Łukasiewicz and modal logic

Josep Maria Font
University of Barcelona

Petr Hájek
Academy of Sciences, Prague

.

Abstract. Łukasiewicz's four-valued modal logic is surveyed and analyzed.

1 Introduction

The Polish philosopher and logician Jan Łukasiewicz (Lwów, 1878 – Dublin, 1956) is one of the fathers of modern many-valued logic, and some of the systems he introduced are presently a topic of deep investigation. In particular his infinitely-valued logic belongs to the core systems of mathematical fuzzy logic as a logic of comparative truth, cf. [3, 6, 7]. It is interesting to recall that modal notions were present in Łukasiewicz's motivation from the start. His 1930 paper [13] on many-valued logics is almost entirely devoted to modal propositions, and in it he says:

> The three-valued system of propositional logic owes its origin to certain enquiries I made into so-called "modal propositions" and the notions of possibility and necessity closely connected with them.
>
> (Quoted from [17], p. 153)

And, as he tells in the preface to the second edition [16] of his book on Aristotle, it was his aim of formalizing (and correcting) Aristotle's modal syllogistic what made him develop his system of modal logic:

> The first edition of this book did not contain an exposition of Aristotle's modal syllogistic. I was not able to examine Aristotle's ideas of necessity and possibility from the standpoint of the known systems of modal logic, as none of them was in my opinion correct. In order to master this difficult subject I had to construct for myself a system of modal logic [which] is different from any other such system, and from this standpoint I was able to explain the difficulties and correct the errors of the Aristotelian modal syllogistic.

In the paper [15] and in Chapter VII of [16] he introduces a propositional logic with two modalities $\Box$ and $\Diamond$ (necessity and possibility)[1] acting as unary

[1] Łukasiewicz uses L or Γ for necessity and M or Δ for possibility; he also uses the Polish notation. Actually, it seems that he was among the first to use L and M in this sense, see [8, Appendix 4]. We use the usual symbolism here.

connectives. Following Aristotle, he concluded that these connectives should be *extensional*, in particular the formulas

$$(1) \qquad (\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \quad \text{and} \quad (\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$$

should be valid. He took a generalized form of the extensionality principle using functional variables as a basis of a syntactical presentation of his logic. Further, he was assuming without discussion that the modal operators (any propositional operators, in fact) had to be truth-functional, which has as a straightforward consequence that they cannot be two-valued. This made him turn to the many-valued logics he had introduced years before by putting *possible* in a par with *true* and *false*. But he said:

> When I had discovered in 1920 a three-valued system of logic, I called the third value, which I denoted by $^1/_2$, "possibility". Later on, having found my n-valued modal systems[2], I thought that only two of them may be of philosophical importance, viz., the 3-valued and the $\aleph_0$-valued system. [...] This opinion, as I see it today, was wrong.
> ([15], quoted from [17], pp. 370–371)

and also:

> I see today that this system [the 3-valued one] does not satisfy all our intuitions concerning modalities and should be replaced by the system described below. ([15], pp. 166–167)

and he presented his four-valued truth-tables, claimed that they completely characterize his logic, and went as far as declaring:

> No serious objection can be maintained against this system. We shall see that this system refutes all false inferences drawn in connexion with modal logic, explains the difficulties of the Aristotelian modal syllogistic, and reveals some unexpected logical facts which are of the greatest importance for philosophy. ([15], p. 169)

However, both the semantics and many theorems of the logic have a difficult interpretation in terms of any intuitive notions of possibility and of necessity. Nowadays the possible worlds semantics is generally recognized to be the standard semantics for modal logic, both for its mathematical elegance and depth, and for its wide applicability; moreover, this semantics has often been considered to be the paradigm of a semantics for *intensional* logics, as opposed to the extensional ones[3]. This semantics was elaborated in full only after Łukasiewicz's

[2] He calls here "modal" his many-valued systems because of the interpretation of the non-classical truth values in terms of "possibility" and because in each of them modal operators were definable, e.g. $\Box\varphi = \neg(\varphi \to \neg\varphi)$ in the 3-valued case (suggested by Tarski in 1921 when he was Łukasiewicz's student).

[3] It is often called "Kripke" semantics, as Kripke has written seminal papers on the domain like [10], where Łukasiewicz's system is very briefly mentioned; but he was preceded by Hintikka and Kanger; see [1] for a historical survey on modal logic. Notice however that it is not the only acceptable semantics for modal logic; let us mention at least neighbourhood frames and the beautiful provability semantics, see [23].

death[4], and clearly Łukasiewicz's system does not fit into it (but see our result at the end of Section 2).

In view of the enormous development of modal logic since Łukasiewicz's book appeared, cf. [1, 2], it is legitimate to ask whether his system still has some value (except as historical rarity) as *modal* logic, and whether it is *natural* in some sense. We shall offer the reader a few easy mathematical facts and refer to relevant literature; everybody is free to take his/her own philosophical consequences. The reader should take a lesson of warning: do not too easily mix many-valuedness and modalities[5] (similarly, one must not as easily mix many-valuedness and probability). After a clear separation, one may build bridges. And even if the reader finds Łukasiewicz's attempt to build modal logic as a many-valued logic doubtful or even a dead end, as the authors are inclined to do, Łukasiewicz will always remain—we stress again—a celebrated pioneer of many-valued logic.

The paper is organized as follows: In Section 2 we present Łukasiewicz's system and offer some mathematical results on it, referring to the literature containing related observations. Section 3 contains several remarks on the oddity of Łukasiewicz's modal logic and on why it is so, partly based on the results in Section 2.

Thanks are due to J. Fiala for calling the attention of one of us (P. H.) to [16] and to L. Godó, S. Gottwald, R. Hähnle, L. Iturrioz, G. Malinowski and D. Vakarelov for valuable e-mail discussion.

2 Łukasiewicz's four-valued modal logic

Modulo inessential notational changes (see above), Łukasiewicz's system, which he denoted by Ł[6], is presented as a propositional calculus with propositional variables, the usual non-modal connectives $\rightarrow, \neg, \&, \vee$ and the specific modal connectives $\Box, \Diamond$ (unary). Łukasiewicz introduced his system syntactically, with both inference and rejection rules, and claimed that it was characterized by the four-element truth-table he also introduced. The rather obscure argument he offered to this effect was later criticized by Smiley, who in [22] fully proved that this truth-table does characterize the system when presented in the usual axiomatic style. Since we are not dealing with the syntax, we are going to take the semantics as our working definition of the logic.

There are four truth-values denoted by $11, 10, 01, 00$ (pairs of zeros and ones) or, as Łukasiewicz did, by $1, 2, 3, 0$. The truth tables of $\rightarrow$, $\neg$, $\Box, \Diamond$ are as follows, & and $\vee$ being defined as in classical logic (i.e., $\varphi \,\&\, \psi = \neg(\varphi \rightarrow \neg\psi)$ and $\varphi \vee \psi = \neg\varphi \rightarrow \psi$).

[4] Actually, [16], the second edition of his [14], was published a few months after he passed away, and he could not proof-read it; this was done by his disciple Cz. Lejewski. Thus, one can consider the chapters on Aristotle's modal logic in [16] as Łukasiewicz's logical testament.

[5] A very different and unrelated attempt to develop a four-valued modal logic is [4]; here the non-modal basis is Belnap's logic, a four-valued logic well-known for its applications.

[6] Warning: The symbol Ł is also used in the literature, often with subscripts, to denote Łukasiewicz's infinitely-valued logic or his usual, linearly-ordered, finitely-valued systems.

$\to$	11	10	01	00	$\neg$	$\Diamond$	$\Box$
11	11	10	01	00	00	11	10
10	11	11	01	01	01	11	10
01	11	10	11	10	10	01	00
00	11	11	11	11	11	01	00

It is immediately seen that according to implication and negation (and hence to conjunction and disjunction as well) this algebra of truth functions is just the cartesian product of the two-element Boolean algebra $\mathbf{B}_0$ with universe $B_0 = \{0,1\}$ with itself, i.e. operations act coordinatewise, if we identify uv with the pair (u,v). From this it follows that modality-free formulas are tautologies of Ł if and only if they are tautologies of CPC (classical propositional calculus), that is:

Proposition 1. *The logic Ł is a conservative expansion of CPC.* ∎

This fact is not explicitly mentioned by Łukasiewicz, who only says that Ł contains CPC. Moreover, the truth functions of $\Box$ and $\Diamond$ satisfy (for $u, v = 0, 1$)

(2) $\qquad \Box(uv) = 10 \wedge uv = u0 \quad , \quad \Diamond(uv) = 01 \vee uv = 10 \to uv = u1.$

Also observe that $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$, and dually $\Box\varphi$ is equivalent to $\neg\Diamond\neg\varphi$; hence only one of the operators needs to be taken as primitive (and so does Łukasiewicz himself).

Definition. For each formula φ of Ł we define its translation φ^* using a new propositional variable L in the following recursive way: $p^* = p$ for each propositional variable p of Ł, * commutes with $\to$ and $\neg$, $(\Box\varphi)^* = L \,\&\, \varphi^*$, and $(\Diamond\varphi)^* = L \to \varphi^*$.

Readers who do not feel at ease with this use of a “new” variable and prefer that the two logics to be compared share a common set of variables should consider the following alternative translation: If the set of variables is ordered as $\{p_0, p_1, \ldots, p_n, \ldots\}$ then put $(p_i)^* = p_{i+1}$ for all i and take p_0 instead of L in the expressions for $(\Box\varphi)^*$ and $(\Diamond\varphi)^*$. With minor adjustments in the proofs they will get the same results.

Theorem 1. *A formula φ is a tautology of Ł if and only if φ^* is a tautology of the classical propositional calculus CPC.*

Proof. Let us assume that φ^* is a tautology of CPC, and let e be any evaluation of the variables of φ with values in $B_0 \times B_0$. Since L does not appear in φ, we can safely extend it to an evaluation e' such that $e'(L) = 10$. Then it is easy to show recursively, using (2), that $e(\psi) = e'(\psi^*)$ for any formula ψ, in particular for φ. Since $\mathbf{B}_0 \times \mathbf{B}_0$ is a Boolean algebra and φ^* is a tautology of CPC, $e'(\varphi^*) = 11$, and therefore $e(\varphi) = 11$ as well. This shows that φ is a tautology of Ł.

Conversely, assume that φ is an Ł-tautology and let e_0 be an evaluation of $L, p_0, p_1, \ldots$ with values in $B_0 = \{0,1\}$. Then define $e_1(p_i) = 11$ if $e_0(p_i) = 1$ and $e_1(p_i) = 00$ if $e_0(p_i) = 0$. Thus e_1 is an evaluation of the variables of φ

with values in $B_0 \times B_0$. Then it is easy to show recursively, using (2) again, that if $e_0(L) = 1$, then for any ψ, $e_0(\psi^*)$ is the projection of $e_1(\psi)$ on the first coordinate, while if $e_0(L) = 0$, then for any ψ, $e_0(\psi^*)$ is the projection of $e_1(\psi)$ on the second coordinate. Since by assumption $e_1(\varphi) = 11$, in both cases we get $e_0(\varphi^*) = 1$. This shows that φ^* is a tautology of CPC. ∎

Corollary 1. *The mapping* * *is a faithful interpretation of* L *into CPC.* ∎

As another corollary we obtain the two-formulas interpretation of Ł first observed by Prior [19]. For each Ł-formula φ, let φ' result from φ by omitting all modalities, and let φ'' result from φ by recursively defining $p'' = p$ for all variables p, letting $''$ commute with $\neg$ and $\rightarrow$, and putting $(\Box\varphi)'' = \bot$ and $(\Diamond\varphi)'' = \top$, where $\top$ denotes any chosen non-modal tautology, and $\bot = \neg\top$. We have

Corollary 2. *A formula* φ *is a tautology of* L *if and only if both* φ' *and* φ'' *are tautologies of CPC.*

Proof. Observe that φ' is equivalent, modulo CPC, to the formula that results from φ^* after substituting $\top$ for L, while φ'' is equivalent, modulo CPC, to the formula that results from φ^* after substituting $\bot$ for L. The (constant) values of $\top$ and $\bot$ are the two possible values of the variable L in B_0. Hence φ^* is a tautology of CPC if and only if both φ' and φ'' are. Now the result follows as a consequence of Theorem 1. ∎

A related question is the extension of the preceding results to the provability from assumptions in Ł. As we show, this is only possible in one direction. First, we obviously have:

Lemma 1. *Modus Ponens is a rule of* L*, that is,* $\{\varphi, \varphi \rightarrow \psi\} \models_L \psi$. ∎

Proposition 2. *Let* Γ *be a set of formulas and put* $\Gamma^* = \{\psi^* : \psi \in \Gamma\}$ *and let* ψ *be any formula. If* $\Gamma^* \models_{CPC} \varphi^*$ *then* $\Gamma \models_L \varphi$.

Proof. If $\Gamma^* \models_{CPC} \varphi^*$ and $\Gamma \neq \emptyset$ then by Compactness and the Deduction Theorem for CPC there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\psi_1^* \rightarrow (\ldots \rightarrow (\psi_n^* \rightarrow \varphi^*))$ is a tautology of CPC. By the previous Theorem 1, $\psi_1 \rightarrow (\ldots \rightarrow (\psi_n \rightarrow \varphi))$ is a tautology of Ł, and by Lemma 1 $\{\psi_1, \ldots, \psi_n\} \models_Ł \varphi$, which shows that $\Gamma \models_Ł \varphi$. ∎

However, the converse implication is not true: If p and q are distinct variables, then trivially $\Box p \models_Ł q$, because no evaluation gives to $\Box p$ the value 11; but it is certainly not the case that $(\Box p)^* = L \,\&\, p \models_{CPC} q = q^*$. The same example shows that the Deduction Theorem in its ordinary form cannot hold for Ł, because $\Box p \rightarrow q$ is not a tautology of Ł (evaluate p to 10 and q to 01).

A natural enquiry concerning Łukasiewicz's modal logic is to look for a Kripke-style semantics. A Kripke model is a structure $K = \langle W, R, e\rangle$ where $W \neq \emptyset$ is a set (the so-called "possible worlds"), e is a mapping assigning to each $w \in W$ and each propositional variable p a truth value $e(p, w) \in B_0 = \{0, 1\}$ and R is a binary relation on W used in the truth conditions for modalized formulas: $e(\Box\varphi, w) = 1$ if and only if $e(\varphi, w') = 1$ for all $w' \in W$ such that

$(w, w') \in R$; from this it follows that $e(\Diamond\varphi, w) = 1$ if and only if there is some $w' \in W$ such that $(w, w') \in R$ and $e(\varphi, w') = 1$. The truth conditions for non-modal connectives are the usual ones of classical logic for each fixed world w. If the evaluation e is clear from context then it is customary to write $w \Vdash \varphi$ for $e(w, \varphi) = 1$.

Let $\mathcal{L}$ be the class of all Kripke models where R is a subrelation of identity: $(w, w') \in R$ implies $w = w'$. Observe that then the general truth conditions become $w \Vdash \Box\varphi$ if and only if $(w, w) \notin R$ or $w \Vdash \varphi$, and $w \Vdash \Diamond\varphi$ if and only if $(w, w) \in R$ and $w \Vdash \varphi$.

We are going to see that $\Box$ behaves in the class $\mathcal{L}$ as $\Diamond$ behaves in the logic Ł and conversely. To this end, let us consider, for each modal formula φ, the formula $\widetilde{\varphi}$ resulting from φ by changing $\Box$ to $\Diamond$ and $\Diamond$ to $\Box$, and leaving everything else untouched.

Theorem 2. *A formula φ is an Ł-tautology if and only if the formula $\widetilde{\varphi}$ is true in all models of the class $\mathcal{L}$.*

Proof. Observe that the satisfaction of φ in w depends only on the evaluation of variables in w and on whether $(w, w) \in R$ or not. It is straightforward to show that $\Diamond\varphi \equiv (\Diamond\top \mathbin{\&} \varphi)$ and $\Box\varphi \equiv (\Diamond\top \to \varphi)$ hold in $\mathcal{L}$. After comparing these expressions with those used in our translation * we see that $\widetilde{\varphi}$ is $\mathcal{L}$-equivalent to the substitution instance of φ^* where $\Diamond\top$ is substituted for L; and these are the only occurrences of modal operators in $\widetilde{\varphi}$. Since $\Diamond\top$ holds in w if and only if $(w, w) \in R$, it follows that $\widetilde{\varphi}$ is true in all models in $\mathcal{L}$ if and only if φ^* is true in all Boolean evaluations, that is, if and only if it is a CPC-tautology. By our previous Theorem 1, this happens if and only if φ is an Ł-tautology. ∎

Certainly $\mathcal{L}$ is a rather unusual class of Kripke models; e.g. $\Diamond\varphi \to \varphi$ and $\varphi \to \Box\varphi$ hold in $\mathcal{L}$ while $\varphi \to \Diamond\varphi$ and $\Box\varphi \to \varphi$ don't. As we see, only after interchanging the roles of $\Box$ and $\Diamond$ does it make sense modally speaking. In fact, if we consider the class $\widetilde{\mathcal{L}}$ of models obtained in this way then from Theorem 2 we obtain:

Corollary 3. *A formula φ is an Ł-tautology if and only if it is true in all models of the class $\widetilde{\mathcal{L}}$.* ∎

Although this result has a more standard appearance (a completeness theorem with respect to a certain class of models), the result in itself is less natural than Theorem 2: in $\widetilde{\mathcal{L}}$ the truth conditions of the modal operators are interchanged, that is, truth of $\Box\varphi$ at w does no longer mean that φ is true at all worlds accessible (i.e., R-related) from w but only in some of them, and dually for $\Diamond$. Hence, the main virtue of Kripke semantics, which is a natural interpretation of the notions of necessity and possibility, has been lost.

Let us close this section by pointing out that a translation provably equivalent to our * also appears in the paper [24]. In it Vakarelov studies a propositional logic with two modal functors intended to mean "probably" and "hardly", the first one being our $\Diamond$, while the second one $D\varphi$ would be equivalent to $\Diamond\bot \to \neg\varphi$. He shows that the translation * (formulated for his primitive operators, which gives $(D\varphi)^* = \varphi^* \to L$) is a faithful interpretation of his logic into CPC. In [25] he shows that his system is definitionally equivalent to Łukasiewicz's system Ł. Connecting the results in the two papers our Theorem 1 follows immediately. This fact is not mentioned explicitly by Vakarelov, and since his papers were published in Bulgarian they have remained rather unknown.

3 Discussion

That Łukasiewicz's four-valued system of modal logic is very difficult to interpret as a modal logic is apparent in our last results in the preceding section, and has been recognized by many scholars, either philosophers or logicians. As a consequence, it is difficult to find any influence of Ł in the mainstream development of modern modal logic. Witness to this is the fact that this system is nowhere mentioned in the historical introduction of Lemmon's [11], originally written in 1966, nor in the historical part of Bull and Segerberg's [1], published in 1984, nor in the historical notes to Chapter 3 of Chagrov and Zacharyaschev's [2], published in 1997; it is also noticeable that while it is discussed in the first book by Hughes and Cresswell [8], published in 1968, it no longer appears in their second book [9] published in 1996. This strange character of Ł was highlighted as early as 1957 by Prior, who on page 3 of his [19] says:

> Ever since this system was put forward in 1953 logicians, including Łukasiewicz himself, have been finding new oddities in it.

Let us look at some of them. From direct verification in the four-valued tables, or by use of the faithful interpretation, it is straightforward to check the next result.

Lemma 2. *The following formulas are Ł-tautologies:*

(3) $(\Box\varphi \,\&\, \Box(\varphi \to \psi)) \to \Box\psi$
(4) $\Box\Box\varphi \equiv \Box\varphi$
(5) $(\Box\varphi \,\&\, \Box\psi) \equiv \Box(\varphi \,\&\, \psi)$
(6) $(\Diamond\varphi \,\&\, \Diamond\psi) \equiv \Diamond(\varphi \,\&\, \psi)$
(7) $(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(8) $(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$
(9) $\Box\varphi \to (\psi \equiv \Box\psi)$ ∎

Here (3)–(5) look familiar; but (6)–(9) are not tautologies for any usual class of Kripke models. This mere fact would not be disastrous; but apparently there is no satisfactory natural intuitive notion of necessity for which (6)–(9) would be tautologies. Tautologicity of (6)–(8) is known to Łukasiewicz and discussed by him. Formula (6), already rejected by Lewis in [12], would account for a very strange notion of "possible", and Łukasiewicz [16, pp. 177–178] takes pains to justify it through some examples and argumentations about two different, yet indistinguishable operators of possibility or contingency, which do not appear to our eyes as clearly concluding. As we said in the Introduction, (7) and

(8) are called by him "laws of extensionality", and he devotes four sections [16, §39–§42] to study them, starting from his finding them in Aristotle's own writings, discussing whether the "$\rightarrow$" in the antecedent should be interpreted as a material or as a strict implication, and deciding this in favour of the material one based on the careful analysis of Aristotle's exact words and argumentations. Notice that solving this in favour of the strict implication would have produced the formulas

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \quad \text{and} \quad \Box(\varphi \rightarrow \psi) \rightarrow (\Diamond\varphi \rightarrow \Diamond\psi)\,,$$

two intuitively acceptable laws that hold in all normal modal logics, i.e., in any class of Kripke models. As to (9), it seems not to have been stated explicitly by Łukasiewicz with this generality, but a particular case of it is discussed in [16, Chapter VIII, formula 113] in the context of his analysis of a controversy concerning modal syllogisms (we refer to them below). Note that in (9) φ and ψ are arbitrary formulas; thus (using Łukasiewicz's examples of sentences) e.g. the following is true: "If necessarily each man is an animal then if the number drawn from the box is divisible by 4 then necessarily the number drawn from the box is divisible by 4". Since Łukasiewicz did not accept that the truth of some proposition might entail that it was necessarily true, he argued that from the particular case of (9) he was considering, which holds in his system, one should draw as a consequence that its premiss should be rejected; but since the premiss states that something is necessary, and in his system no formula like $\Box\varphi$ is a tautology, he found this conclusion entirely acceptable. One can wonder whether Aristotle would accept (9) as a logical truth, for the premiss of our example is a proposition explicitly asserted to hold by him, which would lead him to accept that "if the number drawn from the box is divisible by 4 then necessarily the number drawn from the box is divisible by 4", a plainly counterintuitive sentence. The following strange tautology is a variant of (9): $(\Box\varphi \,\&\, \Diamond\psi) \rightarrow \Box(\varphi \,\&\, \psi)$.

Still, one can try to look for some reasons in favour of Łukasiewicz's system. The first thing one can do for this is to join Prior in his [19] and Hughes and Cresswell in their [8] to acknowledge that the system Ł satisfies certain principles that were regarded by Łukasiewicz himself as minimal requirements for a system of propositional logic to count as a "modal" one. These are:

(10) The system must contain the ordinary, two-valued propositional logic.

(11) It must contain two unary functors $\Box$ and $\Diamond$ not definable in terms of the non-modal ones.

(12) The following should hold: $\Box\varphi \rightarrow \varphi$, $\varphi \rightarrow \Diamond\varphi$, $\Box\varphi \equiv \neg\Diamond\neg\varphi$, $\Diamond\varphi \equiv \neg\Box\neg\varphi$.

(13) The following should not hold: $\varphi \rightarrow \Box\varphi$, $\Diamond\varphi \rightarrow \varphi$, $\neg\Box\varphi$, $\Diamond\varphi$.

Another difficult, yet not impossible, justification for Ł would be to find some senses of the modal notions which may be in accordance with his results. Actually, he closed his 1953 paper [15] thus:

> I am fully aware that other systems of modal logic are possible based on different concepts of necessity and possibility. I firmly believe that we shall never be able to decide which of them is true. Systems of logic are instruments of thought, and the more useful a logical system is, the more valuable it is. I hope that the Ł-modal system expounded above will be a useful instrument, and deserves a further investigation and development.
> (Quoted from [17], pp. 378–379)

While Łukasiewicz's interpretation of the modal operators revolved only around so-to-speak ontological ideas of 'possibility' and 'necessity', we should bear in mind the large variety of interpretations and applications of modalities that have been developed, including some in the field of theoretical computer science. The following remark by Hughes and Cresswell may be relevant here:

> So $\diamond$ and $\square$ in the Ł-modal system do seem to express senses of 'possibility' and 'necessity' considerably different from any of the usual ones; and perhaps, if by a 'modal logic' we mean a logic of possibility and necessity, this system takes us to the limit of what we should regard as a modal logic at all. [8, p. 310]

In this task the idea of 'possibility' seems best suited than that of 'necessity', and our faithful interpretation may be of some help. Observe that in it L is a propositional variable and may be given any suggestive reading, e.g. "Alice is in the wonderland"; thus $\diamond p$ is understood as "If Alice is in the wonderland then p"; or, more generally, we can read "p is possible" as "If something happens then p", that is, as a way of asserting that the truth of p is not actual but may depend on some other, unspecified fact. This interpretation seems to fit well with Prior's idea that one can consider the possibility operator in Łukasiewicz's system as a kind of variable operator, whose value ranges between plain assertion and plain truth:

> I do not know whether anyone has ever used the words 'Possibly' and 'Necessarily' in the way which I am suggesting that the $\diamond$ and $\square$ of the Ł-modal system could be used. It would be rash and indeed definitely erroneous to say that no one ever has [...] Sometimes when a man says 'Possibly p' it does look as if he is trying to convey to some people the idea that he is assenting to the proposition p, and to others that he is not really committing himself to anything at all, and I suppose this would be something like using 'Possibly' as a variable operator capable of taking these two values, and the Ł-modal calculus would serve to show what propositions such a man could assent to without giving the game away.
> [19, p. 5]

We remark here that it was this idea what led him to the two-formulas interpretation of Ł, and that we have obtained it in Corollary 2 as a by-product of our translation.

A final move to support the plausibility of Łukasiewicz's system would be to attribute its oddities directly to Aristotle, and to Łukasiewicz's efforts to obtain a system that could represent Aristotle's ideas about necessity and possibility (reinforced by his commitment to truth-functionality). At this point we would like to recall that Łukasiewicz's analysis of the classical, non-modal Aristotelian theory of the syllogism is a remarkably fine piece of historical reconstruction based on purely scientific grounds. In Chapters I to III of [14, 16] he expounds the Aristotelian doctrines following his texts and those of his contemporaries and disciples as closely as possible, and highlights many errors made by later commentators. Then Chapters IV and V contain a brief and elementary, but sufficient exposition of modern propositional logic and the development of the theory of the syllogism as a formalized theory; the particular atoms of this theory are the four kinds of expressions appearing in Aristotle's syllogisms, which he symbolises using the letters A, I, E, O as medieval logicians did, thus finding the four expressions:

Aab	meaning	'all a is b'
Iab	"	'no a is b'
Eab	"	'some a is b'
Oab	"	'some a is not b'

He uses quantifiers only to better explain the laws of conversion, which justifies taking only Aab and Iab as primitive; but since Aristotle did not use quantifiers in his works he keeps his formal theory strictly inside propositional logic. With the obvious definitions $Eab = \neg Iab$ and $Oab = \neg Aab$ and four axioms (the two laws of identity Aaa and Iaa and the two syllogistic moods known as *Barbara* and *Datisi*) he completely succeeds in proving all 24 valid moods of the syllogism, besides showing that his system is consistent and independent. He also sets up a formal theory for rejection and claims it shows rejection of all 232 invalid moods. Finally, going beyond Aristotle, he investigates the propositional expressions in the new atoms that do not correspond to syllogisms, shows that they cannot be decided on the basis of the formal systems for acceptation and for rejection he has given, and presents a decision procedure based in the reduction to normal forms, essentially due to his pupil Słuplecki.

By contrast with this success, the situation concerning Aristotle's modal logic is quite different; in Łukasiewicz's own words:

> There are two reasons why Aristotle's modal logic is so little known. The first is due to the author himself: in contrast to the assertoric syllogistic which is perfectly clear and nearly free of errors, Aristotle's modal syllogistic is almost incomprehensible because of its many faults and inconsistencies. [...]
>
> The second reason is that modern logicians have not as yet been able to construct a universally acceptable system of modal logic which would yield a solid basis for the interpretation and appreciation of Aristotle's work. I have tried to construct such a system, different from those hitherto known, and built up upon Aristotle's ideas. [16, p. 133]

We see here two facts in support of our claim that the oddities in the Ł system should be attributed to Aristotle rather than to Łukasiewicz himself: The inconsistencies already present in Aristotle, and Łukasiewicz's firm determination of shaping his modal logic upon Aristotle's work as closely as possible. As we recalled before, complete agreement was not always possible, as following Aristotle at some places would have led to a contradiction.

He divided his analysis of Aristotle's modal logic into two parts, namely the propositional modal logic and the modal syllogistic, located respectively in Chapters VI and VIII of [16], and having a different status:

> It is possible to speak of an Aristotelian modal logic of propositions, as some of his theorems are general enough to comprise all kinds of proposition, and some others are expressly formulated by him with propositional variables. I shall begin with Aristotle's modal logic of propositions, which is logically and philosophically far more important than his modal syllogistic of terms.
>
> [...]
>
> Aristotle's modal syllogistic has, in my opinion, less importance in comparison with his assertoric syllogistic or his contributions to propositional modal logic. This system looks like a logical exercise which in spite of its seeming subtlety is full of careless mistakes and does not have any useful application to scientific problems. [16, pp. 133 and 181]

One of the points he discusses, and where he finds more difficulties to agree with Aristotle, is the controversy concerning syllogisms with one assertoric and one apodeictic premiss. Here 'assertoric' means non-modal, and 'apodeictic' means modalized with the necessity operator. The two controversial syllogisms are:

$$(\Box(Aba)\,\&\,Acb) \rightarrow \Box(Aca) \tag{14}$$

$$(Aba\,\&\,\Box(Acb)) \rightarrow \Box(Aca) \tag{15}$$

Aristotle accepts (14) but rejects (15), which has given rise to many philosophical discussions over the centuries. Łukasiewicz shows, syntactically, that both should be accepted by use of his Ł modal system combined with his formal system of non-modal syllogisms. It is in discussing this that he finds the particular case of formula (9) referred to above.

It does not make sense to ask whether (14) and (15) are theses of any modern system of modal logic, unless we re-write the propositional atom Aab, where a, b are terms, as the non-atomic predicate formula $(\forall x)(a(x) \rightarrow b(x))$, where $a(x)$ would mean "x is an a". This is contrary to Aristotle's understanding of universal propositions, and it is a very common mistake, according to Łukasiewicz (page 130), to attribute such view to Aristotle. Nevertheless, it is clear that the resulting formulas

$$(\Box(\forall x)(b(x) \rightarrow a(x))\,\&\,(\forall x)(c(x) \rightarrow b(x))) \rightarrow \Box(\forall x)(c(x) \rightarrow a(x))$$
$$((\forall x)(b(x) \rightarrow a(x))\,\&\,\Box(\forall x)(c(x) \rightarrow b(x))) \rightarrow \Box(\forall x)(c(x) \rightarrow a(x))$$

are not theorems of any of the normal modal logics, as they are not theorems of predicate S5.

Conclusion

We have collected and extended some easy observations on Łukasiewicz's system Ł of modal logic, and seen how its shortcomings result in part from Aristotle and in part from Łukasiewicz's assumption that modal operators should be truth-functional. Repeating our respect to Łukasiewicz's pioneering work in many-valued logic we feel that Ł is *not* a good modal logic since it proves several formulas that appear rather counter-intuitive and it has got (until now) neither any natural intuitive semantics nor any external applications. The following may be (modestly) interesting future research topics:

(1) To revise Łukasiewicz's analysis of Aristotle's modal logic under the light and with the now powerful techniques of intensional logic. The intensional standpoint was explicitly disregarded by him when he wrote:

> This [one of the laws of extensionality] seems perfectly evident, unless modal functions are regarded as intensional functions, i.e. as functions whose truth-values do *not* depend solely on the truth-values of their arguments. But what in this case the necessary and the possible would mean, is for me a mystery as yet. [16, p. 140]

He also rejected (p. 150), with no explicit reason, Quine's objections to the substitutivity of singular terms for variables in modal contexts, which is precisely one of the distinctive features of intensionality. Thus there is here a whole field to be investigated[7].

(2) To find a sounder foundation for the system Ł by giving an intuitively acceptable interpretation of the four values and of the behaviour of the modal operators on them. Given the many and diverse interpretations that modal-like logics have received through the years, it would not be so strange if one were found in accordance with Łukasiewicz's system.

Anyway, Łukasiewicz's Ł remains a remarkable witness of the development of modal logic before it has become "modern modal logic" in our present sense. Due to this development, the following advice of Łukasiewicz himself may be in order:

> Perhaps it would not be impossible to persuade living philosophers that they should cease to write about logic or its history before having acquired a solid knowledge of what is called 'mathematical logic'. It would otherwise be a waste of time for them as well as for their readers. It seems to me that this point is of no small practical importance. [16, p. 47]

[7] Note that we have not made an extensive bibliographic research on recent studies of Aristotle's modal logic as we did for Łukasiewicz's one. McCall's attempt [18], though closer to Aristotle's works, is exclusively syntactic.

References

[1] R. Bull, K. Segerberg: *Basic modal logic,* in: D. Gabbay, F. Guenthner, eds.: *Handbook of Philosophical Logic* vol. II, (Reidel, Dordrecht 1994), 1–88.

[2] A. Chagrov, M. Zacharyaschev: *Modal Logic,* (series Oxford Logic Guides, vol. 35) Clarendon Press, Oxford 1997.

[3] R. Cignoli, I. d'Ottaviano, D. Mundici: *Algebraic foundations of many-valued reasoning,* (series Trends in Logic, Studia Logica Library vol. 7) Kluwer, Dordrecht 1999.

[4] J. M. Font, M. Rius: *An abstract algebraic approach to tetravalent modal logics,* The Journal of Symbolic Logic (to appear).

[5] S. Gottwald: *Mehrwertige Logik,* Akademie-Verlag, Berlin 1988.

[6] S. Gottwald: *Many-valued logic,* (to appear).

[7] P. Hájek: *Metamathematics of Fuzzy Logic,* (series Trends in Logic, Studia Logica Library vol. 4) Kluwer, Dordrecht 1998.

[8] G. E. Hughess, M. J. Cresswell: *An introduction to modal logic,* Methuen and Co., London 1968.

[9] G. E. Hughess, M. J. Cresswell: *A new introduction to modal logic,* Routledge, London 1996.

[10] S. Kripke: *Semantical analysis of modal logic II. Non-normal modal propositional calculi,* in: Addison et al., eds., *The Theory of Models* (series Studies in Logic and the Foundations of Mathematics) North-Holland, Amsterdam 1965, 206–220.

[11] E. J. Lemmon: *An introduction to modal logic,* with the collaboration of D. Scott. (American Philosophical Quarterly Monograph Series, vol. 11) Basil Blackwell, London 1977.

[12] C. I. Lewis, C. H. Langford: *Symbolic Logic,* Appleton-Century-Croft, New York, 1932. 2nd edition published by Dover, New York, 1956.

[13] J. Łukasiewicz: *Philosophische Bemerkungen zu mehrwertigen Systemen der Aussagenlogik,* C. R. Soc. Sci. et Lett. de Varsovie, Cl. III 23 (1930) 51–77. English translation in [17], 153–178.

[14] J. Łukasiewicz: *Aristotle's syllogistic from the standpoint of modern formal logic.* Clarendon Press, Oxford 1951.

[15] J. Łukasiewicz: *A system of modal logic,* The Journal of Computing Systems 1 (1953) 111–149. Reprinted in [17], 352–390.

[16] J. Łukasiewicz: *Aristotle's syllogistic from the standpoint of modern formal logic,* 2nd enlarged edition. Clarendon Press, Oxford 1957.

[17] J. Łukasiewicz: *Selected Works (edited by* L. Borkowski*),* (series Studies in Logic and the Foundations of Mathematics) North-Holland, Amsterdam 1970.

[18] S. McCall: *Aristotle's modal syllogisms,* (series Studies in Logic and the Foundations of Mathematics) North-Holland, Amsterdam 1963.

[19] A. N. Prior: *Time and modality.* Clarendon Press, Oxford 1957.

[20] N. Rescher: *Many-valued logic,* McGraw-Hill, New York 1969.

[21] N. Rescher: *Topics in philosophical logic,* Reidel, Dordrecht 1968.

[22] T. Smiley: *On Łukasiewicz's Ł-modal system,* Notre Dame Journal of Formal Logic 2 (1961) 149–153.

[23] C. Smoryński: *Self-reference and modal logic,* Springer-Verlag, Heidelberg 1985.

[24] D. Vakarelov: *Ein Aussagenkalkül mit Funktionen für "Glaubwürdigkeit" und "Zweifel",* (Bulgarian with German summary) Annuaire de l'Université de Sofia, Fac. Math. vol. 60 (1965–66), Sofia 1967, 83–103.

[25] D. Vakarelov: *Ein Aussagenkalkül mit Funktionen für "Glaubwürdigkeit" und "Zweifel", Part II,* (Bulgarian with German summary) Annuaire de l'Université de Sofia, Fac. Math. vol. 61 (1966–67), Sofia 1968, 47–70.

Faculty of Mathematics, University of Barcelona
Gran Via 585, 08007 Barcelona, Spain
font@mat.ub.es

Institute of Computer Science, Academy of Sciences
Pod vodarenskou vezi 2, 182 07 Prague, Czech Republic
hajek@uivt.cas.cz