Hindawi Publishing Corporation
Journal of Complex Analysis
Volume 2013, Article ID 474231, 3 pages
http://dx.doi.org/10.1155/2013/474231

Research Article

Coefficient Estimate Problem for a New Subclass of Biunivalent Functions

N. Magesh,[1] T. Rosy,[2] and S. Varma[2]

[1] *Postgraduate and Research Department of Mathematics, Government Arts College for Men, Krishnagiri, Tamil Nadu 635001, India*
[2] *Department of Mathematics, Madras Christian College, Thambaram, Chennai, Tamil Nadu 600 059, India*

Correspondence should be addressed to N. Magesh; nmagi_2000@yahoo.co.in

Received 25 May 2013; Accepted 19 September 2013

Academic Editor: Janne Heittokangas

We introduce a unified subclass of the function class Σ of biunivalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this subclass. In addition, many relevant connections with known or new results are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Further, by $\mathcal{S}$, we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include, for example, the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$\begin{gathered} f^{-1}(f(z)) = z \quad (z \in \mathbb{U}), \\ f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right), \end{gathered} \tag{2}$$

where

$$\begin{aligned} f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots. \end{aligned} \tag{3}$$

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of biunivalent functions in $\mathbb{U}$ given by (1).

In 1967, Lewin [1] investigated the biunivalent function class Σ and showed that $|a_2| < 1.51$; on the other hand Brannan and Clunie [2] (see also [3–5]) and Netanyahu [6] made an attempt to introduce various subclasses of biunivalent function class Σ and obtained nonsharp coefficient estimates on the first two coefficients $|a_2|$ and $|a_3|$ of (1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} := \{1, 2, 3, \ldots\}$ is still an open problem. In this line, following Brannan and Taha [4], recently, many researchers have introduced and investigated several interesting subclasses of biunivalent function class Σ and they have found nonsharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$; for details, one can refer to the works of [7–13].

Now, we define $\mathcal{R}_\Sigma(\alpha, \lambda)$ of function $f \in \mathcal{A}$ satisfying the following conditions:

$$\begin{gathered} f \in \Sigma, \quad \left|\arg\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2}, \\ \left|\arg\left(\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2} \quad (z, w \in \mathbb{U};\ \lambda \geq 0) \end{gathered} \tag{4}$$

for some α $(0 < \alpha \leq 1)$, where $g(w)$ is the extension of $f^{-1}(w)$ to $\mathbb{U}$. Similarly, we say that a function $f \in \mathscr{A}$ belongs to the class $\mathscr{R}_{\Sigma}(\beta, \lambda)$ if $f(z)$ satisfies the following inequalities:

$$f \in \Sigma, \quad \Re\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}}\right) > \beta, \qquad \Re\left(\frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}}\right) > \beta \quad (z, w \in \mathbb{U};\ \lambda \geq 0), \tag{5}$$

for some β $(0 \leq \beta < 1)$, where $g(w)$ is the extension of $f^{-1}(w)$ to $\mathbb{U}$. The classes $\mathscr{R}_{\Sigma}(\alpha, \lambda)$ and $\mathscr{R}_{\Sigma}(\beta, \lambda)$ were introduced by Prema and Keerthi [14]; furthermore, for these classes, they have found the following estimates on the first two Taylor-Maclaurin coefficients in (1).

Theorem 1. *If $f \in \mathscr{R}_{\Sigma}(\alpha, \lambda)$, $0 < \alpha \leq 1$, and $\lambda \geq 0$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\alpha + 1 + \lambda)(1 + \lambda)}}, \qquad |a_3| \leq \frac{4\alpha^2}{(1 + \lambda)^2} + \frac{2\alpha}{2 + \lambda}. \tag{6}$$

Theorem 2. *If $f \in \mathscr{R}_{\Sigma}(\beta, \lambda)$, $0 \leq \beta < 1$, and $\lambda \geq 0$, then*

$$|a_2| \leq \sqrt{\frac{2(1 - \beta)}{1 + \lambda}}, \qquad |a_3| \leq \frac{4(1 - \beta)^2}{(1 + \lambda)^2} + \frac{2(1 - \beta)}{2 + \lambda}. \tag{7}$$

Motivated by the works of Xu et al. [12, 13], we introduce the following generalized subclass $\mathscr{R}_{\Sigma}(\varphi, \psi, \lambda)$ of the analytic function class $\mathscr{A}$.

Definition 3. Let $f \in \mathscr{A}$, and let the functions $\varphi, \psi : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\{\Re(\varphi(z)), \Re(\psi(z))\} > 0 \quad (z \in \mathbb{U}), \qquad \varphi(0) = \psi(0) = 1. \tag{8}$$

We say that $f \in \mathscr{R}_{\Sigma}(\varphi, \psi, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} \in \varphi(\mathbb{U}), \qquad \frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} \in \psi(\mathbb{U}) \quad (z, w \in \mathbb{U}), \tag{9}$$

where $\lambda \geq 0$ and the function $g(w)$ is the extension of $f^{-1}(w)$ to $\mathbb{U}$.

We note that by specializing λ, φ, and ψ, we get the following interesting subclasses:

(1) $\mathscr{R}_{\Sigma}(\varphi, \psi, 1) = \mathscr{H}_{\Sigma}^{\varphi,\psi}$; see [12],

(2) $\mathscr{R}_{\Sigma}(((1 + z)/(1 - z))^{\alpha}, ((1 + z)/(1 - z))^{\alpha}, \lambda) = \mathscr{R}_{\Sigma}(\alpha, \lambda)$ $(0 < \alpha \leq 1; \lambda \geq 0)$ and $\mathscr{R}_{\Sigma}((1 + (1 - 2\beta)z)/(1 - z), (1 + (1 - 2\beta)z)/(1 - z), \lambda) = \mathscr{R}_{\Sigma}(\beta, \lambda)$ $(0 \leq \beta < 1; \lambda \geq 0)$; see [14],

(3) $\mathscr{R}_{\Sigma}(((1 + z)/(1 - z))^{\alpha}, ((1 + z)/(1 - z))^{\alpha}, 1) = \mathscr{H}_{\Sigma}^{\alpha}$ $(0 < \alpha \leq 1)$ and $\mathscr{R}_{\Sigma}((1 + (1 - 2\beta)z)/(1 - z), (1 + (1 - 2\beta)z)/(1 - z), 1) = \mathscr{H}_{\Sigma}^{\beta}$ $(0 \leq \beta < 1)$; see [11].

The objective of the present paper is to introduce a new subclass $\mathscr{R}_{\Sigma}(\varphi, \psi, \lambda)$ and to obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in theaforementioned class, employing the techniques used earlier by Xu et al. [12, 13].

2. Main Result

In this section, we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\mathscr{R}_{\Sigma}(\varphi, \psi, \lambda)$.

Theorem 4. *Let $f(z)$ be of the form (1). If $f \in \mathscr{R}_{\Sigma}(\varphi, \psi, \lambda)$, then*

$$|a_2| \leq \sqrt{\frac{|\varphi''(0)| + |\psi''(0)|}{8 + 4\lambda}}, \tag{10}$$

$$|a_3| \leq \frac{|\varphi''(0)|}{4 + 2\lambda}. \tag{11}$$

Proof. Since $f \in \mathscr{R}_{\Sigma}(\varphi, \psi, \lambda)$, from (9), we have,

$$\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}} = \varphi(z) \quad (z \in \mathbb{U}), \qquad \frac{w^{1-\lambda} g'(w)}{(g(w))^{1-\lambda}} = \psi(w) \quad (w \in \mathbb{U}), \tag{12}$$

where

$$\varphi(z) = 1 + \varphi_1 z + \varphi_2 z^2 + \cdots, \qquad \psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \cdots \tag{13}$$

satisfy the conditions of Definition 3. Now, equating the coefficients in (12), we get

$$(1 + \lambda) a_2 = \varphi_1, \tag{14}$$

$$(2 + \lambda) a_3 = \varphi_2, \tag{15}$$

$$-(1 + \lambda) a_2 = \psi_1, \tag{16}$$

$$(2 + \lambda)(2a_2^2 - a_3) = \psi_2. \tag{17}$$

From (14) and (16), we get

$$\varphi_1 = -\psi_1, \qquad 2(1 + \lambda)^2 a_2^2 = \varphi_1^2 + \psi_1^2. \tag{18}$$

From (15) and (17), we obtain

$$a_2^2 = \frac{\varphi_2 + \psi_2}{2(2 + \lambda)}. \tag{19}$$

Since $\varphi(z) \in \varphi(\mathbb{U})$ and $\psi(z) \in \psi(\mathbb{U})$, we immediately have

$$|a_2| \leq \sqrt{\frac{|\varphi''(0)| + |\psi''(0)|}{8 + 4\lambda}}. \tag{20}$$

This gives the bound on $|a_2|$ as asserted in (10).

Next, in order to find the bound on $|a_3|$, by subtracting (17) from (15), we get

$$2(2+\lambda)a_3 - 2(2+\lambda)a_2^2 = \varphi_2 - \psi_2. \quad (21)$$

It follows from (19) and (21) that

$$a_3 = \frac{\varphi_2}{2+\lambda}. \quad (22)$$

Since $\varphi(z) \in \varphi(\mathbb{U})$ and $\psi(z) \in \psi(\mathbb{U})$, we readily get $|a_3| \leq |\varphi''(0)|/(4+2\lambda)$ as asserted in (11). This completes the proof of Theorem 4. □

By setting $\varphi(z) = \psi(z) = ((1+Az)/(1+Bz))^\alpha$, where $-1 \leq B < A \leq 1$ and $0 < \alpha \leq 1$, in Theorem 4, we get the following corollary.

Corollary 5. *Let $f(z)$ be of the form* (1) *and in the class $\mathscr{R}_\Sigma(A,B,\alpha,\lambda)$. Then,*

$$|a_2| \leq \sqrt{\frac{\alpha^2(A-B)^2 - \alpha\left(A^2-B^2\right)}{4+2\lambda}},$$
$$|a_3| \leq \frac{\alpha^2(A-B)^2 - \alpha\left(A^2-B^2\right)}{4+2\lambda}. \quad (23)$$

If we choose $A = 1$ and $B = -1$ in Corollary 5, we have the following corollary.

Corollary 6. *Let $f(z)$ be of the form* (1) *and in the class $\mathscr{R}_\Sigma(\alpha,\lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 0$. Then,*

$$|a_2| \leq \alpha\sqrt{\frac{2}{2+\lambda}}, \qquad |a_3| \leq \frac{2\alpha^2}{2+\lambda}. \quad (24)$$

Remark 7. The estimates found in Corollary 6 would improve the estimates obtained in [14, Theorem 2.2].

If we set $A = 1-2\beta$, $B = -1$, where $0 \leq \beta < 1$ and $\alpha = 1$ in Corollary 5, we readily have the following corollary.

Corollary 8. *Let $f(z)$ be of the form* (1) *and in the class $\mathscr{R}_\Sigma(\beta,\lambda)$, $0 \leq \beta < 1$ and $\lambda \geq 0$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2+\lambda}}, \qquad |a_3| \leq \frac{2(1-\beta)}{2+\lambda}. \quad (25)$$

Remark 9. The estimates found in Corollary 8 would improve the estimates obtained in [14, Theorem 3.2].

Remark 10. For $\lambda = 1$, the bounds obtained in Theorem 4 are coincident with the outcome of Xu et al. [12]. Taking $\lambda = 0$ in Corollaries 6 and 8, the estimates on the coefficients $|a_2|$ and $|a_3|$, are the improvement of the estimates on the first two Taylorû Maclaurin coefficients obtained in [10, Corollaries 2.3 and 3.3]. Also, for the choices of $\lambda = 1$, the results stated in Corollaries 6 and 8 would improve the bounds stated in [11, Theorems 1 and 2], respectively. Furthermore, various other interesting corollaries and consequences of our main result could be derived similarly by specializing φ and ψ.

Acknowledgment

The authors would like to thank the referee for his valuable suggestions.

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