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# Generalized Zalcman Conjecture for Starlike and Typically Real Functions

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Zalcman conjectured that $|a_n^2 - a_{2n-1}| \le (n-1)^2$, $n = 2, 3, \dots$ for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$, the class of normalized holomorphic and univalent functions $f(z)$ in the unit disk $\mathbb{D}$. We propose a generalized conjecture as follows. For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$, $|a_n a_m - a_{n+m-1}| \le (n-1)(m-1)$, $n, m = 2, 3, \dots$. We prove that this conjecture holds for starlike functions and univalent functions with real coefficients. We also obtain sharp bounds on $|a_n a_m - a_{n+m-1}|$ for typically real functions. © 1999 Academic Press

*Key Words:* Zalcman conjecture; coefficient estimates; starlike functions; typically real functions

## 1. INTRODUCTION

Let $S$ be the well known class of holomorphic and univalent functions $f(z)$ in the unit disk $\mathbb{D} = \{z: |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. Zalcman conjectured that

$$|a_n^2 - a_{2n-1}| \le (n-1)^2 \qquad (n = 2, 3, \dots) \tag{1.1}$$

for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$. This conjecture implies the famous Bieberbach conjecture $|a_n| \le n$ (see [2]) and is known to be true for $n = 2$ [3]. Results on various subclasses of $S$ do support the conjecture.

We denote by $S^*$ the subclass of normalized ($f(0) = f'(0) - 1 = 0$) starlike functions. A holomorphic and univalent functions $f(z)$ in $\mathbb{D}$ is called starlike if $f(\mathbb{D})$ is starlike with respect to 0.



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A normalized holomorphic function $f(z)$ defined in $\mathbb{D}$ is called typically real if $\operatorname{Im}\{f(z)\} > 0$ when $\operatorname{Im}\{z\} > 0$ and $\operatorname{Im}\{f(z)\} < 0$ when $\operatorname{Im}\{z\} < 0$. The class of all normalized typically real functions is denoted by $T$.

The Zalcman conjecture is known to be true for the subclass $S_{\mathbb{R}}$ consisting of functions in $S$ with real coefficients (see [2]). Brown and Tsao [2] proved the Zalcman conjecture for $S^*$. They also obtained $a_n^2 - a_{2n-1} \leq (n-1)^2$ for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in T$. Furthermore, this author [5] showed that the Zalcman conjecture holds for close-to-convex functions when $n \geq 4$.

In this paper, we consider the following problem that we state as a conjecture. The Zalcman conjecture is the special case when $m = n$.

Generalized Zalcman conjecture: For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$,

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \qquad (n, m = 2, 3 \ldots). \quad (1.2)$$

We prove that the generalized Zalcman conjecture holds for the subclasses $S^*$ and $S_{\mathbb{R}}$ of $S$. For starlike functions, we use a method similar as the one used in [2]. While our method for typically real functions is different from theirs. We also obtain sharp bounds on $|a_n a_m - a_{n+m-1}|$ for the class $T$, from which part of the result on $S_{\mathbb{R}}$ follows since $S_{\mathbb{R}} \subset T$.

A related problem is to find positive real values of $\lambda$ such that for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$,

$$|\lambda a_n a_m - a_{n+m-1}| \leq \lambda nm - n - m + 1 \qquad (n, m = 2, 3, \ldots). \quad (1.3)$$

Clearly, the Koebe functions $k(z) = z/(1-z)^2$ and its rotations satisfy equality in (1.1), (1.2) and (1.3).

Note that if $\nu > \lambda$ and (1.3) holds, then

$$\begin{aligned} |\nu a_n a_m - a_{n+m-1}| &\leq (\nu - \lambda)|a_n a_m| + |\lambda a_n a_m - a_{n+m-1}| \\ &\leq (\nu - \lambda) nm + \lambda nm - n - m + 1 \\ &= \nu nm - n - m + 1. \end{aligned}$$

Here we have used $|a_n| \leq n$. When $\lambda = 1$, (1.3) becomes (1.2). Actually, we find some of those values of $\lambda$ for the closed convex hull of $S^*$. We did not attempt to determine the least $\lambda$ value.

## 2. GENERALIZED ZALCMAN CONJECTURE FOR STARLIKE FUNCTIONS

Our main goal in this section is to prove that the generalized Zalcman conjecture (1.2) is valid for the subclass $S^*$ of $S$.

Recall that a holomorphic and locally univalent function $f(z) = z + \cdots$ in $\mathbb{D}$ is starlike if and only if $zf'(z)/f(z) \in \mathcal{P}$, where $\mathcal{P}$ is the class of holomorphic functions $p(z)$ in $\mathbb{D}$ with positive real part and $p(0) = 1$.

First, we consider the inequality (1.3) for a larger class $HS^*$, the closed convex hull of $S^*$ in $H(\mathbb{D})$, which is the space of all holomorphic functions in $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$.

We need the following lemma in our proof.

LEMMA 2.1. *Let $\mu(t)$ be a probability measure on $[0, 2\pi]$, then*

$$\left| 2\int_0^{2\pi} e^{int}\, d\mu(t) \int_0^{2\pi} e^{imt}\, d\mu(t) - \int_0^{2\pi} e^{i(n+m)t}\, d\mu(t) \right| \le 1$$

$$(n, m = 1, 2, 3, \dots).$$

*Proof.* Define

$$p(z) = \int_0^{2\pi} \frac{1 + e^{it}z}{1 - e^{it}z}\, d\mu(t),$$

then $p(z) \in \mathcal{P}$. It is clear that

$$p(z) = 1 + 2\sum_{n=1}^{\infty} \int_0^{2\pi} e^{int}\, d\mu(t) z^n.$$

For $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in \mathcal{P}$, we know that [4]

$$|b_n b_m - b_{n+m}| \le 2 \qquad (n, m = 1, 2, 3, \dots). \tag{2.1}$$

That is,

$$\left| 4\int_0^{2\pi} e^{int}\, d\mu(t) \int_0^{2\pi} e^{imt}\, d\mu(t) - 2\int_0^{2\pi} e^{i(n+m)t}\, d\mu(t) \right| \le 2,$$

which is clearly equivalent with the desired inequality. ∎

THEOREM 2.2. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in HS^*$. Then*

$$|\lambda a_n a_m - a_{n+m-1}| \le \lambda nm - n - m + 1,$$

*where $n, m = 2, 3, \dots$ and $\lambda \in [\frac{2(n+m-1)}{nm}, +\infty)$.*

*Proof.* We know that for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in HS^*$, there exists a probability measure $\mu(t)$ on $[0, 2\pi]$ such that [1]

$$f(z) = \int_0^{2\pi} \frac{z}{(1 - e^{it} z)^2} \, d\mu(t).$$

This implies that

$$a_n = n \int_0^{2\pi} e^{i(n-1)t} \, d\mu(t) \qquad (n = 2, 3, \dots).$$

By using Lemma 2.1, we have

$$\begin{aligned}
&|\lambda a_n a_m - a_{n+m-1}| \\
&\le \left( \lambda - \frac{2(n+m-1)}{nm} \right) |a_n a_m| + \left| \frac{2(n+m-1)}{nm} a_n a_m - a_{n+m-1} \right| \\
&\le (\lambda nm - 2(n+m-1)) \left| \int_0^{2\pi} e^{i(n-1)t} \, d\mu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\mu(t) \right| \\
&\quad + (n+m-1) \left| 2 \int_0^{2\pi} e^{i(n-1)t} \, d\mu(t) \int_0^{2\pi} e^{i(m-1)t} \, d\mu(t) \right. \\
&\qquad\qquad \left. - \int_0^{2\pi} e^{i(n+m-2)t} \, d\mu(t) \right| \\
&\le \lambda nm - 2(n+m-1) + n + m - 1 \\
&= \lambda nm - n - m + 1.
\end{aligned}$$

This completes the proof of Theorem 2.2. ∎

Now, we turn to the proof of the generalized Zalcman conjecture for $S^*$. The special case when $n = m$ is the Zalcman conjecture for $S^*$ proved by Brown and Tsao [2].

THEOREM 2.3. *Suppose $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S^*$. Then*

$$|a_n a_m - a_{n+m-1}| \le (n-1)(m-1) \qquad (n, m = 2, 3, \dots).$$

*When $nm \neq 4$, equality holds if and only if $f(z)$ is $k(z) = z/(1-z)^2$ or one of its rotations. Equality holds if and only if $f(z)$ equals $k_\nu(z)$ or one of its rotations when $n = m = 2$, where $k_\nu(z)$ is defined by $k_\nu(0) = k'_\nu(0) - 1 = 0$ and*

$$\frac{k_\nu(z)}{z k'_\nu(z)} = \nu \frac{1+z}{1-z} + (1-\nu) \frac{1-z}{1+z}, \qquad 0 \le \nu \le 1.$$

*Proof.* First, we consider the case when $n \geq 4$ and $m \geq 4$. In this case, $(n-2)(m-2) > 2$, that is, $\frac{2(n+m-1)}{nm} < 1$. Using Theorem 2.2 with $\lambda = \frac{2(n+m-1)}{nm}$, we have

$$\begin{aligned}|a_n a_m - a_{n+m-1}| &\leq (1-\lambda)|a_n a_m| + |\lambda a_n a_m - a_{n+m-1}| \\ &\leq (1-\lambda)nm + \lambda nm - n - m + 1 \\ &= (n-1)(m-1).\end{aligned}$$

Next, we consider the cases when $m = 2$ and $m = 3$ separately. The cases when $n = 2$ or $n = 3$ are covered by interchanging $n$ and $m$. Define

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in \mathscr{P}$ and

$$a_n = \frac{1}{n-1} \sum_{k=1}^{n-1} a_k b_{n-k} \qquad (n = 2, 3, \ldots). \tag{2.2}$$

In particular,

$$a_2 = b_1 \tag{2.3}$$

and

$$a_3 = \tfrac{1}{2}\left(b_1^2 + b_2\right). \tag{2.4}$$

When $m = 2$, using (2.2), (2.3), (2.1) and $|a_k| \leq k$, we have

$$\begin{aligned}|a_n a_2 - a_{n+1}| &= \left| b_1 a_n - \frac{1}{n} \sum_{k=1}^{n} a_k b_{n+1-k} \right| \\ &= \left| \frac{n-1}{n} b_1 a_n - \frac{1}{n} \sum_{k=1}^{n-1} a_k b_{n+1-k} \right| \\ &= \left| \frac{1}{n} b_1 \sum_{k=1}^{n-1} a_k b_{n-k} - \frac{1}{n} \sum_{k=1}^{n-1} a_k b_{n+1-k} \right| \\ &\leq \frac{1}{n} \sum_{k=1}^{n-1} |a_k|\, |b_1 b_{n-k} - b_{n+1-k}| \\ &\leq \frac{1}{n} \sum_{k=1}^{n-1} (2k) = (2-1)(n-1).\end{aligned}$$

To prove the desired inequality when $m = 3$, we need the following estimate. From $|b_1| \le 2$ and (2.1), we see that

$$\begin{aligned}|b_1^2 b_{n-k} - b_{n+2-k}| &= |b_1(b_1 b_{n-k} - b_{n+1-k}) + b_1 b_{n+1-k} - b_{n+2-k}| \\ &\le |b_1|\,|b_1 b_{n-k} - b_{n+1-k}| + |b_1 b_{n+1-k} - b_{n+2-k}| \\ &\le 6.\end{aligned}$$

When $m = 3$, by using (2.2), (2.4), (2.1), $|b_k| \le 2$ and $|a_k| \le k$, we obtain

$$\begin{aligned}&|a_n a_3 - a_{n+2}| \\ &= \left|\frac{1}{2}b_1^2 a_n + \frac{1}{2}b_2 a_n - \frac{1}{n+1}\sum_{k=1}^{n+1} a_k b_{n+2-k}\right| \\ &= \left|\frac{1}{2}b_1^2 a_n + \frac{n-1}{2(n+1)}b_2 a_n - \frac{1}{n+1}b_1 a_{n+1} - \frac{1}{n+1}\sum_{k=1}^{n-1} a_k b_{n+2-k}\right| \\ &= \left|\frac{1}{2}b_1^2 a_n + \frac{n-1}{2(n+1)}b_2 a_n - \frac{1}{(n+1)n}b_1 \sum_{k=1}^{n} a_k b_{n+1-k}\right. \\ &\qquad \left. - \frac{1}{n+1}\sum_{k=1}^{n-1} a_k b_{n+2-k}\right| \\ &= \left|\frac{n^2+n-2}{2(n+1)n}b_1^2 a_n + \frac{n-1}{2(n+1)}b_2 a_n - \frac{1}{(n+1)n}b_1 \sum_{k=1}^{n-1} a_k b_{n+1-k}\right. \\ &\qquad \left. - \frac{1}{n+1}\sum_{k=1}^{n-1} a_k b_{n+2-k}\right| \\ &= \left|\left(\frac{n+2}{2(n+1)n}b_1^2 + \frac{1}{2(n+1)}b_2\right)\sum_{k=1}^{n-1} a_k b_{n-k}\right. \\ &\qquad \left. - \frac{1}{(n+1)n}b_1 \sum_{k=1}^{n-1} a_k b_{n+1-k} - \frac{1}{n+1}\sum_{k=1}^{n-1} a_k b_{n+2-k}\right| \\ &= \left|\left(\frac{1}{2(n+1)} + \frac{1}{(n+1)n}\right)\sum_{k=1}^{n-1} b_1^2 a_k b_{n-k}\right. \\ &\qquad - \frac{1}{(n+1)n}b_1 \sum_{k=1}^{n-1} a_k b_{n+1-k} \\ &\qquad \left. + \frac{1}{2(n+1)}\sum_{k=1}^{n-1} b_2 a_k b_{n-k} - \frac{1}{n+1}\sum_{k=1}^{n-1} a_k b_{n+2-k}\right|\end{aligned}$$

$$
\begin{aligned}
&= \Bigg| \frac{1}{(n+1)n} \sum_{k=1}^{n-1} b_1 a_k (b_1 b_{n-k} - b_{n+1-k}) \\
&\quad + \frac{1}{2(n+1)} \sum_{k=1}^{n-1} a_k (b_2 b_{n-k} - b_{n+2-k}) \\
&\quad + \frac{1}{2(n+1)} \sum_{k=1}^{n-1} a_k (b_1^2 b_{n-k} - b_{n+2-k}) \Bigg| \\
&\le \frac{1}{(n+1)n} \sum_{k=1}^{n-1} 4k + \frac{1}{2(n+1)} \sum_{k=1}^{n-1} 2k + \frac{1}{2(n+1)} \sum_{k=1}^{n-1} 6k \\
&= (3-1)(n-1).
\end{aligned}
$$

Now, we discuss when equality holds.

Note that in each case except when $n = m = 2$, we have used either $|a_k| \le k$ for some $k \ge 2$ or $|b_1| \le 2$. In either case, equality forces that $f(z)$ to be $k(z) = z/(1-z)^2$ or one of its rotations. On the other hand, it is easy to check directly that equality holds for $k(z) = z/(1-z)^2$ and its rotations.

When $n = m = 2$, we have $|a_2^2 - a_3| = \frac{1}{2}|b_1^2 - b_2|$. If we define

$$
q(z) = \frac{1}{p(z)} = 1 + c_1 z + c_2 z^2 + \cdots .
$$

Then $q(z) \in \mathscr{P}$ and $c_2 = b_1^2 - b_2$. $|a_2^2 - a_3| = 1$ means that $|c_2| = 2$, which holds if and only if (see, for example, [6, p. 41])

$$
q(z) = \nu \frac{1+z}{1-z} + (1-\nu)\frac{1-z}{1+z}, \qquad 0 \le \nu \le 1
$$

or one of its rotations. This means that $f(z)$ equals $k_\nu(z)$ or one of its rotations. ∎

## 3. GENERALIZED ZALCMAN CONJECTURE FOR TYPICALLY REAL FUNCTIONS

Now we investigate typically real functions and univalent functions with real coefficients.

It turns out that the generalized Zalcman conjecture does not hold for all typically real functions when one of $n$ and $m$ equals 2 and the other equals $2, 4, \ldots$ . This does not provide a counter example for the generalized Zalcman conjecture since not all typically real functions are univalent.

Recall that a normalized holomorphic function $f(z)$ is typically real in $\mathbb{D}$ if and only if

$$f(z) = \frac{z}{1-z^2}p(z),$$

where $p(z) \in \mathscr{P}$ with real coefficients, (see [6, p. 54]). In particular, all coefficients of typically real functions are real.

THEOREM 3.1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in T$. Then for $n, m = 2, 3, \dots,$ we have sharp estimates*

$$|a_n a_m - a_{n+m-1}| \leq \begin{cases} n+1 & m = 2, n = 2,4,6,\dots, \\ m+1 & n = 2, m = 2,4,6,\dots, \\ (n-1)(m-1) & \text{otherwise}. \end{cases}$$

*Equality holds in the first two cases for $k_1(z) = z(1+z^2)/(1-z^2)^2$ and for $k(z) = z/(1-z)^2$ otherwise.*

*Proof.* As

$$f(z) = \frac{z}{1-z^2}p(z)$$

for some $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in \mathscr{P}$ with real coefficients, we have

$$a_{2k} = b_1 + b_3 + \cdots + b_{2k-1}, \tag{3.1}$$

$$a_{2k+1} = 1 + b_2 + b_4 + \cdots + b_{2k} \tag{3.2}$$

and

$$a_n = a_{n-2} + b_{n-1}. \tag{3.3}$$

In particular,

$$a_2 = b_1,$$

$$a_3 = 1 + b_2$$

and

$$a_4 = b_1 + b_3.$$

First, we consider the case when $m = 2$ and $n = 2k$. By using (3.1), (3.2), and (2.1), we get

$$\begin{aligned}
&|a_2 a_{2k} - a_{2k+1}| \\
&\quad = |b_1(b_1 + b_3 + \cdots + b_{2k-1}) - (1 + b_2 + b_4 + \cdots + b_{2k})| \\
&\quad = |(b_1 b_1 - b_2) + (b_1 b_3 - b_4) + \cdots + (b_1 b_{2k-1} - b_{2k}) - 1| \\
&\quad \le 2k + 1.
\end{aligned}$$

This proves the first desired inequality. The second inequality follows from interchanging the positions of $n$ and $m$. For

$$p(z) = \frac{1 + z^2}{1 - z^2},$$

$b_1 = b_3 = \cdots = b_{2k-1} = 0$ and $b_2 = b_4 = \cdots = b_{2k} = 2$. Therefore, the inequalities become equalities for $k_1(z) = z(1 + z^2)/(1 - z^2)^2$.

Second, we consider the case when $m = 2$ and $n = 2k + 1$. The case when $n = 2$ and $m = 2k + 1$ follows from symmetry. By using (3.1), (3.2), and (2.1), we get

$$\begin{aligned}
&|a_2 a_{2k+1} - a_{2k+2}| \\
&\quad = |b_1(1 + b_2 + b_4 + \cdots + b_{2k}) - (b_1 + b_3 + \cdots + b_{2k+1})| \\
&\quad = |(b_1 b_2 - b_3) + (b_1 b_4 - b_5) + \cdots + (b_1 b_{2k} - b_{2k+1})| \\
&\quad \le 2k = (2 - 1)(2k + 1 - 1).
\end{aligned}$$

Now we consider the case when one of $n$ and $m$ equals 3. Without loss of generality, we assume $m = 3$. Applying (3.1), (3.2), and (2.1), we obtain

$$\begin{aligned}
&|a_3 a_{2k} - a_{2k+2}| \\
&\quad = |(1 + b_2)(b_1 + b_3 + \cdots + b_{2k-1}) - (b_1 + b_3 + \cdots + b_{2k+1})| \\
&\quad = |b_2(b_1 + b_3 + \cdots + b_{2k-3}) + b_2 b_{2k-1} - b_{2k+1}| \\
&\quad \le 4(k - 1) + 2 = (3 - 1)(2k - 1)
\end{aligned}$$

and

$$\begin{aligned}
&|a_3 a_{2k+1} - a_{2k+3}| \\
&\quad = |(1 + b_2)(1 + b_2 + b_4 + \cdots + b_{2k}) - (1 + b_2 + b_4 + \cdots + b_{2k+2})| \\
&\quad = |b_2(1 + b_2 + b_4 + \cdots + b_{2k-2}) + b_2 b_{2k} - b_{2k+2}| \\
&\quad \le 2(1 + 2(k - 1)) + 2 = (3 - 1)(2k + 1 - 1).
\end{aligned}$$

Note that Brown and Tsao showed that $a_4^2 - a_7 \le 9$ [2]. It is clear that $a_4^2 - a_7 \ge -a_7 \ge -7$ as all coefficients of $f(z)$ are real. Thus, $|a_4^2 - a_7| \le 9$.

Next, we prove the desired inequality when one of $n$ and $m$ is 4 and the other is even and at least 6. Again, we can assume $m = 4$ and $n = 2k$, $k \ge 3$. By using (3.1), (3.2), (2.1), and $|a_4 a_4 - a_7| \le 9$, we see that

$$\begin{aligned}
&|a_4 a_{2k} - a_{2k+3}| \\
&\quad = \left|(b_1 + b_3)(b_1 + b_3 + \cdots + b_{2k-1}) - (1 + b_2 + b_4 + \cdots + b_{2k+2})\right| \\
&\quad = \big| b_3(b_5 + b_7 + \cdots + b_{2k-1}) - (b_8 + \cdots + b_{2k+2}) \\
&\qquad + (b_1 + b_3)(b_1 + b_3) - (1 + b_2 + b_4 + b_6) \\
&\qquad + b_1(b_5 + b_7 + \cdots + b_{2k-1}) \big| \\
&\quad = \big|(b_3 b_5 - b_8) + (b_3 b_7 - b_{10}) + \cdots + (b_3 b_{2k-1} - b_{2k+2}) \\
&\qquad\qquad + (a_4 a_4 - a_7) + b_1(b_5 + b_7 + \cdots + b_{2k-1}) \big| \\
&\quad \le 2(k-2) + 9 + 4(k-2) = (4-1)(2k-1).
\end{aligned}$$

Finally, we use what we have proved so far and induction to prove all other cases left. They are either when one of $n$ and $m$ is 4 and the other is odd and at least 5, or when $n, m \ge 5$. In all those cases, using the idea of induction, we can assume $|a_{n-2} a_{m-2} - a_{n+m-5}| \le (n-3)(m-3)$. From (3.3) and $|a_k| \le k$, we have

$$\begin{aligned}
&|a_n a_m - a_{n+m-1}| \\
&\quad = \left|(a_{n-2} + b_{n-1})(a_{m-2} + b_{m-1}) - (a_{n+m-5} + b_{n+m-2} + b_{n+m-4})\right| \\
&\quad = \big|(a_{n-2} a_{m-2} - a_{n+m-5}) + (b_{n-1} b_{m-1} - b_{n+m-2}) \\
&\qquad\qquad + b_{n-1} a_{m-2} + b_{m-1} a_{n-2} - b_{n+m-4} \big| \\
&\quad = \big|(a_{n-2} a_{m-2} - a_{n+m-5}) + (b_{n-1} b_{m-1} - b_{n+m-2}) \\
&\qquad + b_{n-1} a_{m-2} + b_{m-1} a_{n-4} + (b_{m-1} b_{n-3} - b_{n+m-4}) \big| \\
&\quad \le (n-3)(m-3) + 2 + 2(m-2) + 2(n-4) + 2 \\
&\quad = (n-1)(m-1).
\end{aligned}$$

It is easy to see that $|a_n a_m - a_{n+m-1}| = (n-1)(m-1)$ for $k(z) = z/(1-z)^2$. ∎

With the help of Theorem 3.1, we now prove that the generalized Zalcman conjecture is valid for univalent functions with real coefficients.

THEOREM 3.2. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S_{\mathbb{R}}$. *Then for* $n, m = 2, 3, \ldots,$ *we have sharp estimates*

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1).$$

*Equality holds if* $f(z) = k(z) = z/(1-z)^2$ *or* $f(z) = -k(-z)$.

*Proof.* As every univalent function with real coefficients is typically real, Theorem 3.1 implies the desired inequality except when one of $n$ and $m$ is 2 and the other is even. Moreover, $|a_2^2 - a_3| \leq 1$ certainly holds for $S_{\mathbb{R}}$. So we only need to prove the desired inequality when $m = 2$ and $n = 2k$, $k = 2, 3, \ldots$ .

There exists $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in \mathscr{P}$ with real coefficients such that

$$f(z) = \frac{z}{1-z^2} p(z).$$

It is easy to see that

$$\begin{aligned} a_{2k} &= b_1 + b_3 + \cdots + b_{2k-1}, \\ a_{2k+1} &= 1 + b_2 + b_4 + \cdots + b_{2k}, \\ a_3 &= 1 + b_2 \end{aligned}$$

and

$$|b_1^2 - 1 - b_2| = |a_2^2 - a_3| \leq 1.$$

By using (2.1), we get

$$\begin{aligned} &|a_2 a_{2k} - a_{2k+1}| \\ &\quad = |b_1(b_1 + b_3 + \cdots + b_{2k-1}) - (1 + b_2 + b_4 + \cdots + b_{2k})| \\ &\quad = |(b_1^2 - 1 - b_2) + (b_1 b_3 - b_4) + (b_1 b_5 - b_6) \\ &\qquad\qquad + \cdots + (b_1 b_{2k-1} - b_{2k})| \\ &\quad \leq 1 + 2(k-1) = (2-1)(2k-1). \end{aligned}$$

Clearly, equality holds for $k(z) = z/(1-z)^2$ and $-k(-z)$. ∎

Note that even though $|b_1^2 - 1 - b_2| \leq 1$ holds for those $p(z)$ corresponding to $f(z) \in S_{\mathbb{R}}$, it does not hold for all functions $p(z) \in \mathscr{P}$ with real coefficients.

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