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Milnor K-theory and motivic cohomology

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MILNOR K-THEORY AND MOTIVIC COHOMOLOGY

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Abstract.
These are the notes of a talk given at the Oberwolfach Workshop *K-Theory* 2007. We sketch a proof of Beilinson's conjecture relating Milnor K-theory and motivic cohomology. For detailed proofs see [4].

1. Introduction

A — semi-local commutative ring with infinite residue fields
k — field
$\mathbb{Z}(n)$ — Voevodsky's motivic complex [8]

Definition 1.1.

$$K_*^M(A) = \bigoplus_n (A^\times)^{\otimes n}/(a \otimes (1-a)) \quad a, 1-a \in A^\times$$

Beilinson conjectured [1]:

Theorem 1.2. *A/k essentially smooth, $|k| = \infty$. Then:*

$$\eta : K_n^M(A) \longrightarrow H^n_{zar}(A, \mathbb{Z}(n))$$

is an isomorphism for $n > 0$.

The formerly known cases are:

Remark 1.3.
— *A=k a field (Nesterenko-Suslin* [6], *Totaro* [10]*)*
— *surjectivity of η (Gabber* [3], *Elbaz-Vincent/Müller-Stach* [2], *Kerz/Müller-Stach* [5]*)*
— *$\eta \otimes \mathbb{Q}$ is isomorphic (Suslin)*
— *injectivity for A a DVR, $n = 3$ (Suslin-Yarosh* [9]*)*

2. General idea of proof

$X = Spec\, A$
We have a morphism of Gersten complexes which we know to be exact except possibly at $K_n^M(A)$:

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(1)

$$\begin{array}{ccccccc} 0 \longrightarrow & K_n^M(A) & \longrightarrow & \oplus_{x\in X^{(0)}} K_n^M(x) & \longrightarrow & \oplus_{x\in X^{(1)}} K_{n-1}^M(x) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & H^n_{zar}(A),\mathbb{Z}(n)) & \longrightarrow & \oplus_{x\in X^{(0)}} H^n_{zar}(x,\mathbb{Z}(n)) & \longrightarrow & \oplus_{x\in X^{(1)}} H^{n-1}_{zar}(x,\mathbb{Z}(n-1)) \end{array}$$

So it suffices to prove:

Theorem 2.1 (Main Result)**.** *A/k regular, connected, infinite residue fields, $F = Q(A)$. Then:*

$$i : K_n^M(A) \longrightarrow K_n^M(F)$$

is (universally) injective.

3. Applications

Corollary 3.1 (Equicharacteristic Gersten conjecture)**.**
A/k regular, local, $X = Spec\, A$, $|k| = \infty$. Then:

$$0 \longrightarrow K_n^M(A) \longrightarrow \oplus_{x\in X^{(0)}} K_n^M(x) \longrightarrow \oplus_{x\in X^{(1)}} K_{n-1}^M(x) \longrightarrow \cdots$$

is exact.

Proof. Case A/k essentially smooth: Use (1) + Main Result.
Case A general: Use smooth case + Panin's method [7] + Main Result. □

$\mathcal{K}_*^M$ — Zariski sheaf associated to K_*^M

Corollary 3.2 (Bloch formula)**.** *X/k regular excellent scheme, $|k| = \infty$, $n \geq 0$. Then:*

$$H^n_{zar}(X, \mathcal{K}_n^M) = CH^n(X) .$$

Levine and Kahn conjectured:

Corollary 3.3 (Gerneralized Bloch-Kato conjecture)**.** *Assume the Bloch-Kato conjecture. A/k, $|k| = \infty$, $char(k)$ prime to $l > 0$. Then the galois symbol*

$$\chi_n : K_n^M(A)/l \longrightarrow H^n_{et}(A, \mu_l^{\otimes n})$$

is an isomorphism for $n > 0$.

Idea of proof. Case A/k essentially smooth: Use Gersten resolution.
Case A/k general: Use Hoobler's trick + Gabber's rigidity for étale cohomology. □

Corollary 3.4 (Generalized Milnor conjecture)**.** *A/k local, $|k| = \infty$, $char(k)$ prime to 2. Then there exists an isomorphism*

$$K_n^M(A)/2 \longrightarrow I_A^n/I_A^{n+1}$$

where $I_A \subset W(A)$ is the fundamental ideal in the Witt ring of A.

4. New methods in Milnor K-theory

The first new result for K-groups used in the proof of the Main Result states:

Theorem 4.1 (COCA). *$A \subset A'$ local extension of semi-local rings, i.e. $A^\times = A \cap A'^\times$. A, A' factorial, $f \in A$ such that $A/(f) = A'/(f)$. Then:*

$$\begin{array}{ccc} K_n^M(A) & \longrightarrow & K_n^M(A_f) \\ \downarrow & & \downarrow \\ K_n^M(A') & \longrightarrow & K_n^M(A'_f) \end{array}$$

is co-Cartesian.

Remark 4.2. *COCA was proposed by Gabber who proved the surjectivity part at the lower right corner.*

Theorem 4.3 (Local Milnor Theorem)**.** *$q \in A[t]$ monic. There is a split short exact sequence*

$$0 \longrightarrow K_n^M(A) \longrightarrow K_n^t(A,q) \longrightarrow \oplus_{(\pi,q)=1} K_{n-1}^M(A[t]/(\pi)) \longrightarrow 0$$

Explanation: The abelian group $K_n^t(A,q)$ is generated by symbols $\{p_1,\ldots,p_n\}$ with $p_1,\ldots,p_n \in A[t]$ pairwise coprime, $(p_i,q) = 1$ and highest non-vanishing coefficients invertible.
For $A = k$ a field $K_n^t(k,1) = K_n^M(k(t))$.
The standard technique gives:

Theorem 4.4 (Norm Theorem)**.** *Assume A has big residue fields (depending on n). $A \subset B$ finite, étale. Then there exists a norm*

$$N_{B/A} : K_n^M(B) \longrightarrow K_n^M(A)$$

satisfying projection formula, base change.

5. Proof of Main Result

1st step: Reduce to A semi-local with respect to closed points $y_1,\ldots,y_l \in Y/k$ smooth, $|k| = \infty$ and k perfect. For this use Norm Theorem + Popescu desingularization.

2nd step: Induction on $d = dim\, A$ for all n at once.

$$i : K_n^M(A) \longrightarrow K_n^M(F)$$

If $x \in K_n^M(A)$ with $i(x) = 0$ then there exists $f \in A$ such that $i^f(x) = 0$ where

$$i^f : K_n^M(A) \longrightarrow K_n^M(A_f)\ .$$

Gabber's presentation theorem produces $A' \subset A$ a local extension and $f' \in A'$ such that $f'/f \in A^*$ and $A'/(f') = A/(f)$. Here A' is a semi-local ring with respect to closed points $y_1,\ldots,y_l \in \mathbb{A}^d_k$.

The COCA Theorem gives that

$$\begin{array}{ccc} K_n^M(A') & \longrightarrow & K_n^M(A'_f) \\ \downarrow & & \downarrow \\ K_n^M(A) & \longrightarrow & K_n^M(A_f) \end{array}$$

is co-Cartesian. So it suffices to prove that

$$i' : K_n^M(A') \longrightarrow K_n^M(k(t_1, \dots, t_d))$$

is injective. Let $x \in ker(i')$ and $p_1, \dots, p_m \in k[t_1, \dots, t_d]$ be the irreducible, different polynomials appearing in x, $p_i \in A'^{\times}$.
Let

$$W = \bigcup_i \text{sing. loc.}(V(p_i)) \cup \bigcup_{i,j} V(p_i) \cap V(p_j)$$

Then $dim(W) < d - 1$ since k is perfect.
There exists a linear projection

$$p_{\mathbb{A}_k^d} \longrightarrow \mathbb{A}_k^{d-1}$$

such that $p|_{V(p_i)}$ is finite and $p(y_i) \notin p(W)$ for all i.
Let now A'' be the semi-local ring with respect to $p(y_1), \dots, p(y_l) \in \mathbb{A}_k^{d-1}$

$$A'' \subset A''[t] \subset A'$$

Let $q \in A''[t]$ be monic such that

$$V(q) \cap p^{-1}(p(y_i)) = \{y_1, \dots, y_l\} \cap p^{-1}(p(y_i))$$

Under the natural map $K_n^t(A'', q) \to K_n^M(A')$ there exists a preimage $x' \in K_n^t(A'', q)$ of x.
We have a commutative diagram, $F = Q(A'')$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n^M(A'') & \longrightarrow & K_n^M(A'', q) & \longrightarrow & \oplus_\pi K_{n-1}^M(A''[t]/(\pi)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_n^M(F) & \longrightarrow & K_n^M(F(t)) & \longrightarrow & \oplus_\pi K_{n-1}^M(F[t]/(\pi)) & \longrightarrow & 0 \end{array}$$

But since the important summands in the right vertical arrow are injective, a simple diagram chase gives $x' = 0$ and finally $x = 0$. □

References

[1] Beilinson, Alexander Letter to Soulé, 1982, K-theory Preprint Archives, 694

[2] Elbaz-Vincent, Philippe; Müller-Stach, Stefan *Milnor K-theory of rings, higher Chow groups and applications.* Invent. Math. 148, (2002), no. 1, 177–206.

[3] Gabber, Ofer; Letter to Bruno Kahn, 1998

[4] Kerz, Moritz *The Gersten conjecture for Milnor K-theory.* K-theory Preprint Archives, 791

[5] Kerz, Moritz; Müller-Stach, Stefan *The Milnor-Chow homomorphism revisited.* To appear in K-Theory, 2006

[6] Nesterenko, Yu.; Suslin, A. *Homology of the general linear group over a local ring, and Milnor's K-theory.* Math. USSR-Izv. 34 (1990), no. 1, 121–145

[7] Panin, I. A. *The equicharacteristic case of the Gersten conjecture.* Proc. Steklov Inst. Math. 2003, no. 2 (241), 154–163.
[8] Suslin, A.; Voevodsky, V. *Bloch-Kato conjecture and motivic cohomology with finite coefficients* K-theory Preprint Archives, 341
[9] Suslin, A.; Yarosh, V. *Milnor's* K_3 *of a discrete valuation ring.* Algebraic K-theory, 155–170, Adv. Soviet Math., 4
[10] Totaro, Burt *Milnor* K*-theory is the simplest part of algebraic* K*-theory.* K-Theory 6 (1992), no. 2, 177–189

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