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THE STARLIKENESS, CONVEXITY, COVERING THEOREM AND EXTREME POINTS OF p-HARMONIC MAPPINGS

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Abstract. The main aim of this paper is to introduce three classes $H^0_{p,q}$, $H^1_{p,q}$ and TH^*_p of p-harmonic mappings and discuss the properties of mappings in these classes. First, we discuss the starlikeness and convexity of mappings in $H^0_{p,q}$ and $H^1_{p,q}$. Then establish the covering theorem for mappings in $H^1_{p,q}$. Finally, we determine the extreme points of the class TH^*_p.

1. Introduction

A $2p$ times continuously differentiable complex-valued mapping $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\underbrace{\Delta \cdots \Delta}_{p} F = 0$, where p (≥ 1) is an integer and Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is known that a mapping F is p-harmonic in a simply connected domain D if and only if F has the following representation:

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$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

where $\Delta G_{p-k+1}(z) = 0$, i.e., $G_{p-k+1}(z)$ is harmonic in D for each $k \in \{1, \cdots, p\}$ (see [4]).

Obviously, when $p = 1$ (respectively 2), F is harmonic (respectively biharmonic). The properties of harmonic mappings have been investigated by many authors, see [1, 4, 6, 7, 9, 16, 17]. Biharmonic mappings arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology. See [12, 14] for the details. Nowadays, the study of biharmonic mappings attracts much attention, see [1, 2, 3, 5, 8, 15].

Let $\mathbb{D}_r = \{z : |z| < r\}$ $(r > 0)$. In particular, we use $\mathbb{D}$ to denote the unit disk $\mathbb{D}_1$. Throughout this paper, we consider p-harmonic mappings in $\mathbb{D}$.

In [6], Clunie and Sheil-Small introduced the class S_H^0 of univalent harmonic mappings in $\mathbb{D}$, consisting of all harmonic mappings F with the series expansion:

$$F(z) = z + \sum_{n=2}^{\infty} (a_n z^n + \overline{b}_n \overline{z}^n).$$

The main aim of Ganczar [9] was to discuss the starlikeness and convexity of mappings F in S_H^0 under the coefficient condition:

$$\sum_{n=2}^{\infty} n^q \big(|a_n| + |b_n|\big) \leq 1 \tag{1.1}$$

for $q > 0$. For convenience, we denote by $H_{1,q}^0$ the subclass of S_H^0 with the coefficient condition (1.1).

Let H_1^* denote the set of all mappings in S_H^0 mapping $\mathbb{D}$ onto starlike domains, and let TH_1^* denote the subclass of H_1^* whose elements satisfy that $F = h + \overline{g}$, where

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \text{ with } a_n \geq 0 \text{ and } g(z) = -\sum_{n=2}^{\infty} b_n z^n \text{ with } b_n \geq 0.$$

In [18], Silverman obtained many properties of mappings in [18]. For example, he proved that $F \in TH_1^*$ if and only if $\sum_{n=2}^{\infty} n\big(|a_n| + |b_n|\big) \leq 1$ (cf. [18, Theorems 2]). Also the extreme points of TH_1^* were determined (cf. [18, Theorem 4(a)]). See [10, 11, 13] for other discussions in this line.

We use $H^1_{p,1}$ to denote the set of all p-harmonic mappings F in $\mathbb{D}$ with the following series expansion:

$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}\Big(\sum_{j=1}^{\infty}a_{j,p-k+1}z^j+\sum_{j=1}^{\infty}\overline{b}_{j,p-k+1}\bar{z}^j\Big),$$

where $a_{1,p}=1$ and $b_{1,p}=0$, and satisfying the following coefficient condition:

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)\big(|a_{j,p-k+1}|+|b_{j,p-k+1}|\big)\leq 2.$$

In [16], Qiao and Wang proved that the mappings F in $H^1_{p,1}$ is sense-preserving, univalent and starlike in $\mathbb{D}$ (cf. [16, Theorems 3.1 and 3.2]).

In Section 2, we will introduce three classes of p-harmonic mappings: $H^0_{p,q}$, $H^1_{p,q}$ and TH^*_p. When $p=1$, $H^0_{p,q}$, and TH^*_p coincide with $H^0_{1,q}$ and TH^*_1, respectively, and when $q=1$, $H^1_{p,q}$ is $H^1_{p,1}$.

The first aim of this paper is to discuss the starlikeness and convexity of p-harmonic mappings in $H^0_{p,q}$. Our results are Theorems 3.3 and 3.5, where Theorem 3.3 extends [9, Theorems 1 and 4] to the setting of p-harmonic mappings, and Theorem 3.5 is a generalization of [9, Theorems 2 and 3]). Also we consider the univalence, starlikeness and convexity of mappings belonging to $H^1_{p,q}$ with $q\in(0,1]$. Our result is Theorem 3.7 which is a generalization of [16, Theorems 3.1 and 3.2]. The proofs of the mentioned theorems will be presented in Section 3.

As the second aim of this paper, we investigate the covering theorem for mappings in $H^1_{p,q}$. Our result is Theorem 4.1 which is a generalization of [9, Theorem 5]. We will prove this theorem in Section 4.

Finally, we get a necessary and sufficient condition for a p-harmonic mapping to be in TH^*_p and then determine the extreme points of TH^*_p. Our main results are Theorems 5.1 and 5.2, where Theorems 5.1 and 5.2 are generalizations of [18, Theorem 2] and [18, Theorem 4(a)], respectively. Theorems 5.1, 5.2 proved in Section 5.

2. Necessary notions and notations

Let

$$
\begin{aligned}
(2.1)\qquad F(z) &= \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)} \big(h_{p-k+1} + \overline{g}_{p-k+1}\big) \\
&= \sum_{k=1}^{p} |z|^{2(k-1)} \big(\sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \overline{b}_{j,p-k+1} \bar{z}^j\big)
\end{aligned}
$$

be a p-harmonic mapping with $a_{1,p} = 1$ and $b_{1,p} = 0$.

We denote by $H^0_{p,q}$ with $q > 0$ the class of all univalent mappings satisfying the form (2.1) and the following condition:

$$
(2.2)\qquad \sum_{k=1}^{p} \sum_{j=1}^{\infty} j^q \big(|a_{j,p-k+1}| + |b_{j,p-k+1}|\big) \leq 2,
$$

and the class $H^1_{p,q}$ with $q > 0$ the class of all mappings satisfying the form (2.1) and the following condition:

$$
(2.3)\qquad \sum_{k=1}^{p} \sum_{j=1}^{\infty} (2(k-1) + j)^q \big(|a_{j,p-k+1}| + |b_{j,p-k+1}|\big) \leq 2.
$$

Proposition 2.1. *If $f \in H^1_{p,q}$ and f is univalent, then $f \in H^0_{p,q}$.*

We use J_F to denote the Jacobian of F, that is,

$$
J_F = |F_z|^2 - |F_{\bar{z}}|^2.
$$

Then it is known that F is sense-preserving and locally univalent if $J_F > 0$.

Definition 2.2. *We say that a univalent p-harmonic mapping F with $F(0) = 0$ is starlike of order $\alpha \in [0, 1)$ with respect to the origin if the curve $F(re^{i\theta})$ is starlike of order α with respect to the origin for each $r \in (0, 1)$. In other words, F is starlike of order α if $\frac{\partial}{\partial\theta}(argF(re^{i\theta})) \geq \alpha$ for all $z = re^{i\theta} \neq 0$.*

Definition 2.3. *A univalent p-harmonic mapping F with $F(0) = 0$ and $\frac{\partial}{\partial\theta}F(re^{i\theta}) \neq 0$ whenever $0 < r < 1$ is said to be convex of order $\beta \in [0, 1)$ if the curve $F(re^{i\theta})$ is convex of order β for each $r \in (0, 1)$. In other words, F is convex of order β if $\frac{\partial}{\partial\theta}(arg\frac{\partial}{\partial\theta}F(re^{i\theta})) \geq \beta$ for all $z = re^{i\theta} \neq 0$.*

Definition 2.4. *Let X be a topological vector space over the field of complex numbers, and let D be a subset of X. A point $x \in D$ is called an extreme point of D if it has no representation of the form $x = ty+(1-t)z$ $(t \in (0, 1))$ as a proper convex combination of two distinct points y and z in D.*

Furthermore, we introduce following notions and notations.

Let TH_p^* denote the class of all p-harmonic mappings F which are univalent, starlike and has the form (2.1), where $a_{1,p} = 1$ and $b_{1,p} = 0$, with an additional restriction that all the other coefficients are nonpositive.

3. **Starlikeness and convexity**

We start this section with two lemmas which will be useful for the following proofs.

Lemma 3.1. *Let*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}\big(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j\big)$$

be a univalent p-harmonic mapping with $a_{1,p} = 1$ and $b_{1,p} = 0$. If

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(\frac{j-\alpha}{1-\alpha}|a_{j,p-k+1}| + \frac{j+\alpha}{1-\alpha}|b_{j,p-k+1}|\big) \leq 2 \tag{3.1}$$

for some $\alpha \in [0, 1)$, then F is starlike of order α.

Proof. Note that

$$\frac{\partial}{\partial\theta}(\arg F(re^{i\theta})) = \mathrm{Re}\Big\{\frac{z\frac{\partial}{\partial z}F(z) - \overline{z}\frac{\partial}{\partial \overline{z}}F(z)}{F(z)}\Big\} = \mathrm{Re}\frac{1+A(z)}{1+B(z)}$$

for $r \neq 0$, where

$$A(z) = -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty}j|z|^{2(k-1)}(a_{j,p-k+1}z^j - \overline{b}_{j,p-k+1}\overline{z}^j)$$

and

$$B(z) = -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j).$$

Let

$$w_1(z) = \frac{A(z) - B(z)}{2 - 2\alpha + A(z) + (1-2\alpha)B(z)}.$$

Then

$$\frac{1 + A(z)}{1 + B(z)} = \frac{1 + (1-2\alpha)w_1(z)}{1 - w_1(z)}.$$

An elementary calculation shows that

$$\mathrm{Re}\frac{1 + A(z)}{1 + B(z)} = \mathrm{Re}\frac{1 + (1-2\alpha)w_1(z)}{1 - w_1(z)} \geq \alpha$$

if and only if

$$|w_1(z)| \leq 1.$$

Obviously, a sufficient condition of

$$|w_1(z)| \leq 1$$

is

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\Big((j-1)|a_{j,p-k+1}| + (j+1)|b_{j,p-k+1}|\Big)$$
$$\leq 4 - 4\alpha - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\Big((j+1-2\alpha)|a_{j,p-k+1}| + (j-1+2\alpha)|b_{j,p-k+1}|\Big),$$

which is equivalent to (3.1).

The proof of the lemma is complete. □

Lemma 3.2. *Let*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}\big(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j\big)$$

be a univalent p-harmonic mapping with $a_{1,p} = 1$ and $b_{1,p} = 0$. If

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(\frac{j(j-\beta)}{1-\beta}|a_{j,p-k+1}| + \frac{j(j+\beta)}{1-\beta}|b_{j,p-k+1}|\big) \leq 2 \tag{3.2}$$

for some $\beta \in [0, 1)$, then F is convex of order β.

Proof. Note that

$$\frac{\partial}{\partial\theta}(\arg\frac{\partial}{\partial\theta}F(re^{i\theta})) = \mathrm{Re}\frac{1+P(z)}{1+Q(z)}$$

for $r \neq 0$, where

$$\begin{aligned} P(z) =& \\ & z\frac{\partial}{\partial z}F(z) + z^2\frac{\partial^2}{\partial z^2}F(z) - 2|z|^2\frac{\partial^2}{\partial z\partial\overline{z}}F(z) + \overline{z}\frac{\partial}{\partial\overline{z}}F(z) \\ & + \overline{z}^2\frac{\partial^2}{\partial\overline{z}^2}F(z) - 1 \\ =& -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty} j^2|z|^{2(k-1)}(a_{j,p-k+1}z^j + \overline{b}_{j,p-k+1}\overline{z}^j) \end{aligned}$$

and

$$\begin{aligned} Q(z) =& z\frac{\partial}{\partial z}F(z) - \overline{z}\frac{\partial}{\partial\overline{z}}F(z) - 1 \\ =& -1 + \sum_{k=1}^{p}\sum_{j=1}^{\infty} j|z|^{2(k-1)}(a_{j,p-k+1}z^j - \overline{b}_{j,p-k+1}\overline{z}^j). \end{aligned}$$

Let

$$w_2(z) = \frac{P(z) - Q(z)}{2 - 2\beta + P(z) + (1-2\beta)Q(z)}.$$

Then

$$\frac{1+P(z)}{1+Q(z)} = \frac{1+(1-2\beta)w_2(z)}{1-w_2(z)}.$$

It is easy to deduce that

$$\mathrm{Re}\frac{1+P(z)}{1+Q(z)} = \mathrm{Re}\frac{1+(1-2\beta)w_2(z)}{1-w_2(z)} \geq \beta$$

if and only if

$$|w_2(z)| \leq 1.$$

Obviously, a sufficient condition of

$$|w_2(z)| \leq 1$$

is

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big((j^2-j)|a_{j,p-k+1}|+(j^2+j)|b_{j,p-k+1}|\big)$$
$$\leq 4-4\beta-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big((j^2+j-2\beta j)|a_{j,p-k+1}|+(j^2-j+2\beta j)|b_{j,p-k+1}|\big),$$

which is equivalent to (3.2). □

Now we are ready to state and prove the results concerning the geometric properties of mappings in $H^0_{p,q}$.

Theorem 3.3. *Suppose $F\in H^0_{p,q}$ and $b_{1,p-k+1}=0$ for $k\in\{2,\cdots,p\}$.*

(1) *If $q\in[1,2)$, then F is starlike of order $\alpha(q)$, where $\alpha(q)=\frac{2^q-2}{2^q+1}$;*
(2) *If $q\in[2,+\infty)$, then F is convex of order $\beta(q)$, where $\beta(q)=\frac{2^{q-1}-2}{2^{q-1}+1}$.*

Proof. By Lemma 3.1, for a fixed $q\in[1,2)$ and any $j\in\{2,3,\cdots\}$, we know that F is p-harmonic starlike of order $\alpha=\alpha(q)$ if

$$j^q\geq\frac{j+\alpha}{1-\alpha},$$

which is equivalent to

$$\alpha\leq\frac{j^q-j}{j^q+1}.$$

Since $\{S_q(j)=\frac{j^q-j}{j^q+1}\}$ is an increasing sequence about j for any fixed $q\in[1,2)$, it follows that

$$\frac{j^q-j}{j^q+1}\geq\frac{2^q-2}{2^q+1}=S_q(2)=\alpha,$$

which proves (1).

Next, we prove (2). By Lemma 3.2, for a fixed $q\in[2,+\infty)$, F will be p-harmonic and convex of order $\beta=\beta(q)$ if

$$j^q\geq\frac{j(j+\beta)}{1-\beta},$$

which is equivalent to

$$\beta\leq\frac{j^q-j^2}{j^q+j}.$$

It is easy to know that $\{T_q(j)=\frac{j^q-j^2}{j^q+j}\}$ is an increasing sequence about j for any fixed $q\in[2,\infty)$. Hence

$$\frac{j^q-j^2}{j^q+j}\geq\frac{2^{q-1}-2}{2^{q-1}+1}=T_q(2)=\beta(q),$$

which shows that (2) holds. □

Corollary 3.4. *If $F\in H^0_{p,1}$ (respectively $F\in H^0_{p,2}$), then F is starlike (respectively convex) in $\mathbb{D}$.*

By taking $\alpha=0$ (respectively $\beta=0$), Lemma 3.1 (respectively Lemma 3.2) implies that if $F\in H^0_{p,q}$ with $q\geq 1$ (respectively $q\geq 2$), then F is starlike (respectively convex) in $\mathbb{D}$. However, when $q\in(0,1)$ (respectively $q\in(0,2)$), $F\in H^0_{p,q}$ need not be starlike (respectively convex). For instance, the harmonic polynomials

$$f^*_q(z)=z-2^{-q}\overline{z}^2\ \ (\text{respectively } f^c_q(z)=z+2^{-q}\overline{z}^2)$$

with $q\in(0,1)$ (respectively $q\in(0,2)$). Upon choosing the value of z in the interval $z\in(-1,-2^{q-1})$ (respectively $z\in(-1,-2^{q-2})$) , it is easy to know that

$$\frac{\partial}{\partial\theta}\big(\arg f^*_q(re^{i\theta})\big)<0\ \ (\text{respectively } \frac{\partial}{\partial\theta}\arg\big(\frac{\partial}{\partial\theta}f^c_q(re^{i\theta})\big)<0).$$

By replacing $\mathbb{D}$ by some subdisk, in this case, we can prove the following result.

Theorem 3.5. *Suppose $F\in H^0_{p,q}$ for $k\in\{2,\cdots,p\}$.*

(1) *If $q\in(0,1]$, then F is starlike in $\mathbb{D}_{\frac{1}{2^{1-q}}}$;*

(2) *If $q\in(0,2]$, then F is convex in $\mathbb{D}_{\frac{1}{2^{2-q}}}$.*

And the results are sharp with extremal functions

$$F_1(z)=z+2^{-q}\alpha\overline{z}^2\ \ \textit{and}\ \ F_2(z)=z+2^{-q}\beta\overline{z}^2,$$

respectively, where α, β are constants with $|\alpha|=|\beta|=1$.

Proof. Let

$$F^*(z)=2^{1-q}F\big(\frac{z}{2^{1-q}}\big).$$

Then

$$F^*(z)=\sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}\Big(\frac{a_{j,p-k+1}}{2^{(1-q)(2k+j-3)}}z^j+\frac{\overline{b}_{j,p-k+1}}{2^{(1-q)(2k+j-3)}}\overline{z}^j\Big).$$

By (2.2) and the inequality

$$\frac{j}{2^{(1-q)(2k+j-3)}}\cdot\frac{1}{j^q}\le\frac{(2j)^{1-q}}{(2^j)^{1-q}}\le 1$$

for any $j\in\{1,2,\cdots\}$, $k\in\{1,\cdots,p\}$ and fixed $q\in(0,1)$, it follows that

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}j\Big(\frac{|a_{j,p-k+1}|+|b_{j,p-k+1}|}{2^{(1-q)(2k+j-3)}}\Big)\le\sum_{k=1}^{p}\sum_{j=1}^{\infty}j^q(|a_{j,p-k+1}|+|b_{j,p-k+1}|)\le 2.$$

Then Lemma 3.1 implies that F^* is starlike in $\mathbb{D}$, which shows that F is starlike in $\mathbb{D}_{\frac{1}{2^{1-q}}}$.

Let

$$F^c(z)=2^{2-q}F\big(\frac{z}{2^{2-q}}\big).$$

By similar arguments as in the proof of (1), we know that (2) holds. □

Corollary 3.6. *If $F\in H^0_{p,1}$, then F maps the disk $\mathbb{D}_{\frac{1}{2}}$ onto a convex domain.*

Next, we consider the starlikeness and convexity of $F\in H^1_{p,q}$ and prove

Theorem 3.7. *If $F\in H^1_{p,q}$ is a p-harmonic mapping with $q\in(0,1]$, then F is sense-preserving and univalent in $\mathbb{D}_{\frac{1}{2^{1-q}}}$. Moreover, F is starlike in $\mathbb{D}_{\frac{1}{2^{1-q}}}$ and convex in $\mathbb{D}_{\frac{1}{2^{2-q}}}$, and the extremal functions are*

$$F_3(z)=z+2^{-q}\alpha_1\overline{z}^2\ \text{ and }\ F_4(z)=z+2^{-q}\beta_1\overline{z}^2,$$

respectively, where α_1, β_1 are constants with $|\alpha_1|=|\beta_1|=1$.

Proof. First, we prove that F is sense-preserving in $\mathbb{D}_{\frac{1}{2^{1-q}}}$. Let $0\le|z|=r<\frac{1}{2^{1-q}}$. Then

$$\begin{aligned}\Big|\frac{\partial}{\partial z}F(z)\Big|-\Big|\frac{\partial}{\partial\overline{z}}F(z)\Big|\ge&2-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(2(k-1)+j\big)r^{2k+j-3}(|a_{j,p-k+1}|\\&+|b_{j,p-k+1}|)\\>&2-\sum_{k=1}^{p}\sum_{j=1}^{\infty}\frac{2(k-1)+j}{2^{(1-q)(2k+j-3)}}(|a_{j,p-k+1}|+|b_{j,p-k+1}|)\\\ge&0,\end{aligned}$$

since

$$\frac{2(k-1)+j}{2^{(1-q)(2k+j-3)}} \leq (2(k-1)+j)^q$$

for $j \in \{1, \cdots\}$, $k \in \{1, \cdots, p\}$ and $q \in (0,1]$. Therefore, F is sense-preserving in $\mathbb{D}_{\frac{1}{2^{1-q}}}$.

Next, we show that $F(z_1) \neq F(z_2)$ if $z_1 \neq z_2$. Suppose $z_1, z_2 \in \mathbb{D}_{\frac{1}{2^{1-q}}}$ such that $z_1 \neq z_2$ and $|z_1| \geq |z_2|$. Then

$$\begin{aligned}\Big|\frac{F(z_1)-F(z_2)}{z_1-z_2}\Big| &\geq 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(2(k-1)+j\big)(|a_{j,p-k+1}| \\ &\quad + |b_{j,p-k+1}|)|z_1|^{2k+j-3} \\ &> 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\frac{2(k-1)+j}{2^{(1-q)(2k+j-3)}}(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \\ &\geq 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(2(k-1)+j\big)^q(|a_{j,p-k+1}| + |b_{j,p-k+1}|)| \\ &\geq 0.\end{aligned}$$

Hence F is univalent in $\mathbb{D}_{\frac{1}{2^{1-q}}}$.

The remaining part of the proof easily follows from the similar reasoning as in Theorem 3.5. □

4. **Covering theorem**

Theorem 4.1. *Let $F \in H^1_{p,q}$ be a p-harmonic mapping with $q \in (0, \infty)$. Then*

(1) $\{\omega : |\omega| < \frac{1}{2^{2-q}}\} \subseteq F(\mathbb{D}_{\frac{1}{2^{1-q}}}) \subseteq \{\omega : |\omega| < \frac{3}{2^{2-q}}\}$ *if* $q \in (0,1]$;

(2) $\{\omega : |\omega| < 1 - \frac{1}{2^q}\} \subseteq F(\mathbb{D}) \subseteq \{\omega : |\omega| < 1 + \frac{1}{2^q}\}$ *if* $q \in [1,\infty)$.

Proof. Since $F \in H^1_{p,q}$, it is easy to show that

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \leq 1 + \frac{1}{2^q}.$$

Then for $0 < r < 1$, we have

$$\begin{aligned}|F(re^{i\theta})| &\geq 2r - \sum_{k=1}^{p}\sum_{j=1}^{\infty} r^{2(k-1)+j}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)\\ &\geq r + r^2 - r^2\sum_{k=1}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)\\ &\geq r - \frac{1}{2^q}r^2\end{aligned}$$

and

$$\begin{aligned}|F(re^{i\theta})| &\leq \sum_{k=1}^{p}\sum_{j=1}^{\infty} r^{2(k-1)+j}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)\\ &\leq r - r^2 + r^2\sum_{k=1}^{p}\sum_{j=1}^{\infty}(|a_{j,p-k+1}| + |b_{j,p-k+1}|)\\ &\leq r + \frac{1}{2^q}r^2.\end{aligned}$$

Hence

$$r - \frac{1}{2^q}r^2 \leq |F(re^{i\theta})| \leq r + \frac{1}{2^q}r^2.$$

By Theorem 3.7, if $0 < q \leq 1$, then F is univalent in $\mathbb{D}_{\frac{1}{2^{1-q}}}$. Letting $r \to \frac{1}{2^{1-q}}$ in the above inequality gives (1). By [16, Theorem 3.1], if $q \geq 1$, then F is univalent in $\mathbb{D}$. By letting $r \to 1$ in the above inequality, (2) easily follows. These complete the proof. □

5. Extreme points of TH_p^*

In this section, we consider the mappings in TH_p^*. First, we give a characterization for a p-harmonic mapping to be in TH_p^*.

Theorem 5.1. *Let F be a p-harmonic mapping with the form* (2.3). *Then $F \in TH_p^*$ if and only if*

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)(|a_{j,p-k+1}| + |b_{j,p-k+1}|) \leq 2.$$

Proof. The sufficiency easily follows from [16, Theorems 3.1 and 3.2]. To prove the necessity, it suffices to show that $F \notin TH_p^*$ if

$$\sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)(|a_{j,p-k+1}|+|b_{j,p-k+1}|) > 2.$$

Under this assumption, it suffices to prove that F is not univalent. Setting $z = r > 0$ gives

$$F(r) = 2r - \sum_{k=1}^{p}\sum_{j=1}^{\infty} r^{2(k-1)+j}(|a_{j,p-k+1}|+|b_{j,p-k+1}|)$$

and

$$F'(r) = 2 - \sum_{k=1}^{p}\sum_{j=1}^{\infty}(2(k-1)+j)r^{2k-3+j}(|a_{j,p-k+1}|+|b_{j,p-k+1}|).$$

Since $F'(0) = 1$ and $F'(1) < 0$, there must exist some r_0 with $r_0 < 1$ such that $F'(r_0) = 0$. Hence $F(r)$ is not one-to-one on the real interval (0, 1) which implies $F \notin TH_p^*$. □

From Theorem 5.1, we know that TH_p^* is closed under the convex combination. Now we use Theorem 5.1 to determine the extreme points in TH_p^*.

Theorem 5.2. *Let*

$$h_{1,p}(z) = z, \quad h_{j,p-k+1}(z) = z - \frac{|z|^{2(k-1)}z^j}{2(k-1)+j}$$

and

$$g_{1,p}(z) = 0 \text{ and } g_{j,p-k+1}(z) = z - \frac{|z|^{2(k-1)}\overline{z}^j}{2(k-1)+j},$$

where $j \in \{1, \cdots\}$, $k \in \{1, \cdots, p\}$ *and* $|j-1|+|k-1| \neq 0$. *Then*

(1) $F \in TH_p^*$ *if and only if it can be expressed in the form*

$$F(z) = \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(\lambda_{j,p-k+1}h_{j,p-k+1}(z) + \gamma_{j,p-k+1}g_{j,p-k+1}(z)\big),$$

where $\lambda_{j,p-k+1} \geq 0$, $\gamma_{j,p-k+1} \geq 0$, $\gamma_{1,p} = 0$ *and* $\sum_{k=1}^{p}\sum_{j=1}^{\infty}(\lambda_{j,p-k+1}+\gamma_{j,p-k+1}) = 1$.

(2) *The set of all extreme points of TH_p^* are the union of the sets $\{h_{j,p-k+1}\}$ and $\{g_{j,p-k+1}\}$.*

Proof. Suppose

$$\begin{aligned} F(z) =& \sum_{k=1}^{p}\sum_{j=1}^{\infty}\big(\lambda_{j,p-k+1}h_{j,p-k+1}(z)+\gamma_{j,p-k+1}g_{j,p-k+1}(z)\big) \\ =&(1+\lambda_{1,p})z-\sum_{k=1}^{p}\sum_{j=1}^{\infty}|z|^{2(k-1)}\Big(\frac{\lambda_{j,p-k+1}z^j}{2(k-1)+j}+\frac{\gamma_{j,p-k+1}\overline{z}^j}{2(k-1)+j}\Big). \end{aligned}$$

Since $\lambda_{j,p-k+1}\geq 0$, $\gamma_{j,p-k+1}\geq 0$, $\gamma_{1,p}=0$ and $\sum_{k=1}^{p}\sum_{j=1}^{\infty}(\lambda_{j,p-k+1}+\gamma_{j,p-k+1})=1$, the starlikeness of F follows from Theorem 5.1. Hence, $F\in TH_p^*$.

Conversely, if $F\in TH_p^*$, then by Theorem5.1,

$$|a_{j,p-k+1}|\leq\frac{1}{2(k-1)+j}\ \text{ and }\ |b_{j,p-k+1}|\leq\frac{1}{2(k-1)+j}.$$

Set

$$\lambda_{j,p-k+1}=-(2(k-1)+j)a_{j,p-k+1},\ \ \gamma_{j,p-k+1}=-(2(k-1)+j)b_{j,p-k+1},$$

$$\lambda_{1,p}=1-\sum_{j\cdot k>1}(\lambda_{j,p-k+1}+\gamma_{j,p-k+1})$$

and

$$\gamma_{1,p}=0.$$

Then

$$F(z)=\sum_{k=1}^{p}\sum_{j=1}^{\infty}(\lambda_{j,p-k+1}h_{j,p-k+1}(z)+\gamma_{j,p-k+1}g_{j,p-k+1}(z)).$$

Hence (1) holds.

The proof of (2) easily follows from (1). Hence we complete the proof of Theorem 5.2. □

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