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SOME REMARKS ON THE SUPERCUSPIDAL REPRESENTATIONS OF p-ADIC SEMISIMPLE GROUPS

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1. Let $\boldsymbol{G}$ be a semisimple group over a local nonarchimedean field K with ring of integers $\mathcal{O}$ and finite residue field k. We define $G = \boldsymbol{G}(K)$, $\tilde{G} = \boldsymbol{G}(\tilde{K})$ where $\tilde{K}$ is a maximal unramified extension of K (with ring of integers $\tilde{\mathcal{O}}$ and residue field $\bar{k}$, an algebraic closure of k).

Let $\boldsymbol{T} \subset \boldsymbol{G}$ be a maximal torus, defined and anisotropic over K. We shall assume that $\boldsymbol{T}$ satisfies the following condition. There exists a Borel subgroup $\boldsymbol{B}$ of $\boldsymbol{G}$ containing $\boldsymbol{T}$ and defined over $\tilde{K}$. (This condition is certainly satisfied if $\boldsymbol{T}$ is split over $\tilde{K}$.) Let $\check{T}$ be the set of homomorphisms of $T = \boldsymbol{T}(K)$ into $\bar{Q}_l^*$ (l a prime $\neq$ char k) which factor through a finite quotient of T. One expects that to any $\theta \in \check{T}$ satisfying some regularity condition, there corresponds an irreducible admissible supercuspidal representation of G (over $\bar{Q}_l$). Such a correspondence has been established by Gérardin (for $\boldsymbol{T}$ split over $\tilde{K}$, with certain restrictions, see **[2]**) using methods of Shintani, Howe and Corwin. (For SL_2 the correspondence was established by Gelfand, Graev, Shalika and others.)

I would like to suggest another possible approach to the question of constructing this correspondence. This would use l-adic cohomology (or homology) of a certain infinite dimensional variety X over $\bar{k}$. (The fact that l-adic cohomology might be used to construct representations of G is made plausible by the work of Drinfeld and by that of Deligne and myself, in the case of finite fields **[1]**. Note that, even in Gérardin's approach, one has to appeal, in the case where the conductor is very small, to the representation theory of a reductive group over a finite field, which is, itself, based on l-adic cohomology.)

Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{B}$ and let $\tilde{U} = \boldsymbol{U}(\tilde{K})$. Consider the Frobenius map $F: \tilde{G} \to \tilde{G}$ defined by the Frobenius element $\phi \in \mathrm{Gal}(\tilde{K}/K)$, so that $G = \tilde{G}^F$.

Let $X = \{g \in \tilde{G} \mid g^{-1} F(g) \in \tilde{U}\}/\tilde{U} \cap F^{-1}\tilde{U}$. (The action of $\tilde{U} \cap F^{-1}\tilde{U}$ on $\tilde{G}$ is by right multiplication.) Now $G \times T$ acts on X by $(g_0, t): g \to g_0 g t^{-1}$ ($g_0 \in G$, $t \in T$, $g \in \tilde{G}$).

I believe that, by regarding X as an infinite dimensional variety over $\bar{k}$, one can define l-adic homology groups $H_i(X)$ on which $G \times T$ acts in such a way that $H_i(X) = \bigoplus_{\theta \in \check{T}} H_i(X)_\theta$ (where $H_i(X)_\theta$ is the subspace of $H_i(X)$ on which T acts according to θ). Moreover, for θ fixed, $H_i(X)_\theta$ should be zero for large i, while if θ is not fixed

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by any nontrivial element in the small Weyl group of T, $H_i(X)_\theta$ should be nonzero for exactly one index $i = i_0$ (depending on θ) and the resulting representation of G on $H_{i_0}(X)_\theta$ should be an irreducible admissible supercuspidal representation. This should establish the required correspondence between characters of T and representations of G.

2. Consider, for example, the case where $\tilde{G} = \mathrm{SL}_n(\tilde{K})$ and let $F\colon \tilde{G} \to \tilde{G}$ be the homomorphism defined by

$$(2.1) \qquad F(A) = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & 0 & \ddots & 1 \\ \pi & & & 0 \end{pmatrix} A^\phi \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & 0 & \ddots & 1 \\ \pi & & & 0 \end{pmatrix}^{-1}$$

where π is a uniformizing element of $\mathcal{O}$ and, for any matrix A over $\tilde{K}$, A^ϕ is obtained by applying ϕ to each entry of A. The fixed point set $\tilde{G}^F$ is the group of elements of reduced norm one in a central division algebra of dimension n^2 over K. Let $\tilde{T}$ be the group of diagonal matrices in $\mathrm{SL}_n(\tilde{K})$; it is invariant under F. Let $\tilde{U} \subset \mathrm{SL}_n(\tilde{K})$ (resp. $\tilde{U}^-$) be the subgroup consisting of all upper (lower) triangular matrices with 1's on the diagonal. If A is a matrix in $\mathrm{SL}_n(\tilde{K})$ satisfying $A^{-1}F(A) \in \tilde{U}$, we can find a unique $B \in \tilde{U} \cap F^{-1}\tilde{U}$ such that $(AB)^{-1}F(AB) \in \tilde{U} \cap F\tilde{U}^-$. Thus,

$$\{A \in \mathrm{SL}_n(\tilde{K}) \mid A^{-1}F(A) \in \tilde{U}\}/\tilde{U} \cap F^{-1}(\tilde{U})$$

can be identified with

$$X = \{A \in \mathrm{SL}_n(\tilde{K}) \mid A^{-1}F(A) \in \tilde{U} \cap F\tilde{U}^-\}$$

on which $\tilde{G}^F$ acts by left multiplication and $\tilde{T}^F$ acts by right multiplication. X is just the set of all $n \times n$ matrices of the form

$$(2.2) \qquad \begin{pmatrix} a_1 & a_n^{\phi^{-1}} & \cdots & a_2^{\phi^{-(n-1)}} \\ \pi a_2 & a_1^{\phi^{-1}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_n^{\phi^{-(n-1)}} \\ \pi a_n & \cdots & \pi a_2^{\phi^{-(n-2)}} & a_1^{\phi^{-(n-1)}} \end{pmatrix}$$

with $a_i \in \tilde{K}$ and determinant equal to 1. For such a matrix, we have automatically $a_i \in \tilde{\mathcal{O}}$ $(1 \leqq i \leqq n)$ and $a_1 \notin \pi\tilde{\mathcal{O}}$.

It follows that X may be regarded as the projective limit $\operatorname{proj\,lim}_h X_h$ of the algebraic varieties X_h over $\bar{k}$, where X_h $(h \geqq 1)$ is the set of all $n \times n$ matrices of the form (2.2) with $a_1 \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$, $a_i \in \tilde{\mathcal{O}}/\pi^{h-1}\tilde{\mathcal{O}}$ $(2 \leqq i \leqq n)$, with determinant equal to 1. (We regard πa_i $(2 \leqq i \leqq n)$ as elements of $\pi\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$; the determinant of such a matrix has an obvious meaning as an element of $\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$.)

Let G_h be the set of all $n \times n$ matrices (a_{ij}) with $a_{ij} \in \pi\,\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$ $(\forall i > j)$, $a_{ij} \in \tilde{\mathcal{O}}/\pi^{h-1}\tilde{\mathcal{O}}$ $(\forall i < j)$, $a_{ii} \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$ $(\forall i)$, with determinant equal to 1, as an element of $\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}}$. Define $F\colon G_h \to G_h$ by the formula (2.1). G_h is an algebraic group over $\bar{k}$ and $F\colon G_h \to G_h$ is the Frobenius map for a k-rational structure on G_h. Let T_h be the group of diagonal matrices in G_h and let U_h (resp. U_h^-) be the group of upper (lower) triangular matrices in G_h with 1's on the diagonal. We may identify X_h with the variety

$$\{A \in G_h \mid A^{-1}F(A) \in U_h \cap FU_h^-\}.$$

G_h^F acts on X_h by left multiplication and T_h^F acts on X_h by right multiplication. (Note that $\tilde{G}^F = \text{proj lim}_h G_h^F$, $\tilde{T}^F = \text{proj lim}_h T_h^F$). The following lemma is crucial.

LEMMA. *The action of $K_h = \ker(T_h \to T_{h-1})$ $(h \geqq 2)$, on X_h preserves each fibre of the natural map $X_h \to X_{h-1}$, and each fibre of $X_h/K_h \to X_{h-1}$ is isomorphic to the affine space of dimension $(n-1)$ over k.*

Let us define $H_i(Y)$ for any smooth algebraic variety Y over k of pure dimension d, to be $H_c^{2d-i}(Y, \bar{Q}_l(d))$, where l is a fixed prime $\neq$ char $\bar{k}$. The previous lemma shows that, for $h \geqq 2$, $H_i(X_{h-1})$ is canonically isomorphic to $H_i(X_h)^{K_h}$ (fixed points of K_h on $H_i(X_h)$). In particular, we have a well-defined embedding $H_i(X_{h-1}) \to H_i(X_h)$. Using these embeddings, we form the direct limit inj $\lim_h H_i(X_h)$. We define $H_i(X)$ to be this direct limit. $H_i(X)$ decomposes naturally in a direct sum $\bigoplus H_i(X)_\theta$, where θ runs through the set $(\tilde{T}^F)^\vee$. It is clear that, for fixed θ, $H_i(X)_\theta$ is of finite dimension and is zero for large i. On the other hand, it is in a natural way a $\tilde{G}^F$-module ($\tilde{G}^F$ acting via a finite quotient). Let $R_\theta = \sum_i (-1)^i H_i(X)_\theta$.

THEOREM. *For each $\theta \in (\tilde{T}^F)^\vee$, $\pm R_\theta$ is an irreducible $\tilde{G}^F$-module. If $\theta \neq \theta'$, then $\pm R_\theta$, $\pm R_\theta'$ are distinct.*

It suffices to prove that, if $\Sigma = G_h^F \backslash (X_h \times X_h)$ (with G_h^F acting diagonally on $X_h \times X_h$) we have

$$\sum_i (-1)^i \dim H_i(\Sigma)_{\theta, \theta'^{-1}} = 1 \quad \text{if } \theta' = \theta,$$
$$= 0 \quad \text{if } \theta' \neq \theta.$$

(Here $T_h^F \times T_h^F$ acts on Σ by right multiplication and $H_i(\Sigma)_{\theta, \theta'^{-1}}$ denotes the (θ, θ'^{-1})-eigenspace of $T_h^F \times T_h^F$.)

To compute the (equivariant) Euler characteristic of Σ we use the principle that the Euler characteristic of a space can be computed from the zeros of a nice vector field.

The map $(g, g') \to (x, x', y)$, $x = g^{-1}F(g)$, $x' = g'^{-1}F(g')$, $y = g^{-1}g'$ defines an isomorphism

$$\Sigma \simeq \{(x, x', y) \in (U_h \cap FU_h^-) \times (U_h \cap FU_h^-) \times G_h \mid xF(y) = yx'\}$$

(compare [**1**, 6.6]). Now, any $y \in G_h$ can be written uniquely in the form

$$y = y_1' y_2' y_1'' y_2'', \qquad y_1' \in U_h^- \cap F^{-1}U_h,\ y_2' \in T_h(U_h^- \cap F^{-1}U_h^-),$$
$$y_1'' \in U_h \cap F^{-1}U_h^-,\ y_2'' \in U_h \cap F^{-1}U_h.$$

We now make the substitution $xF(y_1') = \bar{x} \in U_h$, so the equation of Σ becomes

$$\bar{x}F(y_2')F(y_1'')F(y_2'') = y_1' y_2' y_1'' y_2'' x'.$$

Any element $z \in U_h$ can be written uniquely in the form $z = y_2'' x' F(y_2'')^{-1}$ with $y_2'' \in U_h \cap F^{-1}U_h$, $x' \in U_h \cap FU_h^-$, so the equation of Σ becomes

$$\tilde{x}F(y_2')F(y_1'') \in y_1' y_2' y_1'' U_h = y_1' y_2' U_h.$$

Thus, we may identify

$$\Sigma = \{(\tilde{x}, y_1', y_2' y_i'') \in (U_h \cap FU_h^-) \times (U_h^- \cap F^{-1}U_h) \times T_h(U_h^- \cap F^{-1}U_h^-) \times (U_h \cap F^{-1}U_h^-) \mid y_2'^{-1} y_1'^{-1} \tilde{x}F(y_2') F(y_1'') \in U_h\}.$$

The action of $T_h^F \times T_h^F$ is given by

$$(t, t'): (\tilde{x}, y_1', y_2', y_1'') \to (t\tilde{x}t^{-1}, ty_1' t^{-1}, ty_2' t'^{-1}, t'y_1'' t'^{-1}).$$

This action extends to an action of a larger group H, consisting of all pairs (t, t') of $n \times n$-diagonal matrices with diagonal entries in $(\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$ such that $\det t = \det t'$ and $t^{-1}F(t) = t'^{-1}F(t') \in$ centralizer of $T_h(U_h^- \cap F^{-1}U_h^-)$; the action of H is given by

$$(t, t'): (\tilde{x}, y_1', y_2', y_1'') \to (F(t)\tilde{x}F(t)^{-1}, F(t)y_1'F(t)^{-1}, ty_2't'^{-1}, t'y_1''t'^{-1}).$$

A diagonal matrix of the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \ddots & & \\ 0 & & & 1 & \\ & & & & a \end{pmatrix}, \qquad a \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*,$$

certainly centralizes $T_h(U_h^- \cap F^{-1}U_h^-)$. Thus, any pair (t, t) with t of the form

$$\begin{pmatrix} \xi^{\phi^{n-1}} & & 0 \\ & \ddots & \\ & & \xi^{\phi} & \\ 0 & & & \xi \end{pmatrix}$$

(with $\xi \in (\tilde{\mathcal{O}}/\pi^h\tilde{\mathcal{O}})^*$ a root of 1 of order prime to char k) is in H. The set of all such pairs (t, t) is a one dimensional subtorus $\mathcal{T}$ of H (over $\bar{k}$). This torus acts on Σ commuting with $T_h^F \times T_h^F$. Its fixed point set on Σ is the finite set given by $\tilde{x} = y_1' = y_1'' = e$, $y_2' \in T_h^F$, hence it is isomorphic to T_h^F with $T_h^F \times T_h^F$ acting by left and right multiplication. It follows that

$$\begin{aligned} \sum(-1)^i \dim H_i(\Sigma)_{\theta,\theta'^{-1}} &= \sum(-1)^i \dim H_i(\Sigma^\tau)_{\theta,\theta'^{-1}} \\ &= \dim H_0(T_h^F)_{\theta,\theta'^{-1}} = 1 \quad \text{if } \theta = \theta', \\ &\qquad\qquad\qquad\quad = 0 \quad \text{if } \theta \neq \theta', \end{aligned}$$

as required.

3. Let V be a 2-dimensional vector space over K and let $\tilde{V} = V \otimes_k \tilde{K}$. Let F: $\tilde{V} \to \tilde{V}$ be defined by $F = 1 \otimes \phi$. Assume that we are given a nonzero element $\omega \in \Lambda^2 V$. The set $X = \{x \in \tilde{V} \mid x \wedge Fx = \omega\}$ is invariant under the obvious action of $\mathrm{SL}_2(V)$ and under the action of the group $T = \{\lambda \in \tilde{K}^* \mid \lambda \cdot \lambda^\phi = 1\}$ which acts by scalar multiplication. (This set X can be identified with the set X defined in §1 for $\boldsymbol{G} = \mathrm{SL}_2$ and $\boldsymbol{T}$ a maximal torus associated to the unramified quadratic extension of K.) Objects similar to this X appear in the work of Drinfeld. We now show what is the meaning of the homology groups $H_i(X)$ in the present case. It can be easily checked that if $x \in X$, the $\tilde{\mathcal{O}}$-module spanned by x and Fx is invariant under F, hence it comes from a lattice L in V.

Thus X is a disjoint union $X = \coprod_L X_L$ over all lattices $L \subset V$ with determinant $\mathcal{O}^* \omega$, where

$$X_L = \{x \in L \otimes_{\mathcal{O}} \tilde{\mathcal{O}} \,|\, x \wedge Fx = \omega\}.$$

Each X_L can be regarded in a natural way as proj $\lim_h X_{L,h}$ where $X_{L,h}$ is the h-dimensional variety over $\bar{k}$ defined by

$$X_{L,h} = \{x \in L \otimes_{\mathcal{O}} (\tilde{\mathcal{O}}/\pi^L \tilde{\mathcal{O}}) \,|\, x \wedge Fx = \omega\}.$$

The fibres of $X_{L,h} \to X_{L,h-1}$ have a property similar to that in the Lemma in §2. This allows us to define $H_i(X_L) = \text{inj} \lim_h H_i(X_{L,h})$ as in §2. We also define $H_i(X) = \bigoplus_L H_i(X_L)$.

Similar arguments should apply in the general case.

4. Let $\boldsymbol{G}$, $\boldsymbol{T}$, $\boldsymbol{B}$, $\boldsymbol{U}$ be as in §1. Assume that $\boldsymbol{G}$ comes from a Chevalley group over $\boldsymbol{Z}$ by extension of scalars so that $\boldsymbol{G}(\mathcal{O})$, $\boldsymbol{G}(\tilde{\mathcal{O}})$ are well defined. Let $\tilde{G}_h = \boldsymbol{G}(\tilde{\mathcal{O}}/\pi^h \tilde{\mathcal{O}})$. Assume that $\boldsymbol{T}(\tilde{K}) \subset \boldsymbol{G}(\tilde{\mathcal{O}})$. Let $\tilde{T}_h$, $\tilde{U}_h$ be the images of $\boldsymbol{T}(\tilde{K})$, $\boldsymbol{U}(\tilde{K}) \cap \boldsymbol{G}(\tilde{\mathcal{O}})$ under $\boldsymbol{G}(\tilde{\mathcal{O}}) \to \tilde{G}_h$. $F\colon \boldsymbol{G}(\tilde{K}) \to \boldsymbol{G}(\tilde{K})$ induces $F\colon \tilde{G}_h \to \tilde{G}_h$. This gives a $\bar{k}$-rational structure on the $\bar{k}$-algebraic group $\tilde{G}_h$. Define

$$X_h = \{g \in \tilde{G}_h \,|\, g^{-1} F(g) \in \tilde{U}_h\}/\tilde{U}_h \cap F^{-1}\tilde{U}_h.$$

The finite group $\tilde{G}_h^F \times \tilde{T}_h^F$ acts on X_h, as before, by left and right multiplication. For each character $\theta\colon \tilde{T}_h^F \to \boldsymbol{Q}_l^{-1}$ we form

$$R_\theta = \sum_i (-1)^i H_i(X_h)_\theta.$$

THEOREM. *If θ is sufficiently regular, the virtual $\tilde{G}_h^F$-module $\pm R_\theta$ is irreducible. It is independent of the choice of $\boldsymbol{B}$.*

In the case $h = 1$, this follows from **[1]**.

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