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Harmonic Univalent Mappings into a Half-Plane with Nonreal Vertical Slits

W. Majchrzak

Department of Mathematics, University of Łódź, ul. Banacha 22, 90-238 Łódź, Poland
E-mail: wmajch@math.uni.lodz.pl

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A class of harmonic univalent mappings is constructed by applying the method of Clunie and Sheil-Small. These mappings, assuming their values in a half-plane with a vertical boundary, omit two vertical half-lines symmetric w.r.t. the real axis. Several basic properties are proved.

Key Words: planar univalent harmonic mappings; mappings convex in one direction.

In papers [4, 7, 8] the authors studied the classes of harmonic, univalent, and sense-preserving mappings that transform the unit disk $\Delta = \{z : |z| < 1\}$ onto the complex plane $\mathbb{C}$ slit along one or two half-lines included in the real axis. The closure of these classes (in the topology of locally uniform convergence) was also considered, and some extremal problems concerning mappings of the closure were solved.

Our aim is to construct a class of harmonic, univalent, and sense-preserving mappings defined on Δ, which, assuming their values in a half-plane with a vertical boundary, omit two vertical half-lines symmetric w.r.t. the real axis.

For the construction, we shall apply the well-known construction of Clunie and Sheil-Small (cf. [2, 5]), also used in [4, 7, 8], taking for a generating function a conformal mapping p_0 that is defined as follows.

Definition 0.1. For given constants α, m, M such that $0 < m < 1 < M$, $0 < \alpha < \alpha_0 \leq 1/2$, where $\alpha_0 = (1-m)/(M-m)$, consider a function

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$p_0\colon \Delta \to \mathbb{C}$ of the form

$$p_0(z;\alpha,m,M) = (1-\alpha)m + \alpha M + (\alpha_0 - \alpha)(M-m)\frac{1+z}{1-z} + i\frac{M-m}{\pi}\log\frac{1-e^{i\alpha\pi}z}{1-e^{-i\alpha\pi}z}, \tag{1}$$

where $\log[(1-e^{i\alpha\pi}z)/(1-e^{-i\alpha\pi}z)]|_{z=0} = 0$.

From (1) we immediately have $p_0(0) = 1$ and $p_0'(0) > 0$.

The function p_0 is a conformal mapping of Δ onto Ω, which is a half-plane $\{w : \operatorname{Re} w > m\}$ from which two half-lines,

$$l_1 = \left\{M + iv : v \geq \frac{1}{\pi}(M-m)d\right\} \quad \text{and} \quad l_2 = \{w : \bar{w} \in l_1\}, \tag{2}$$

are removed, where $d > 0$ has the form

$$d = d(\alpha, m, M) = \ln\frac{\delta+1}{\delta-1} + \frac{2\delta}{\delta^2-1} \tag{3}$$

with

$$\delta = \delta(\alpha, m, M) = \left(\frac{2 + (\alpha_0-\alpha)\pi\cot(\alpha(\pi/2))}{(\alpha_0-\alpha)\pi\cot(\alpha(\pi/2))}\right)^{1/2}, \qquad \delta > 1. \tag{4}$$

For fixed m, M, by (3) and (4), we get that $d(\alpha)$ decreases for $\alpha \in (0, \alpha_0)$ and $d(\alpha) \to 0$ as $\alpha \to \alpha_0^-$ and $d(\alpha) \to +\infty$ as $\alpha \to 0^+$.

Let ζ_1, ζ_2 be arbitrary points of the unit circle $\partial\Delta$. Then denote by $\widehat{\zeta_1\zeta_2}$ an open arc of $\partial\Delta$ going from ζ_1 to ζ_2 in the counterclockwise direction.

Next, put $\hat{p}_0(e^{it}) = \lim_{r\to 1^-} p_0(re^{it})$ for $t \in (0, 2\pi)\setminus\{\pm\alpha\pi\}$ and $\hat{p}_0(\zeta) = \infty$, $\zeta \in \{1, e^{\pm i\alpha\pi}\}$.

If $\hat{p}_0(z) := p_0(z)$, $z \in \Delta$, then $\hat{p}_0$ is a continuous extension of p_0 to $\overline{\Delta} = \{z : |z| \leq 1\}$.

Using (2), notice that $\hat{p}_0^{-1}(l_1) = \widehat{1\, e^{i\alpha\pi}}$, $\hat{p}_0^{-1}(l_2) = \widehat{e^{-i\alpha\pi} 1}$, and $\hat{p}_0^{-1}(\{w : \operatorname{Re} w = m\}) = \widehat{e^{i\alpha\pi}e^{-i\alpha\pi}}$.

Thus we have

$$\operatorname{Re}\hat{p}_0(\zeta) = \begin{cases} M & \text{for } \zeta \in \Im_\alpha^- \cup \Im_\alpha^+, \\ m & \text{for } \zeta \in \partial\Delta\setminus\overline{(\Im_\alpha^- \cup \Im_\alpha^+)}, \end{cases}$$

where $\Im_\alpha^- = \{e^{it} : t \in (-\alpha\pi, 0)\}$, $\Im_\alpha^+ = \{e^{it} : t \in (0, \alpha\pi)\}$, and $\overline{(\Im_\alpha^- \cup \Im_\alpha^+)} = \Im_\alpha^- \cup \Im_\alpha^+ \cup \{1, e^{\pm i\alpha\pi}\}$.

In the sequel, the points ζ_0, $\overline{\zeta_0}$ that are preimages under $\hat{p}_0$ of finite ends of “conformal” slits l_1, l_2 play a significant role. We can compute that

$$\begin{aligned}\zeta_0 &= \zeta_0(\alpha, m, M) \\ &= (1+b)^{-1}\left(b + \cos(\alpha\pi) + 2i\left(b + \cos^2\left(\alpha\frac{\pi}{2}\right)\right)^{1/2}\sin\left(\alpha\frac{\pi}{2}\right)\right),\end{aligned} \tag{5}$$

where

$$b = b(\alpha, m, M) = \sin(\alpha\pi)/((\alpha_0 - \alpha)\pi), \tag{6}$$

and, in (5), $(\cdot)^{1/2}$ is positive.

For convenience, we present here the graph of p_0 (Fig. 1).

DEFINITION 0.2. For fixed α, m, M satisfying the restrictions of Definition 0.1, let $\mathcal{F}(\alpha, m, M)$ denote the class of all harmonic mappings $f\colon \Delta \to \mathbb{C}$ of the form $f = \operatorname{Re} F + i \operatorname{Re} G$ where

$$F = p_0 \qquad \text{and} \qquad G = -i\int_0^z p_0'(\xi)p(\xi)d\xi, \tag{7}$$

and $p \in \wp$ is any holomorphic function in Δ, $\operatorname{Re} p(z) > 0$, $z \in \Delta$, $p(0) = 1$.

If $f \in \mathcal{F}(\alpha, m, M)$, then $f(0) = 1$, $f_z(0) > 0$, $f_{\bar{z}}(0) = 0$, and $\operatorname{Re} f(z) > m$, $z \in \Delta$.

It is well known that if we put $h = 1/2(F + iG)$ and $g = 1/2(F - iG)$, then any harmonic function $\operatorname{Re} F + i \operatorname{Re} G$ admits an equivalent form $h + \bar{g}$.

In particular, this concerns $f \in \mathcal{F}(\alpha, m, M)$. Notice that, by (7),

$$a = \frac{g'}{h'} = \frac{F' - iG'}{F' + iG'} = \frac{1 - p}{1 + p},$$

hence $a(0) = 0$, and, by the subordination $p(z) \prec (1 + z)/(1 - z)$, $z \in \Delta$, $p \in \wp$, we also have $|a(z)| < 1$, $z \in \Delta$.

Thus the Jacobian $J_f = |h'|^2 - |g'|^2 > 0$ in Δ, and, by the well-known theorem of Lewy [6], any $f = h + \bar{g} \in \mathcal{F}(\alpha, m, M)$ is locally univalent and sense-preserving.

For the sake of completeness, we employ here the well-known theorem.

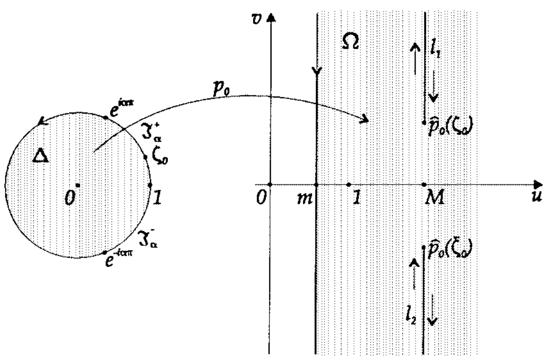

FIGURE 1

THEOREM 0.1 (Clunie and Sheil-Small [2]). *A harmonic $f = \bar{g} + h$ locally univalent in Δ is a univalent mapping of Δ onto a domain convex in the direction of the real axis if and only if $h - g$ is a conformal univalent mapping of Δ onto a domain convex in the direction of the real axis.*

For any $f = h + \bar{g} \in \mathcal{F}(\alpha, m, M)$, from (7) we have $p_0 = h + g$, and, since p_0 is convex in the direction of the imaginary axis, $-ip_0 = -ih - ig$ is convex in the direction of the real axis. Thus, by the theorem cited above, $-ih + \overline{ig} = -if$ is also univalent and convex in the direction of the real axis.
Summing up, we have

PROPOSITION 0.1. *All mappings f in $\mathcal{F}(\alpha, m, M)$ are harmonic, univalent, sense-preserving, and convex in the direction of the imaginary axis.*

For $z \in \Delta$ and $|\eta| = 1$, define

$$K(z, \eta) = 2(M - m) \int_0^z \left[\frac{\sin(\alpha\pi)}{\pi} \frac{1}{(e^{-i\alpha\pi} - \xi)(e^{i\alpha\pi} - \xi)} + \frac{\alpha_0 - \alpha}{(1 - \xi)^2} \right] \frac{1 + \eta\xi}{1 - \eta\xi} d\xi. \tag{8}$$

Let $\mathfrak{M}$ denote the family of all probability measures on all Borel sets of the circle $|\eta| = 1$.

Using the Riesz–Herglotz integral representation of $p \in \wp$, by (1), (7), and (8) we obtain

PROPOSITION 0.2. *If $f \in \mathcal{F}(\alpha, m, M)$, then there is a unique $\mu \in \mathfrak{M}$ such that*

$$f(z) = \operatorname{Re} p_0(z) + i \operatorname{Im} \int_{|\eta|=1} K(z, \eta) d\mu(\eta), \tag{9}$$

where

$$\operatorname{Re} p_0(z) = (1 - \alpha)m + \alpha M - \frac{1}{\pi}(M - m)\left(\arg \frac{1 - e^{i\alpha\pi} z}{1 - z} - \arg \frac{1 - e^{-i\alpha\pi} z}{1 - z} \right) + (\alpha_0 - \alpha)(M - m) \operatorname{Re} \frac{1 + z}{1 - z}; \tag{10}$$

conversely, for any $\mu \in \mathfrak{M}$, function (9) is in $\mathcal{F}(\alpha, m, M)$.
We assume that $\arg(\cdot) \in (-\pi, \pi]$ *in* (10) *and further on.*

Integral representation (9) allows one to obtain

PROPOSITION 0.3. *$\mathcal{F}(\alpha, m, M)$ is convex and compact (in the topology of locally uniform convergence), and all of its extreme points are*

$$f_{(\eta)}(z) = \operatorname{Re} p_0(z) + i \operatorname{Im} K(z, \eta), \qquad |\eta| = 1, \quad z \in \Delta. \tag{11}$$

Proof. As we know, $\mathfrak{M}$ is a *-weak compact and convex set, and all of its extreme points are unit point masses. Since $\mu \to \operatorname{Im} \int_{|\eta|=1} K(z,\eta)\,d\mu$ is a linear homeomorphism, we get what has been asserted. ∎

More properties of $f \in \mathcal{F}(\alpha, m, M)$ will be obtained by studying the sets $f_{(\eta)}(\Delta)$ where $f_{(\eta)}$ are extreme points (11).

Let us denote $k(z,\eta) = 1/[2(M-m)] \operatorname{Im} K(z,\eta)$. After integration and a suitable arrangement, from (8) we get

$$k(z,\eta) = \begin{cases} \dfrac{\cot(\alpha\pi/2)}{2\pi}\left(\arg\dfrac{1-e^{i\alpha\pi}z}{1-z} + \arg\dfrac{1-e^{-i\alpha\pi}z}{1-z}\right) \\ \quad + (\alpha_0-\alpha)\operatorname{Im}\dfrac{z}{(1-z)^2} \qquad \text{for } \eta = 1, & (12) \\ \dfrac{\cot(\alpha\pi)}{2\pi}\left(\arg\dfrac{1-e^{-i\alpha\pi}z}{1-z} - \arg\dfrac{1-e^{i\alpha\pi}z}{1-z}\right) \\ \quad - \dfrac{1}{\pi}\operatorname{Re}\dfrac{e^{i\alpha\pi}z}{1-e^{i\alpha\pi}z} + (\alpha_0-\alpha) \\ \quad \times\left(\dfrac{2}{|1-e^{i\alpha\pi}|^2}\arg\dfrac{1-e^{i\alpha\pi}z}{1-z} + \cot\dfrac{\alpha\pi}{2}\operatorname{Re}\dfrac{z}{1-z}\right) \\ \qquad \text{for } \eta = e^{i\alpha\pi}, & (13) \\ \dfrac{\cot(\alpha\pi)}{2\pi}\left(\arg\dfrac{1-e^{i\alpha\pi}z}{1-z} - \arg\dfrac{1-e^{-i\alpha\pi}z}{1-z}\right) \\ \quad + \dfrac{1}{\pi}\operatorname{Re}\dfrac{e^{-i\alpha\pi}z}{1-e^{-i\alpha\pi}z} + (\alpha_0-\alpha) \\ \quad \times\left(\dfrac{2}{|1-e^{i\alpha\pi}|^2}\arg\dfrac{1-e^{-i\alpha\pi}z}{1-z} - \cot\dfrac{\alpha\pi}{2}\operatorname{Re}\dfrac{z}{1-z}\right) \\ \qquad \text{for } \eta = e^{-i\alpha\pi}, & (14) \\ \left[\dfrac{\sin(\alpha\pi)}{\pi}\dfrac{-2\bar\eta}{(\bar\eta-e^{i\alpha\pi})(\bar\eta-e^{-i\alpha\pi})} + \dfrac{2(\alpha_0-\alpha)}{|1-\eta|^2}\right] \\ \quad \times\arg\dfrac{1-\eta z}{1-z} - \dfrac{i}{2\pi}\dfrac{\bar\eta+e^{i\alpha\pi}}{\bar\eta-e^{i\alpha\pi}}\arg\dfrac{1-e^{-i\alpha\pi}z}{1-z} \\ \quad + \dfrac{i}{2\pi}\dfrac{\bar\eta+e^{-i\alpha\pi}}{\bar\eta-e^{-i\alpha\pi}}\arg\dfrac{1-e^{i\alpha\pi}z}{1-z} - i(\alpha_0-\alpha) \\ \quad \times\dfrac{1+\eta}{1-\eta}\operatorname{Re}\dfrac{z}{1-z} \qquad \text{for other } \eta, \quad \eta \in \partial\Delta. & (15) \end{cases}$$

Remark 0.1. For $\eta = \zeta_0$ and $\eta = \bar{\zeta}_0$, where ζ_0 is point (5), we have

$$\frac{\sin(\alpha\pi)}{\pi} \frac{\bar{\eta}}{(\bar{\eta} - e^{i\alpha\pi})(\bar{\eta} - e^{-i\alpha\pi})} - \frac{\alpha_0 - \alpha}{|1-\eta|^2} = 0.$$

Then $\arg[(1-\eta z)/(1-z)]$ does not appear in (15).

This holds only for the above-mentioned values of η with all admissible α, m, M. Hence, by (15), in particular for $\eta = \zeta_0$ and $\eta = \bar{\zeta}_0$, we have

$$k(z,\eta) = \frac{i}{2\pi}\left(\frac{\bar{\eta} + e^{-i\alpha\pi}}{\bar{\eta} - e^{-i\alpha\pi}} \arg \frac{1 - e^{i\alpha\pi} z}{1-z} - \frac{\bar{\eta} + e^{i\alpha\pi}}{\bar{\eta} - e^{i\alpha\pi}} \arg \frac{1 - e^{-i\alpha\pi} z}{1-z}\right) - i(\alpha_0 - \alpha)\frac{1+\eta}{1-\eta} \operatorname{Re} \frac{z}{1-z}. \tag{16}$$

Let h^p denote the standard Hardy space of harmonic functions on Δ. Since each $f \in \mathcal{F}(\alpha, m, M)$ is close to convex, $f \in h^p$ for $p \in (0, 1/3)$ (see [10]). Therefore we may put

$$\hat{f}(e^{it}) := \lim_{r\to 1^-} f(re^{it}) \qquad \text{a.e. in } \langle 0, 2\pi).$$

In particular, for the extreme points $f_{(\eta)}$, formulae (10), (11), and (12)–(16) yield that among $\zeta \in \partial\Delta$, only the three points $\zeta = 1, e^{\pm i\alpha\pi}$ correspond to ∞.

Now, we can determine the following cluster sets of the functions mentioned below:

(i) for $\arg(1-\eta\zeta)/(1-\zeta)$ at $\zeta = \bar{\eta}$, $\eta = e^{i\beta}$, it is the interval $[\beta/2 - \pi, \beta/2]$ if $\beta \in (0, \pi]$, and it is the interval $[\beta/2, \beta/2 + \pi]$ if $\beta \in (-\pi, 0)$;

(ii) for $\arg(1 - e^{i\alpha\pi}\zeta)/(1-\zeta)$ at $\zeta = e^{-i\alpha\pi}$, it is the interval $[\alpha\pi/2 - \pi, \alpha\pi/2]$;

(iii) for $\arg(1 - e^{-i\alpha\pi}\zeta)/(1-\zeta)$ at $\zeta = e^{i\alpha\pi}$, it is the interval $[-\alpha\pi/2, -\alpha\pi/2 + \pi]$.

Thus, at $e^{is} \in \{\bar{\eta}, 1, e^{\pm i\alpha\pi}\}$ only, the limits $\lim_{t\downarrow s} \hat{f}_{(\eta)}(e^{it})$ and $\lim_{t\uparrow s} \hat{f}_{(\eta)}(e^{it})$ can be distinct.

Taking account of (10)–(16), we can state that $\hat{f}_{(\eta)}$ assume four finite values, or three only if $\eta \in \{1, \zeta_0, \bar{\zeta}_0, e^{\pm i\alpha\pi}\}$.

First, we shall examine the distribution of values of $\hat{f}_{(\eta)}$ for $\beta = \arg\eta \in [0, \pi]$. Notice that the distribution of values of $\hat{f}_{(\eta)}$ for $\beta \in (-\pi, 0)$ compared with the one for $-\beta \in (0, \pi)$ is symmetric w.r.t. the real axis.

To describe the distribution of values of $\hat{f}_{(\eta)}$, we shall use the functions following from (12)–(16),

$$\widetilde{v}_1(\beta) := \begin{cases} \phi(\beta;\alpha,\alpha_0)\dfrac{\beta}{2} + \dfrac{1}{4}\Big(\alpha\cot\dfrac{\alpha\pi-\beta}{2} + (2-\alpha)\cot\dfrac{\alpha\pi+\beta}{2}\Big) \\ \quad -\dfrac{\alpha_0-\alpha}{2}\cot\dfrac{\beta}{2} \qquad \text{for } \beta\in(0,\pi]\backslash\{\alpha\pi\}, \\ \dfrac{1}{2}\cot\big(\alpha\dfrac{\pi}{2}\big) \qquad \text{for } \beta=0, \\ \dfrac{1}{2\pi} + \dfrac{1-\alpha}{2}\cot(\alpha\pi) + \dfrac{\alpha_0-\alpha}{4}\dfrac{\alpha\pi-\sin(\alpha\pi)}{\sin^2(\alpha\pi/2)} \quad \text{for } \beta=\alpha\pi; \end{cases} \tag{17}$$

$$\tilde{v}_2(\beta) := \begin{cases} \phi(\beta;\alpha,\alpha_0)\dfrac{\beta}{2} + \dfrac{\alpha}{4}\Big(\cot\dfrac{\alpha\pi-\beta}{2} - \cot\dfrac{\alpha\pi+\beta}{2}\Big) \\ \quad -\dfrac{\alpha_0-\alpha}{2}\cot\dfrac{\beta}{2} \qquad \text{for } \beta\in(0,\pi]\backslash\{\alpha\pi\}, \\ 0 \qquad \text{for } \beta=0, \\ \dfrac{1}{2\pi} - \dfrac{\alpha}{2}\cot(\alpha\pi) + \dfrac{\alpha_0-\alpha}{4}\dfrac{\alpha\pi-\sin(\alpha\pi)}{\sin^2(\alpha(\pi/2))} \quad \text{for } \beta=\alpha\pi; \end{cases} \tag{18}$$

$$\tilde{v}_3(\beta) := \begin{cases} \phi(\beta;\alpha,\alpha_0)\dfrac{\beta}{2} - \dfrac{1}{4}\Big((2-\alpha)\cot\dfrac{\alpha\pi-\beta}{2} + \alpha\cot\dfrac{\alpha\pi+\beta}{2}\Big) \\ \quad -\dfrac{\alpha_0-\alpha}{2}\cot\dfrac{\beta}{2} \qquad \text{for } \beta\in(0,\alpha\pi), \\ -\dfrac{1}{2}\cot\Big(\alpha\dfrac{\pi}{2}\Big) \qquad \text{for } \beta=0; \end{cases} \tag{19}$$

$$\tilde{v}_4(\beta) := \begin{cases} \phi(\beta;\alpha,\alpha_0)\Big(\dfrac{\beta}{2}-\pi\Big) - \dfrac{1}{4}\Big((2-\alpha)\cot\dfrac{\alpha\pi-\beta}{2} \\ \qquad +\alpha\cot\dfrac{\alpha\pi+\beta}{2}\Big) - \dfrac{\alpha_0-\alpha}{2}\cot\dfrac{\beta}{2} \\ \qquad \text{for } \beta\in(0,\pi]\backslash\{\alpha\pi\}, \\ \dfrac{1}{2\pi} + \dfrac{1-\alpha}{2}\cot(\alpha\pi) - \dfrac{\alpha_0-\alpha}{4}\dfrac{(2-\alpha)\pi+\sin(\alpha\pi)}{\sin^2(\alpha(\pi/2))} \\ \qquad \text{for } \beta=\alpha\pi; \end{cases} \tag{20}$$

$$\tilde{v}_5(\beta) := \phi(\beta;\alpha,\alpha_0)\left(\frac{\beta}{2}-\pi\right) + \frac{\alpha}{4}\left(\cot\frac{\alpha\pi-\beta}{2} - \cot\frac{\alpha\pi+\beta}{2}\right) - \frac{\alpha_0-\alpha}{2}\cot\frac{\beta}{2} \qquad \text{for } \beta\in(\alpha\pi,\pi], \tag{21}$$

where, in (17)–(21), we use the notation

$$\phi(\beta;\alpha,\alpha_0) = \frac{\sin(\alpha\pi)}{\pi}\frac{1}{\cos(\alpha\pi)-\cos\beta} + \frac{\alpha_0-\alpha}{1-\cos\beta}.$$

Moreover, put

$$v_k := 2(M-m)\tilde{v}_k, \qquad k = 1,2,\dots,5. \tag{22}$$

For $\eta = e^{i\beta}$, $\beta\in[0,\pi]$, particularly for $\eta = \zeta_0 = e^{i\beta_0}$ (cf. (5)), using (17)–(21), we obtain the 1° – 6° results

1° for $\beta = 0$,

$$\lim_{r\to 1^-} k(r\zeta,1) = \begin{cases} \tilde{v}_1(0) & \text{if } \zeta\in\widehat{1\ e^{i\alpha\pi}},\\ \tilde{v}_2(0)=0 & \text{if } \zeta\in\widehat{e^{i\alpha\pi}e^{-i\alpha\pi}},\\ \tilde{v}_3(0) = -\tilde{v}_1(0) & \text{if } \zeta\in\widehat{e^{-i\alpha\pi}\ 1}; \end{cases} \tag{23}$$

2° for $\beta\in(0,\beta_0)$,

$$\lim_{r\to 1^-} k(r\zeta,\eta) = \begin{cases} \tilde{v}_1(\beta) & \text{if } \zeta\in\widehat{1\ e^{i\alpha\pi}},\\ \tilde{v}_2(\beta) & \text{if } \zeta\in\widehat{e^{i\alpha\pi}e^{-i\alpha\pi}},\\ \tilde{v}_3(\beta) & \text{if } \zeta\in\widehat{e^{-i\alpha\pi}\ \bar{\eta}},\\ \tilde{v}_4(\beta) & \text{if } \zeta\in\widehat{\bar{\eta}\ 1}; \end{cases} \tag{24}$$

3° for $\beta = \beta_0$,

$$\lim_{r\to 1^-} k(r\zeta,\zeta_0) = \begin{cases} \tilde{v}_1(\beta_0) & \text{if } \zeta\in\widehat{1\ e^{i\alpha\pi}},\\ \tilde{v}_2(\beta_0) & \text{if } \zeta\in\widehat{e^{i\alpha\pi}e^{-i\alpha\pi}},\\ \tilde{v}_3(\beta_0) = \tilde{v}_4(\beta_0) & \text{if } \zeta\in\widehat{e^{-i\alpha\pi}\ 1}; \end{cases} \tag{25}$$

4° for $\beta\in(\beta_0,\alpha\pi)$,

$$\lim_{r\to 1^-} k(r\zeta,\eta) = \begin{cases} \tilde{v}_1(\beta) & \text{if } \zeta\in\widehat{1\ e^{i\alpha\pi}},\\ \tilde{v}_2(\beta) & \text{if } \zeta\in\widehat{e^{i\alpha\pi}e^{-i\alpha\pi}},\\ \tilde{v}_3(\beta) & \text{if } \zeta\in\widehat{e^{-i\alpha\pi}\ \bar{\eta}},\\ \tilde{v}_4(\beta) & \text{if } \zeta\in\widehat{\bar{\eta}\ 1}; \end{cases} \tag{26}$$

5° for $\beta = \alpha\pi$,

$$\lim_{r\to 1^-} k(r\zeta,e^{i\alpha\pi}) = \begin{cases} \tilde{v}_1(\alpha\pi) & \text{if } \zeta\in\widehat{1\ e^{i\alpha\pi}},\\ \tilde{v}_2(\alpha\pi) & \text{if } \zeta\in\widehat{e^{i\alpha\pi}e^{-i\alpha\pi}},\\ \tilde{v}_4(\alpha\pi) & \text{if } \zeta\in\widehat{e^{-i\alpha\pi}\ 1}; \end{cases} \tag{27}$$

6° for $\beta \in (\alpha\pi, \pi]$,

$$\lim_{r\to 1^-} k(r\zeta, \eta) = \begin{cases} \tilde{v}_1(\beta) & \text{if } \zeta \in \widehat{1e^{i\alpha\pi}}, \\ \tilde{v}_2(\beta) & \text{if } \zeta \in \widehat{e^{i\alpha\pi}\bar{\eta}}, \\ \tilde{v}_5(\beta) & \text{if } \zeta \in \widehat{\bar{\eta}e^{-i\alpha\pi}}, \\ \tilde{v}_4(\beta) & \text{if } \zeta \in \widehat{e^{-i\alpha\pi}1}. \end{cases} \tag{28}$$

To determine what the distribution of boundary values of $\hat{f}_{(\eta)}$ is like, besides (23)–(28) we must examine the monotonicity and interrelations of all v_k (see (22)), mostly for $k = 1, 3, 4$. Since v_k are differentiable and $\alpha_0 \le 1/2$, in a well-known but tiresome way one can obtain

LEMMA 0.1. (a) v_1 *is decreasing in* $[0, \pi]$ *from* $(M - m)\cot(\alpha\frac{\pi}{2})$ *to* $\frac{1}{2}(\alpha_0 - \alpha)(M - m)\pi$, *and, for any* α, m, M,

$$\frac{1}{2}(\alpha_0 - \alpha)(M - m)\pi < \operatorname{Im} \hat{p}_0(\zeta_0) < (M - m)\cot\left(\alpha\frac{\pi}{2}\right); \tag{29}$$

(b) v_2 *is continuous for* $\beta \in [0, \pi]$; *hence* v_2 *is bounded, and, for any* α, m, M, *the relation* $v_5(\beta) < v_2(\beta)$, $\beta \in (\alpha\pi, \pi]$, *holds* (*cf.* (e));

(c) v_3 *is decreasing in* $[0, \alpha\pi)$ *from* $-(M - m)\cot(\alpha\frac{\pi}{2})$ *to* $-\infty$; *what is more*, $v_3(\beta_0) = v_4(\beta_0)$ *only* (*cf.* (5) *and* (d) *below*);

(d) v_4 *increases in* $(0, \pi]$ *from* $-\infty$ *to* $\max_\beta v_4(\beta) = v_4(\beta_1)$, $\beta_1 \in (\alpha\pi, \pi)$, *and decreases near* $\beta = \pi$, *but* $v_4(\pi) > \operatorname{Im} \hat{p}_0(\bar{\zeta}_0)$, *where* $v_4(\pi) = -v_1(\pi)$; *moreover*, $v_4(\beta) < v_1(\beta_1)$ *holds for* $\beta \in (0, \pi]$, *and* $\min_\beta(v_1 - v_4)(\beta) = (v_1 - v_4)(\pi)$, *although* $\max_\beta v_4(\beta)$ *can be positive, depending on* α;

(e) v_5 *increases in* $(\alpha\pi, \pi]$ *from* $-\infty$ *to* $v_5(\pi)$, $v_5(\beta) < v_2(\beta)$ *holds for* $\beta \in (\alpha\pi, \pi]$, *and* $\min_\beta(v_2 - v_5)(\beta) = (v_2 - v_5)(\pi) = (\alpha_0 - \alpha)(M - m)\pi + 2(M - m)\tan(\alpha\frac{\pi}{2})$.

To complete our observations concerning $\hat{f}_{(\eta)}(\partial\Delta)$, we shall prove that the cluster sets of $\hat{f}_{(\eta)}$ at 1, $e^{-i\alpha\pi}$ and $e^{i\alpha\pi}$ are line segments or two parallel half-lines.

For that goal, it is convenient to replace $z \in \Delta$ by $\omega = (1 + z)/(1 - z) := \theta + i\nu$, where $\omega \in \{\omega : \operatorname{Re} \omega > 0\}$ (cf. [2, 7]).

Since the mappings from $\mathcal{F}(\alpha, m, M)$ are convex in the direction of the imaginary axis, suppose that

$$\operatorname{Re} f_{(\eta)}(\omega) = c, \qquad c > m, \tag{30}$$

and, depending on c, find the bounds of $\operatorname{Im} f_{(\eta)}$ for all η.

It is sufficient to do this for $\eta = \exp(i\beta)$, $\beta \in [0, \pi]$.

Using (10), transform (30) to the following form:

$$\arg \frac{1 + i\omega\tan(\alpha(\pi/2))}{1 - i\omega\tan(\alpha(\pi/2))} = \frac{c - m}{M - m}\pi - (\alpha_0 - \alpha)\pi\theta. \tag{31}$$

The solution of (31) is an implicit function $\nu = \nu(\theta)$ with two real branches,

$$\nu_n(\theta) = \pm\left[-\theta^2 - 2\theta \cot\left(\alpha\frac{\pi}{2}\right)\cot\left(\pi\left(\frac{c-m}{M-m} - (\alpha_0-\alpha)\theta\right)\right) + \cot^2\left(\alpha\frac{\pi}{2}\right)\right]^{1/2}, \tag{32}$$

$n = 1, 2$, where, if $c \in (m, M)$, then ν_n are defined on $(0, \theta^*]$ and θ^* is the smallest positive root of the equation

$$\theta^2 - 2\theta\cot\left(\alpha\frac{\pi}{2}\right)\cot\left[\pi\left(\frac{c-m}{M-m} - (\alpha_0-\alpha)\theta\right)\right] = \cot^2\left(\alpha\frac{\pi}{2}\right).$$

The mappings from $\mathscr{F}(\alpha, m, M)$ are open; hence the functions $(\operatorname{Im} f_{(\eta)}) \circ \nu_n$, $n = 1, 2$, cannot assume their boundary values at points of $(0, \theta^*]$. Therefore we obtain those having

$$\lim_{\substack{\theta\to 0^+ \\ (\nu_2\to\cot(\alpha\pi/2))}} (\operatorname{Im} f_{(\eta)} \circ \nu_1) := \lambda_1(c; \eta)$$

and

$$\lim_{\substack{\theta\to 0^+ \\ (\nu_2\to -\cot(\alpha\pi/2))}} (\operatorname{Im} f_{(\eta)} \circ \nu_2)(\theta) := \lambda_2(c; \eta). \tag{33}$$

Limits (33) are finite for all η, $\arg\eta \in [0, \pi]$, besides $\eta = \exp(i\alpha\pi)$ when $\lambda_2(c; \eta) = -\infty$ for all $c \in (m, M)$ (cf. (13)).

If $\lambda_n(c; \eta)$, $c \in (m, M)$, is finite, then, for $n = 1, 2$, we get the equations $w(c) = c + i\lambda_n(c; \eta)$, $c \in (m, M)$, that represent, since $\lambda_n(c; \eta)$ is a linear function w.r.t. c, the line segments joining suitable boundary values on the lines $\operatorname{Re} w = m$ and $\operatorname{Re} w = M$. These are the cluster sets of extreme points $f_{(\eta)}$ at $e^{\pm i\alpha\pi}$ for $n = 1, 2$ resp.; for $\eta = \exp(i\alpha\pi)$, the cluster set at $\zeta = \exp(-i\alpha\pi)$ is a pair of parallel half-lines included in the lines $\operatorname{Re} w = m$ and $\operatorname{Re} w = M$ (Fig. 6).

To determine the cluster set of $f_{(\eta)}$ at $\zeta = 1$, we proceed in a similar way by taking $c > M$.

In that case, functions (32) satisfying (31) are defined for $\theta \in (\theta', \theta^*]$, where $\theta' = (c - M)/[(\alpha_0 - \alpha)(M - m)]$.

By (32), note that if $\theta \to \theta'$, then $\nu_1(\theta) \to +\infty$, while $\nu_2(\theta) \to -\infty$. Thus, as in the preceding case, to find the cluster set of $f_{(\eta)}$ at 1, it is sufficient to have

$$\lim_{\substack{\theta\to \theta'^+ \\ (\nu_1\to+\infty)}} (\operatorname{Im} f_{(\eta)} \circ \nu_1)(\theta) := \tilde{\lambda}_1(c; \eta),$$

$$\lim_{\substack{\theta\to \theta'^+ \\ (\nu_2\to-\infty)}} \operatorname{Im} f_{(\eta)} \circ \nu_2)(\theta) := \tilde{\lambda}_2(c; \eta). \tag{34}$$

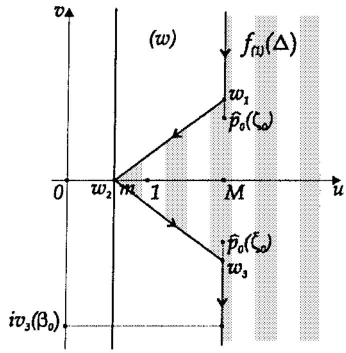

FIGURE 2

Limits (34) are finite for all η and c, $\arg\eta \in (0, \pi]$, $c \in (M, +\infty)$, and are linear functions w.r.t. c.

Hence, we obtain two parallel half-lines $w(c) = c + i\tilde{\lambda}_n(c;\eta)$, $c \in (M, +\infty)$, η-fixed, $n = 1, 2$, that start from suitable boundary values on the line $\operatorname{Re} w = M$. The equations of these half-lines can be ordered to the form

$$w(c) = w_0(\eta, \alpha, m, M, n) + \left(1 + i\cot\left(\frac{\beta}{2}\right)\right)c, \quad c \in (M, +\infty); \tag{35}$$

therefore the direction of half-lines (35) depends on $\beta = \arg\eta$ only.

Thus if $\arg\eta \in (0, \pi]$, then the cluster set of $f_{(\eta)}$ at $\zeta = 1$ consists of two parallel inclined (horizontal for $\beta = \pi$) half-lines (35).

If $\eta = 1$, then $\tilde{\lambda}_n(c;\eta) = \pm\infty$ for all $c \in (M, +\infty)$ and $n = 1, 2$, resp., and the cluster set of $f_{(1)}$ at $\zeta = 1$ is a pair of half-lines included in the line $\operatorname{Re} w = M$ and symmetric w.r.t. the real axis (cf. (23) and Fig. 2).

Summing up all of the considerations concerning $f_{(\eta)}(\Delta)$ under extreme points from $\mathcal{F}(\alpha, m, M)$, we represent these sets in Figs. 2–8. Let us recall that the sets $f_{(\bar{\eta})}(\Delta)$ and $f_{(\eta)}(\Delta)$ are symmetric w.r.t. the real axis.

1° $\beta = 0$

where $w_1 = M + iv_1(0)$, $w_2 = m$, $w_3 = M + iv_3(0)$, $v_3(0) = -v_1(0)$, (Fig. 2).

2° $\beta \in (0, \beta_0)$

where $w_1 = M + iv_1$, $w_2 = m + iv_2$, $w_3 = M + iv_3$, $w_4 = M + iv_4$, (Fig. 3).

Remark 0.2. If $\beta \in (0, \beta_0)$, then

(a) $\operatorname{Im} w_1$ decreases from $v_1(0)$ to $v_1(\beta_0)$;

(b) $\operatorname{Im} w_3$ decreases from $v_3(0)$ to $v_3(\beta_0)$;

(c) $\operatorname{Im} w_4$ increases from $-\infty$ to $v_3(\beta_0) = v_4(\beta_0)$.

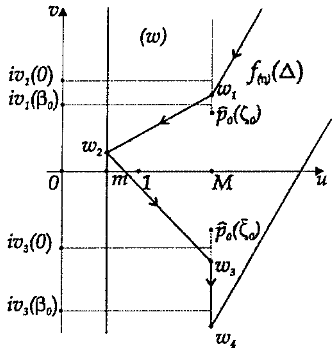

FIGURE 3

3° $\beta = \beta_0$
where $w_1 = M + iv_1(\beta_0)$, $w_2 = m + iv_2(\beta_0)$, $w_3 = M + iv_3(\beta_0)$, (Fig. 4).

4° $\beta \in (\beta_0, \alpha\pi)$
where $w_1 = M + iv_1$, $w_2 = m + iv_2$, $w_3 = M + iv_3$, $w_4 = M + iv_4$, (Fig. 5).

Remark 0.3. If $\beta \in (\beta_0, \alpha\pi)$, then

(a) $\operatorname{Im} w_1$ decreases from $v_1(\beta_0)$ to $v_1(\alpha\pi)$;

(b) $\operatorname{Im} w_3$ decreases from $v_3(\beta_0)$ to $-\infty$;

(c) $\operatorname{Im} w_4$ increases from $v_4(\beta_0) = v_3(\beta_0)$ to $v_4(\alpha\pi)$.

5° $\beta = \alpha\pi$

where $w_1 = M + iv_1(\alpha\pi)$, $w_2 = m + iv_2(\alpha\pi)$, $w_3 = M + iv_4(\alpha\pi)$, (Fig. 6).

Remark 0.4. sgn $\operatorname{Im} w_2$ changes depending on α.

6° $\beta \in (\alpha\pi, \pi)$
where $w_1 = M + iv_1$, $w_2 = m + iv_2$, $w_3 = m + iv_5$, $w_4 = M + iv_4$, (Fig. 7).

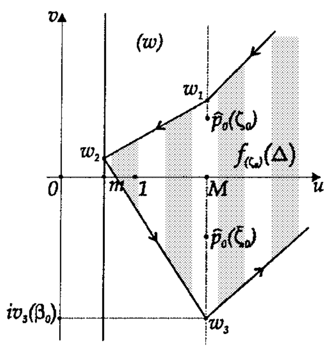

FIGURE 4

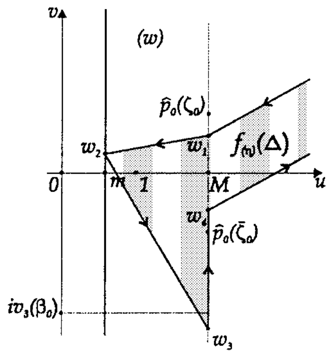

FIGURE 5

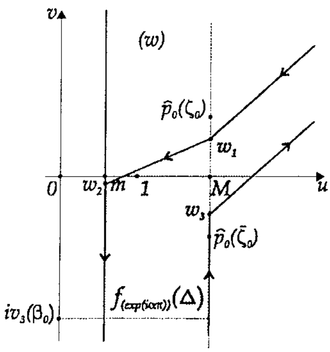

FIGURE 6

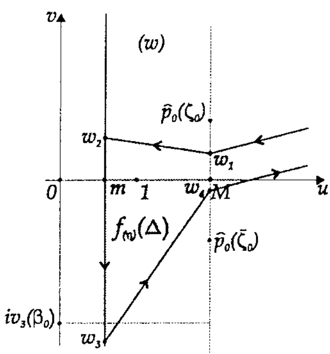

FIGURE 7

Remark 0.5. If $\beta \in (\alpha\pi, \pi)$, then

(a) Im w_1 decreases from $v_1(\alpha\pi)$ to $v_1(\pi)$;

(b) Im w_3 increases from $-\infty$ to $v_5(\pi)$, and Im $\{w_2 - w_3\}(\beta)$ is positive and decreasing to Im $\{w_2 - w_3\}(\pi)$;

(c) Im w_4 increases from $v_4(\alpha\pi)$ to its maximum, which may be positive, depending on α, and, next, decreases near $\beta = \pi$ to $v_4(\pi)$; however, $\mathrm{Im}\{w_1 - w_4\}(\beta)$ is positive and decreases to $\mathrm{Im}\{w_1 - w_4\}(\pi)$.

7° $\beta = \pi$
where $w_1 = M + \frac{i}{2}(\alpha_0 - \alpha)(M - m)\pi$, $w_2 = m + \frac{i}{2}((M - m)(\alpha_0 - \alpha)\pi + 2\tan(\alpha\frac{\pi}{2}))$, $w_3 = \bar{w}_2$, $w_4 = \bar{w}_1$, (Fig. 8).

Remark 0.6. Case 7° can be joined to the preceding one. Among $f_{(\eta)}(\Delta)$, only the regions $f_{(\pm 1)}(\Delta)$ are symmetric w.r.t. the real axis.

Now,

$$
\begin{aligned}
l_1^* &= \left\{M + iv : v \le \mathrm{Im}\, \hat{f}_{(\zeta_0)}(\zeta),\ \zeta \in \widehat{e^{-i\alpha\pi}}\right\}, \\
l_2^* &= \{w : \bar{w} \in l_1^*\}.
\end{aligned} \tag{36}
$$

Otherwise, $l_1^* = \{M + iv : v \le v_3(\beta_0)\}$, $\beta_0 = \arg\zeta_0$ (see (5)). Half-lines (36) will be called harmonic slits.

By all interrelations among the boundary values of $f_{(\eta)}(\zeta)$, $\zeta \in \partial\Delta$, and the symmetry of $f_{(\eta)}(\Delta)$ and $f_{(\bar{\eta})}(\Delta)$ w.r.t. the real axis for all η, $|\eta| = 1$, we get

$$
\bigcup_{\eta} f_{(\eta)}(\Delta) = \Omega^*(\alpha, m, M), \tag{37}
$$

where $\Omega^*(\alpha, m, M) = \{w : \mathrm{Re}\, w > m\}/(l_1^* \cup l_2^*)$ and l_n^*, $n = 1, 2$, are harmonic slits (36). We have

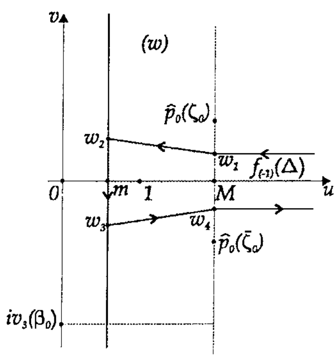

FIGURE 8

THEOREM 0.2. *For all admissible α, m, M and for $f \in \mathcal{F}(\alpha, m, M)$, we obtain*

$$\bigcup_f f(\Delta) = \Omega^*(\alpha, m, M).$$

Proof. Notice that if f is a convex linear combination of extreme points $f_{(\eta)}$, then $\operatorname{Re} f = \operatorname{Re} p_0$. Now, use the Krein–Milman Theorem. ∎

What is more, the results concerning $f_{(\eta)}(\Delta)$, $|\eta| = 1$, prove

PROPOSITION 0.4. 1° *For all extreme points $f_{(\eta)} \in \mathcal{F}(\alpha, m, M)$, the relation $f_{(\eta)}(\Delta) \neq \Omega^*(\alpha, m, M)$ holds.*

2° *If $\Omega(\alpha, m, M) = p_0(\Delta)$, then $\Omega(\alpha, m, M) \subsetneqq \Omega^*(\alpha, m, M)$.*

3° *There is $\eta^* \in \partial\Delta$, $\beta_0 < \arg \eta^* < \pi$, such that, for all $\eta \in \partial\Delta \setminus (\widehat{\bar{\eta}^* \eta^*})$, we have $f_{(\eta)}(\Delta) \subset \Omega(\alpha, m, M)$; however, if $\eta \in \widehat{\bar{\eta}^* \eta^*}$, then $f_{(\eta)}$ cuts at least one of the conformal slits l_1, l_2 (cf. (2)), which means that $f_{(\eta)}(\Delta) \cap l_n \neq \emptyset$ for $n = 1$ or $n = 2$.*

If $\alpha \to 0^+$, then harmonic (also conformal) slits vanish, and, as a limit case, we obtain the class of harmonic maps related to a half-plane [1]. On the other hand, the harmonic maps considered in [3, 9] are a limit case if $\alpha \to \alpha_0^-$.

The condition $\alpha_0 \leq 1/2$ (cf. Definition 0.1) is essential in Lemma 0.1. The case $1/2 < \alpha_0 < 1$ is left open.

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