(x) has a simple root at $r \in D$ and $x_0$ is an initial guess sufficiently close to r, then the formula defined by (3.10) satisfies the following error equation*

$$e_{n+1} = C_2 e_n^2 + o(e_n^3), \tag{4.1}$$

*where* $C_j = (1/j!)\dfrac{f^{(j)}(r)}{f'(r)}, \quad j = 1, 2, \ldots$

**Proof.** The suggested (3.10) technique is

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) \pm \sqrt{f'^2(x_n) + 4p^2 f^2(x_n)}}, \quad n = 0, 1 \ldots, \tag{4.2}$$

where $p \in R$. If $r$ be a simple root and $e_n$ be the error at $n$th iteration, then

$$f(r) = 0, \quad f'(r) \neq 0 \quad \text{and} \quad x_n = r + e_n.$$

Using Taylor's expansion

$$f(x_n) = f(r + e_n) = f'(r)\lfloor e_n + C_2 e_n^2 + o(e_n^3)\rfloor. \tag{4.3}$$

Furthermore, we have

$$f'(x_n) = f'(r + e_n) = f'(r)\lfloor 1 + 2C_2 e_n + 3C_3 e_n^2 + o(e_n^3)\rfloor. \tag{4.4}$$

On dividing (4.3) by (4.4), after some simplifications, we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - C_2 e_n^2 + o(e_n^3). \tag{4.5}$$

Also

$$\left(\frac{f(x_n)}{f'(x_n)}\right)^2 = e_n^2 + o(e_n^3). \tag{4.6}$$

From (4.2), we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left[1 + p^2 \frac{f^2(x_n)}{f'^2(x_n)}\right]}. \tag{4.7}$$

Eq. (4.7) upon using (4.5) and (4.6), finally gives

$$e_{n+1} = (C_2 + p^2)e_n^2 + o(e_n^3), \tag{4.8}$$

which proves that given technique is quadratically convergent for every value of $p \in R$.

Table 1
Comparison of present method with Newton's method

| Equation | Initial guess | Number of Iterations | | Roots | |
|---|---|---|---|---|---|
| | | Newton | Case(b) | Newton | Case(b) |
| $x^{10}-1=0$ | 0.0 | fails | 1 | fails | 1 |
| | 0.5 | 42 | 9 | 1 | 1 |
| $x^2-4=0$ | 0.0 | fails | 5 | fails | 2 |
| $4x^4-4x^2=0$ | $\pm\frac{\sqrt{21}}{7}$ | divergent | 31 | divergent | 0 |
| $\tan^{-1}(x)=0$ | −1 | 5 | 4 | 0 | 0 |
| | 3 | divergent | 6 | divergent | 0 |
| $\sin(x)=0$ | 1.5 | 3 | 4 | −12.5663709641 | 0.0 |
| $\ln(x)=0$ | 0.5 | 4 | 5 | 1 | 1 |
| | 5 | divergent | 8 | divergent | 1 |
| $\exp(x^2+7x-30)-1=0$ | 2 | divergent | 2 | divergent | 3 |
| | 3.5 | 11 | 11 | 3 | 3 |
| $e^x-1-\cos(\pi x)=0$ | −0.10 | 71 | 5 | −7.3182411194 | 0.3692564070 |

## 5. Numerical experimentation

A comparison of the method proposed in Case (b) with Newton's method is presented in Table 1 with the help of various examples. Here the formulae of Case (b) are tested for $p = 1$ with termination criterion $|f(x)| < 1.0\times10^{-11}$.

## 6. Conclusions

In applying Newton's method to solve the equation $4x^4-4x^2=0$, problems arise if the points cycle back and forth from one to another. The points $\pm\frac{\sqrt{21}}{7}$ cycle, each leading to the other and back. Further, if we include the term containing the second derivative in Taylor's expansion of Case (a) and Case (b), then the resulting techniques are faster but with a drawback that in each iteration three functional values viz. $f(x)$, $f'(x)$ and $f''(x)$ are evaluated unlike Newton's technique. Therefore, these techniques can be used as an alternative to Newton's technique or in the cases where Newton's technique is not successful.

## References

[1] M. Dowell, P. Jarrat, A modified Regula-Falsi method for computing the root of an equation, BIT 11 (1971) 168–174.
[2] X. Wu, H.W. Wu, On a class of quadratic convergence iteration formulae without derivatives, Applied Mathematics and Computation 10 (7) (2000) 77–80.
[3] X. Wu, D. Fu, New high-order convergence iteration methods without employing derivatives for solving nonlinear equations, Computers and Mathematics with Applications 41 (2001) 489–495.
[4] A.M. Ostrowski, Solution of Equations in Euclidean and Banach Space, third ed., Academic Press, New York, 1973.