Differential Geometry in Physics

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Preface

These notes were developed as a supplement to a course on Differential Geometry at the advanced undergraduate, first year graduate level, which the author has taught for several years. There are many excellent texts in Differential Geometry but very few have an early introduction to differential forms and their applications to Physics. It is the purpose of these notes to bridge some of these gaps and thus help the student get a more profound understanding of the concepts involved. When appropriate, the notes also correlate classical equations to the more elegant but less intuitive modern formulation of the subject.

These notes should be accessible to students who have completed traditional training in Advanced Calculus, Linear Algebra, and Differential Equations. Students who master the entirety of this material will have gained enough background to begin a formal study of the General Theory of Relativity.

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Chapter 1

Vectors and Curves

1.1 Tangent Vectors

1.1 Definition Euclidean n-space $\mathbf{R}^n$ is defined as the set of ordered n-tuples $\mathbf{p} = (p^1, \dots, p^n)$, where $p^i \in \mathbf{R}$, for each $i = 1, \dots, n$.

Given any two n-tuples $\mathbf{p} = (p^1, \dots, p^n)$, $\mathbf{q} = (q^1, \dots, q^n)$ and any real number c, we define two operations:

$$\begin{aligned} \mathbf{p} + \mathbf{q} &= (p^1 + q^1, \dots, p^n + q^n) \\ c\mathbf{p} &= (cp^1, \dots, cp^n) \end{aligned} \tag{1.1}$$

With the sum and the scalar multiplication of ordered n-tuples defined this way, Euclidean space acquires the structure of a vector space of n dimensions [1].

1.2 Definition Let x^i be the real valued functions in $\mathbf{R}^n$ such that $x^i(\mathbf{p}) = p^i$ for any point $\mathbf{p} = (p^1, \dots, p^n)$. The functions x^i are then called the natural *coordinates* of the the point $\mathbf{p}$. When the dimension of the space $n = 3$, we often write: $x^1 = x$, $x^2 = y$ and $x^3 = z$.

1.3 Definition A real valued function in $\mathbf{R}^n$ is of class C^r if all the partial derivatives of the function up to order r exist and are continuous. The space of infinitely differentiable (smooth) functions will be denoted by $C^\infty(\mathbf{R}^n)$.

In advanced calculus, vectors are usually regarded as arrows characterized by a direction and a length. Vectors as thus considered as independent of their location in space. Because of physical and mathematical reasons, it is advantageous to introduce a notion of vectors that does depend on location. For example, if the vector is to represent a force acting on a rigid body, then the resulting equations of motion will obviously depend on the point at which the force is applied.

In a later chapter we will consider vectors on curved spaces. In these cases the positions of the vectors are crucial. For instance, a unit vector pointing north at the earth's equator is not at all the same as a unit vector pointing north at the tropic of Capricorn. This example should help motivate the following definition.

1.4 Definition A **tangent vector** X_p in $\mathbf{R}^n$, is an ordered pair $(\mathbf{X}, \mathbf{p})$. We may regard $\mathbf{X}$ as an ordinary advanced calculus vector and $\mathbf{p}$ is the position vector of the foot the arrow.

[1] In these notes we will use the following index conventions:
Indices such as i, j, k, l, m, n, run from 1 to n.
Indices such as μ, ν, ρ, σ, run from 0 to n.
Indices such as $\alpha, \beta, \gamma, \delta$, run from 1 to 2.

The collection of all tangent vectors at a point $\mathbf{p} \in \mathbf{R}^n$ is called the **tangent space** at $\mathbf{p}$ and will be denoted by $T_p(\mathbf{R}^n)$. Given two tangent vectors X_p, Y_p and a constant c, we can define new tangent vectors at $\mathbf{p}$ by $(X+Y)_p = X_p + Y_p$ and $(cX)_p = cX_p$. With this definition, it is easy to see that for each point $\mathbf{p}$, the corresponding tangent space $T_p(\mathbf{R}^n)$ at that point has the structure of a vector space. On the other hand, there is no natural way to add two tangent vectors at different points.

Let U be a open subset of $\mathbf{R}^n$. The set $T(U)$ consisting of the union of all tangent vectors at all points in U is called the **tangent bundle**. This object is not a vector space, but as we will see later it has much more structure than just a set.

1.5 Definition A vector field X in $U \in \mathbf{R}^n$ is a smooth function from U to $T(U)$.

We may think of a vector field as a smooth assignment of a tangent vector X_p to each point in in U. Given any two vector fields X and Y and any smooth function f, we can define new vector fields $X+Y$ and fX by

$$\begin{aligned} (X+Y)_p &= X_p + Y_p \\ (fX)_p &= fX_p \end{aligned} \tag{1.2}$$

Remark Since the space of smooth functions is not a field but only a ring, the operations above give the space of vector fields the structure of a ring module. The subscript notation X_p to indicate the location of a tangent vector is sometimes cumbersome. At the risk of introducing some confusion, we will drop the subscript to denote a tangent vector. Hopefully, it will be clear from the context whether we are referring to a vector or to a vector field.

Vector fields are essential objects in physical applications. If we consider the flow of a fluid in a region, the velocity vector field indicates the speed and direction of the flow of the fluid at that point. Other examples of vector fields in classical physics are the electric, magnetic and gravitational fields.

1.6 Definition Let X_p be a tangent vector in an open neighborhood U of a point $\mathbf{p} \in \mathbf{R}^n$ and let f be a C^∞ function in U. The directional derivative of f at the point $\mathbf{p}$, in the direction of X_p, is defined by

$$\nabla_X(f)(p) = \boldsymbol{\nabla} f(p) \cdot \mathbf{X}(p), \tag{1.3}$$

where $\boldsymbol{\nabla} f(p)$ is the gradient of the function f at the point $\mathbf{p}$. The notation

$$X_p(f) = \nabla_X(f)(p)$$

is also often used in these notes. We may think of a tangent vector at a point as an operator on the space of smooth functions in a neighborhood of the point. The operator assigns to a function the directional derivative of the function in the direction of the vector. It is easy to generalize the notion of directional derivatives to vector fields by defining $X(f)(p) = X_p(f)$.

1.7 Proposition If $f, g \in C^\infty \mathbf{R}^n$, $a, b \in \mathbf{R}$, and X is a vector field, then

$$\begin{aligned} X(af+bg) &= aX(f) + bX(g) \\ X(fg) &= fX(g) + gX(f) \end{aligned} \tag{1.4}$$

The proof of this proposition follows from fundamental properties of the gradient, and it is found in any advanced calculus text.

Any quantity in Euclidean space which satisfies relations 1.4 is a called a **linear derivation** on the space of smooth functions. The word *linear* here is used in the usual sense of a linear operator in linear algebra, and the word derivation means that the operator satisfies Leibnitz' rule.

The proof of the following proposition is slightly beyond the scope of this course, but the proposition is important because it characterizes vector fields in a coordinate-independent manner.

1.8 Proposition Any linear derivation on $C^\infty(\mathbf{R}^n)$ is a vector field.

This result allows us to identify vector fields with linear derivations. This step is a big departure from the usual concept of a "calculus" vector. To a differential geometer, a vector is a linear operator whose inputs are functions. At each point, the output of the operator is the directional derivative of the function in the direction of X.

Let $\mathbf{p} \in U$ be a point and let x^i be the coordinate functions in U. Suppose that $X_p = (\mathbf{X}, \mathbf{p})$, where the components of the Euclidean vector $\mathbf{X}$ are $a^1, \ldots, a^n$. Then, for any function f, the tangent vector X_p operates on f according to the formula

$$X_p(f) = \sum_{i=1}^{n} a^i (\frac{\partial f}{\partial x^i})(p). \tag{1.5}$$

It is therefore natural to identify the tangent vector X_p with the differential operator

$$\begin{aligned} X_p &= \sum_{i=1}^{n} a^i (\frac{\partial}{\partial x^i})(p) &(1.6)\\ X_p &= a^1(\frac{\partial}{\partial x^1})_p + \ldots + a^n(\frac{\partial}{\partial x^n})_p. \end{aligned}$$

Notation: We will be using Einstein's convention to suppress the summation symbol whenever an expression contains a repeated index. Thus, for example, the equation above could be simply written

$$X_p = a^i (\frac{\partial}{\partial x^i})_p. \tag{1.7}$$

This equation implies that the action of the vector X_p on the coordinate functions x^i yields the components a^i of the vector. In elementary treatments, vectors are often identified with the components of the vector and this may cause some confusion.

The difference between a tangent vector and a vector field is that in the latter case, the coefficients a^i are smooth functions of x^i. The quantities

$$(\frac{\partial}{\partial x^1})_p, \ldots, (\frac{\partial}{\partial x^n})_p$$

form a basis for the tangent space $T_p(\mathbf{R}^n)$ at the point $\mathbf{p}$, and any tangent vector can be written as a linear combination of these basis vectors. The quantities a^i are called the **contravariant** components of the tangent vector. Thus, for example, the Euclidean vector in $\mathbf{R}^3$

$$\mathbf{X} = 3\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

located at a point $\mathbf{p}$, would correspond to the tangent vector

$$X_p = 3(\frac{\partial}{\partial x})_p + 4(\frac{\partial}{\partial y})_p - 3(\frac{\partial}{\partial z})_p.$$

1.2 Curves in $\mathbf{R}^3$

1.9 Definition A **curve** $\alpha(t)$ in $\mathbf{R}^3$ is a C^∞ map from an open subset of $\mathbf{R}$ into $\mathbf{R}^3$. The curve assigns to each value of a parameter $t \in \mathbf{R}$, a point $(x^1(t), x^2(t), x^2(t))$ in $\mathbf{R}^3$

$$\begin{aligned} U \in \mathbf{R} &\overset{\alpha}{\longmapsto} \mathbf{R}^3 \\ t &\longmapsto \alpha(t) = (x^1(t), x^2(t), x^2(t)) \end{aligned}$$

One may think of the parameter t as representing time, and the curve α as representing the trajectory of a moving point particle.

1.10 Example Let
$$\alpha(t) = (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3).$$
This equation represents a straight line passing through the point $\mathbf{p} = (b_1, b_2, b_3)$, in the direction of the vector $\mathbf{v} = (a_1, a_2, a_3)$.

1.11 Example Let
$$\alpha(t) = (a \cos \omega t, a \sin \omega t, bt).$$
This curve is called a circular helix. Geometrically, we may view the curve as the path described by the hypotenuse of a triangle with slope b, which is wrapped around a circular cylinder of radius a. The projection of the helix onto the xy-plane is a circle and the curve rises at a constant rate in the z-direction.

Occasionally, we will revert to the position vector notation
$$\mathbf{x}(t) = (x^1(t), x^2(t), x^3(t)) \tag{1.8}$$
which is more prevalent in vector calculus and elementary physics textbooks. Of course, what this notation really means is
$$x^i(t) = (x^i \circ \alpha)(t), \tag{1.9}$$
where x^i are the coordinate slot functions in an open set in $\mathbf{R}^3$.

1.12 Definition The derivative $\alpha'(t)$ of the curve is called the **velocity** vector and the second derivative $\alpha''(t)$ is called the **acceleration**. The length $v = \|\alpha'(t)\|$ of the velocity vector is called the speed of the curve. The components of the velocity vector are simply given by
$$\mathbf{V}(t) = \frac{d\mathbf{x}}{dt} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right), \tag{1.10}$$
and the speed is
$$v = \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2} \tag{1.11}$$
The differential $d\mathbf{x}$ of the classical position vector given by
$$d\mathbf{x} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right) dt \tag{1.12}$$
is called an **infinitesimal tangent vector**, and the norm $\|d\mathbf{x}\|$ of the infinitesimal tangent vector is called the differential of arclength ds. Clearly, we have
$$ds = \|d\mathbf{x}\| = v dt \tag{1.13}$$

As we will see later in this text, the notion of infinitesimal objects needs to be treated in a more rigorous mathematical setting. At the same time, we must not discard the great intuitive value of this notion as envisioned by the masters who invented Calculus, even at the risk of some possible confusion! Thus, whereas in the more strict sense of modern differential geometry, the velocity vector is really a tangent vector and hence it should be viewed as a linear derivation on the space of functions, it is helpful to regard $d\mathbf{x}$ as a traditional vector which, at the infinitesimal level, gives a linear approximation to the curve.

If f is any smooth function on $\mathbf{R}^3$, we formally define $\alpha'(t)$ in local coordinates by the formula

$$\alpha'(t)(f) \mid_{\alpha(t)} = \frac{d}{dt}(f \circ \alpha) \mid_t . \tag{1.14}$$

The modern notation is more precise, since it takes into account that the velocity has a vector part as well as point of application. Given a point on the curve, the velocity of the curve acting on a function, yields the directional derivative of that function in the direction tangential to the curve at the point in question.

The diagram below provides a more geometrical interpretation of the the velocity vector formula (1.14). The map $\alpha(t)$ from $\mathbf{R}$ to $\mathbf{R}^3$ induces a map α_* from the tangent space of $\mathbf{R}$ to the tangent space of $\mathbf{R}^3$. The image $\alpha_*(\frac{d}{dt})$ in $T\mathbf{R}^3$ of the tangent vector $\frac{d}{dt}$ is what we call $\alpha'(t)$

$$\alpha_*(\frac{d}{dt}) = \alpha'(t).$$

Since $\alpha'(t)$ is a tangent vector in $\mathbf{R}^3$, it acts on functions in $\mathbf{R}^3$. The action of $\alpha'(t)$ on a function f on $\mathbf{R}^3$ is the same as the action of $\frac{d}{dt}$ on the composition $f \circ \alpha$. In particular, if we apply $\alpha'(t)$ to the coordinate functions x^i, we get the components of the the tangent vector, as illustrated

$$\begin{array}{ccccc} \frac{d}{dt} \in T\mathbf{R} & \stackrel{\alpha_*}{\longmapsto} & T\mathbf{R}^3 \ni \alpha'(t) & & \\ \downarrow & & \downarrow & & \\ \mathbf{R} & \stackrel{\alpha}{\longmapsto} & \mathbf{R}^3 & \stackrel{x^i}{\longmapsto} & \mathbf{R} \end{array}$$

$$\alpha'(t)(x^i) \mid_{\alpha(t)} = \frac{d}{dt}(x^i \circ \alpha) \mid_t . \tag{1.15}$$

The map α_* on the tangent spaces induced by the curve α is called the **push-forward**. Many authors use the notation $d\alpha$ to denote the push-forward, but we prefer to avoid this notation because most students fresh out of advanced calculus have not yet been introduced to the interpretation of the differential as a linear map on tangent spaces.

1.13 Definition

If $t = t(s)$ is a smooth, real valued function and $\alpha(t)$ is a curve in $\mathbf{R}^3$, we say that the curve $\beta(s) = \alpha(t(s))$ is a **reparametrization** of α.

A common reparametrization of curve is obtained by using the arclength as the parameter. Using this reparametrization is quite natural, since we know from basic physics that the rate of change of the arclength is what we call speed

$$v = \frac{ds}{dt} = \|\alpha'(t)\|. \tag{1.16}$$

The arc length is obtained by integrating the above formula

$$s = \int \|\alpha'(t)\| \, dt = \int \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2} \, dt \tag{1.17}$$

In practice it is typically difficult to actually find an explicit arclength parametrization of a curve since not only does one have calculate the integral, but also one needs to be able to find the inverse function t in terms of s. On the other hand, from a theoretical point of view, arclength parametrizations are ideal since any curve so parametrized has unit speed. The proof of this fact is a simple application of the chain rule and the inverse function theorem.

$$\beta'(s) = [\alpha(t(s))]'$$

$$
\begin{aligned}
&= \alpha'(t(s))t'(s) \\
&= \alpha'(t(s))\frac{1}{s'(t(s))} \\
&= \frac{\alpha'(t(s))}{\|\alpha'(t(s))\|},
\end{aligned}
$$

and any vector divided by its length is a unit vector. Leibnitz notation makes this even more self evident

$$
\begin{aligned}
\frac{d\mathbf{x}}{ds} &= \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\frac{d\mathbf{x}}{dt}}{\frac{ds}{dt}} \\
&= \frac{\frac{d\mathbf{x}}{dt}}{\|\frac{d\mathbf{x}}{dt}\|}
\end{aligned}
$$

1.14 Example Let $\alpha(t) = (a\cos\omega t, a\sin\omega t, bt)$. Then

$$\mathbf{V}(t) = (-a\omega\sin\omega t, a\omega\cos\omega t, b),$$

$$
\begin{aligned}
s(t) &= \int_0^t \sqrt{(-a\omega\sin\omega u)^2 + (a\omega\cos\omega u)^2 + b^2}\; du \\
&= \int_0^t \sqrt{a^2\omega^2 + b^2}\; du \\
&= ct, \quad \text{where,} \quad c = \sqrt{a^2\omega^2 + b^2}.
\end{aligned}
$$

The helix of unit speed is then given by

$$\beta(s) = (a\cos\frac{\omega s}{c}, a\sin\frac{\omega s}{c}, b\frac{\omega s}{c}).$$

Frenet Frames

Let $\beta(s)$ be a curve parametrized by arc length and let T(s) be the vector

$$T(s) = \beta'(s). \tag{1.18}$$

The vector $T(s)$ is tangential to the curve and it has unit length. Hereafter, we will call T the unit **unit tangent** vector. Differentiating the relation

$$T \cdot T = 1, \tag{1.19}$$

we get

$$2T \cdot T' = 0, \tag{1.20}$$

so we conclude that the vector T' is orthogonal to T. Let N be a unit vector orthogonal to T, and let κ be the scalar such that

$$T'(s) = \kappa N(s). \tag{1.21}$$

We call N the **unit normal** to the curve, and κ the **curvature**. Taking the length of both sides of last equation, and recalling that N has unit length, we deduce that

$$\kappa = \|T'(s)\| \tag{1.22}$$

It makes sense to call κ the curvature since, if T is a unit vector, then $T'(s)$ is not zero only if the direction of T is changing. The rate of change of the direction of the tangent vector is precisely what one would expect to measure how much a curve is curving. In particular, it $T' = 0$ at a particular point, we expect that at that point, the curve is locally well approximated by a straight line.

We now introduce a third vector

$$B = T \times N, \tag{1.23}$$

which we will call the **binormal** vector. The triplet of vectors (T, N, B) forms an orthonormal set; that is,

$$\begin{aligned} T \cdot T = N \cdot N = B \cdot B = 1 \\ T \cdot N = T \cdot B = N \cdot B = 0. \end{aligned} \tag{1.24}$$

If we differentiate the relation $B \cdot B = 1$, we find that $B \cdot B' = 0$, hence B' is orthogonal to B. Furthermore, differentiating the equation $T \cdot B = 0$, we get

$$B' \cdot T + B \cdot T' = 0.$$

rewriting the last equation

$$B' \cdot T = -T' \cdot B = -\kappa N \cdot B = 0,$$

we also conclude that B' must also be orthogonal to T. This can only happen if B' is orthogonal to the TB-plane, so B' must be proportional to N. In other words, we must have

$$B'(s) = -\tau N(s) \tag{1.25}$$

for some quantity τ, which we will call the **torsion**. The torsion is similar to the curvature in the sense that it measures the rate of change of the binormal. Since the binormal also has unit length, the only way one can have a non-zero derivative is if B is changing directions. The quantity B' then measures the rate of change in the up and down direction of an observer moving with the curve always facing forward in the direction of the tangent vector. The binormal B is something like the flag in the back of sand dune buggy.

The set of basis vectors $\{T, N, B\}$ is called the **Frenet Frame** or the **repere mobile** (moving frame). The advantage of this basis over the fixed (**i**,**j**,**k**) basis is that the Frenet frame is naturally adapted to the curve. It propagates along with the curve with the tangent vector always pointing in the direction of motion, and the normal and binormal vectors pointing in the directions in which the curve is tending to curve. In particular, a complete description of how the curve is curving can be obtained by calculating the rate of change of the frame in terms of the frame itself.

1.15 Theorem Let $\beta(s)$ be a unit speed curve with curvature κ and torsion τ. Then

$$\begin{array}{rcccc} T' & = & & \kappa N & \\ N' & = & -\kappa T & & \tau B \\ B' & = & & -\tau B & \end{array} . \tag{1.26}$$

Proof: We need only establish the equation for N'. Differentiating the equation $N \cdot N = 1$, we get $2N \cdot N' = 0$, so N' is orthogonal to N. Hence, N' must be a linear combination of T and B.

$$N' = aT + bB.$$

Taking the dot product of last equation with T and B respectively, we see that

$$a = N' \cdot T, \text{ and } \quad b = N' \cdot B.$$

On the other hand, differentiating the equations $N \cdot T = 0$, and $N \cdot B = 0$, we find that

$$\begin{aligned} N' \cdot T &= -N \cdot T' = -N \cdot (\kappa N) = -\kappa \\ N' \cdot B &= -N \cdot B' = -N \cdot (-\tau N) = \tau. \end{aligned}$$

We conclude that $a = -\kappa$, $b = \tau$, and thus

$$N' = -\kappa T + \tau B.$$

The Frenet frame equations (1.26) can also be written in matrix form as shown below.

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{1.27}$$

The group-theoretic significance of this matrix formulation is quite important and we will come back to this later when we talk about general orthonormal frames. At this time, perhaps it suffices to point out that the appearance of an antisymmetric matrix in the Frenet equations is not at all coincidental.

The following theorem provides a computational method to calculate the curvature and torsion directly from the equation of a given unit speed curve.

1.16 Proposition Let $\beta(s)$ be a unit speed curve with curvature $\kappa > 0$ and torsion τ. Then

$$\begin{aligned} \kappa &= \|\beta''(s)\| \\ \tau &= \frac{\beta' \cdot [\beta'' \times \beta''']}{\beta'' \cdot \beta''} \end{aligned} \tag{1.28}$$

Proof: If $\beta(s)$ is a unit speed curve, we have $\beta'(s) = T$. Then

$$\begin{aligned} T' = \beta''(s) &= \kappa N, \\ \beta'' \cdot \beta'' &= (\kappa N) \cdot (\kappa N), \\ \beta'' \cdot \beta'' &= \kappa^2 \\ \kappa^2 &= \|\beta''\|^2 \end{aligned}$$

$$\begin{aligned} \beta'''(s) &= \kappa' N + \kappa N' \\ &= \kappa' N + \kappa(-\kappa T + \tau B) \\ &= \kappa' N + -\kappa^2 T + \kappa \tau B. \end{aligned}$$

$$\begin{aligned} \beta' \cdot [\beta'' \times \beta'''] &= T \cdot [\kappa N \times (\kappa' N + -\kappa^2 T + \kappa \tau B)] \\ &= T \cdot [\kappa^3 B + \kappa^2 \tau T] \\ &= \kappa^2 \tau \\ \tau &= \frac{\beta' \cdot [\beta'' \times \beta''']}{\kappa^2} \\ &= \frac{\beta' \cdot [\beta'' \times \beta''']}{\beta'' \cdot \beta''} \end{aligned}$$

1.17 Example Consider a circle of radius r whose equation is given by

$$\alpha(t) = (r \cos t, r \sin t, 0).$$

Then,

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t, 0) \\
\|\alpha'(t)\| &= \sqrt{(-r\sin t)^2 + (r\cos t)^2 + 0^2} \\
&= \sqrt{r^2(\sin^2 t + \cos^2 t)} \\
&= r.
\end{aligned}$$

Therefore, $ds/dt = r$ and $s = rt$, which we recognize as the formula for the length of an arc of circle of radius t, subtended by a central angle whose measure is t radians. We conclude that

$$\beta(s) = (-r\sin\frac{s}{r}, r\cos\frac{s}{r}, 0)$$

is a unit speed reparametrization. The curvature of the circle can now be easily computed

$$\begin{aligned}
T &= \beta'(s) = (-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0) \\
T' &= (\frac{1}{r}\sin\frac{s}{r}, -\frac{1}{r}\cos\frac{s}{r}, 0) \\
\kappa &= \|\beta''\| = \|T'\| \\
&= \sqrt{\frac{1}{r^2}\sin^2\frac{s}{r} + \frac{1}{r^2}\cos^2\frac{s}{r} + 0^2} \\
&= \sqrt{\frac{1}{r^2}(\sin^2\frac{s}{r} + \cos^2\frac{s}{r})} \\
&= \frac{1}{r}
\end{aligned}$$

This is a very simple but important example. The fact that for a circle of radius r the curvature is $\kappa = 1/r$ could not be more intuitive. A small circle has large curvature and a large circle has small curvature. As the radius of the circle approaches infinity, the circle locally looks more and more like a straight line, and the curvature approaches 0. If one were walking along a great circle on a very large sphere (like the earth) one would be perceive the space to be locally flat.

1.18 Proposition Let $\alpha(t)$ be a curve of velocity $\mathbf{V}$, acceleration A, speed v and curvature κ, then

$$\begin{aligned}
\mathbf{V} &= vT, \\
\mathbf{A} &= \frac{dv}{dt}T + v^2\kappa N. \qquad (1.29)
\end{aligned}$$

Proof: Let $s(t)$ be the arclength and let $\beta(s)$ be a unit speed reparametrization. Then $\alpha(t) = \beta(s(t))$ and by the chain rule

$$\begin{aligned}
\mathbf{V} &= \alpha'(t) \\
&= \beta'(s(t))s'(t) \\
&= vT
\end{aligned}$$

$$\begin{aligned}
\mathbf{A} &= \alpha''(t) \\
&= \frac{dv}{dt}T + vT'(s(t))s'(t) \\
&= \frac{dv}{dt}T + v(\kappa N)v \\
&= \frac{dv}{dt}T + v^2\kappa N
\end{aligned}$$

Equation 1.29 is important in physics. The equation states that a particle moving along a curve in space feels a component of acceleration along the direction of motion whenever there is a change of speed, and a centripetal acceleration in the direction of the normal whenever it changes direction. The **centripetal acceleration** and any point is

$$a = v^3\kappa = \frac{v^2}{r}$$

where r is the radius of a circle which has maximal tangential contact with the curve at the point in question. This tangential circle is called the **osculating circle**. The osculating circle can be envisioned by a limiting process similar to that of the tangent to a curve in differential calculus. Let $\mathbf{p}$ be point on the curve, and let $\mathbf{q}_1$ and $\mathbf{q}_2$ be two nearby points. The three points uniquely determine a circle. This circle is a "secant" approximation to the tangent circle. As the points $\mathbf{q}_1$ and $\mathbf{q}_2$ approach the point $\mathbf{p}$, the "secant" circle approaches the osculating circle. The osculating circle always lies in the the TN-plane, which by analogy is called the **osculating plane**.

1.19 Example (Helix)

$$\begin{aligned}
\beta(s) &= (a\cos\frac{\omega s}{c}, a\sin\frac{\omega s}{c}, \frac{bs}{c}), \quad \text{where } c = \sqrt{a^2\omega^2+b^2} \\
\beta'(s) &= (-\frac{a\omega}{c}\sin\frac{\omega s}{c}, \frac{a\omega}{c}\cos\frac{\omega s}{c}, \frac{b}{c}) \\
\beta''(s) &= (-\frac{a\omega^2}{c^2}\cos\frac{\omega^2 s}{c}, -\frac{a\omega^2}{c^2}\sin\frac{\omega s}{c}, 0) \\
\beta'''(s) &= (-\frac{a\omega^3}{c^3}\cos\frac{\omega^2 s}{c}, -\frac{a\omega^3}{c^3}\sin\frac{\omega s}{c}, 0) \\
\kappa^2 &= \beta''\cdot\beta'' \\
&= \frac{a^2\omega^4}{c^4} \\
\kappa &= \pm\frac{a\omega^2}{c^2} \\
\tau &= \frac{(\beta'\beta''\beta''')}{\beta''\cdot\beta''} \\
&= \frac{b}{c}\begin{bmatrix} -\frac{a\omega^2}{c^2}\cos\frac{\omega s}{c} & -\frac{a\omega^2}{c^2}\sin\frac{\omega s}{c} \\ \frac{a\omega^3}{c^2}\sin\frac{\omega s}{c} & -\frac{a\omega^3}{c^2}\cos\frac{\omega s}{c} \end{bmatrix}\frac{c^4}{a^2\omega^4}. \\
&= \frac{b}{c}\frac{a^2\omega^5}{c^5}\frac{c^4}{a^2\omega^4}
\end{aligned}$$

Simplifying the last expression and substituting the value of c, we get

$$\begin{aligned}
\tau &= \frac{b\omega}{a^2\omega^2+b^2} \\
\kappa &= \pm\frac{a\omega^2}{a^2\omega^2+b^2}
\end{aligned}$$

Notice that if $b = 0$, the helix collapses to a circle in the xy-plane. In this case, the formulas above reduce to $\kappa = 1/a$ and $\tau = 0$. The ratio $\kappa/\tau = a\omega/b$ is particularly simple. Any curve where $\kappa/\tau = constant$ is called a helix, of which the circular helix is a special case.

1.20 Example (Plane curves) Let $\alpha(t) = (x(t), y(t), 0)$. Then

$$\alpha' = (x', y', 0)$$

$$
\begin{aligned}
\alpha'' &= (x'', y'', 0) \\
\alpha''' &= (x''', y''', 0) \\
\kappa &= \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3} \\
&= \frac{|\, x'y'' - y'x'' \,|}{(x'^2 + y'^2)^{3/2}} \\
\tau &= 0
\end{aligned}
$$

1.21 Example (Cornu Spiral) Let $\beta(s) = (x(s), y(s), 0)$, where

$$
\begin{aligned}
x(s) &= \int_0^s \cos \frac{t^2}{2c^2}\, dt \\
y(s) &= \int_0^s \sin \frac{t^2}{2c^2}\, dt.
\end{aligned} \tag{1.30}
$$

Then, using the fundamental theorem of calculus, we have

$$
\beta'(s) = (\cos \frac{s^2}{2c^2}, \sin \frac{t^2}{2c^2}, 0),
$$

Since $\|\beta' = v = 1\|$, the curve is of unit speed, and s is indeed the arc length. The curvature of the Cornu spiral is given by

$$
\begin{aligned}
\kappa &= |\, x'y'' - y'x'' \,| = (\beta' \cdot \beta')^{1/2} \\
&= \| -\frac{s}{c^2} \sin \frac{t^2}{2c^2}, \frac{s}{c^2} \cos \frac{t^2}{2c^2}, 0 \| \\
&= \frac{s}{c^2}.
\end{aligned}
$$

The integrals (1.30) defining the coordinates of the Cornu spiral are the classical **Frenel Integrals**. These functions, as well as the spiral itself, arise in the computation of the diffraction pattern of a coherent beam of light by a straight edge.

In cases where the given curve $\alpha(t)$ is not of unit speed, the following proposition provides formulas to compute the curvature and torsion in terms of α.

1.22 Proposition If $\alpha(t)$ is a regular curve in $\mathbf{R}^3$, then

$$
\kappa^2 = \frac{\|\alpha' \times \alpha''\|^2}{\|\alpha'\|^6} \tag{1.31}
$$

$$
\tau = \frac{(\alpha'\alpha''\alpha''')}{\|\alpha' \times \alpha''\|^2}, \tag{1.32}
$$

where $(\alpha'\alpha''\alpha''')$ is the triple vector product $[\alpha \times' \alpha''] \cdot \alpha'''$.
Proof:

$$
\begin{aligned}
\alpha' &= vT \\
\alpha'' &= v'T + v^2 \kappa N \\
\alpha''' &= (v^2\kappa) N'((s(t)) s'(t) + \dots \\
&= v^3 \kappa N' + \dots \\
&= v^3 \kappa \tau B + \dots
\end{aligned}
$$

The other terms are unimportant here because $\alpha' \times \alpha''$ is proportional to B.

$$\begin{aligned}
\alpha' \times \alpha'' &= v^3\kappa(T \times N) = v^3\kappa B \\
\|\alpha' \times \alpha''\| &= v^3\kappa \\
\kappa &= \frac{\|\alpha' \times \alpha''\|}{v^3} \\
(\alpha' \times \alpha'') \cdot \alpha''' &= v^6\kappa^2\tau \\
\tau &= \frac{(\alpha'\alpha''\alpha''')}{v^6\kappa^2} \\
&= \frac{(\alpha'\alpha''\alpha''')}{\|\alpha' \times \alpha''\|^2}
\end{aligned}$$

1.3 Fundamental Theorem of Curves

Some geometrical insight into the significance of the curvature and torsion can be gained by considering the Taylor series expansion of an arbitrary unit speed curve $\beta(s)$ about $s = 0$.

$$\beta(s) = \beta(0) + \beta'(0)s + \frac{\beta''(0)}{2!}s^2 + \frac{\beta'''(0)}{3!}s^3 + \dots \tag{1.33}$$

Since we are assuming that s is an arclength parameter,

$$\begin{aligned}
\beta'(0) &= T(0) = T_0 \\
\beta''(0) &= (\kappa N)(0) = \kappa_0 N_0 \\
\beta'''(0) &= (-\kappa^2 T + \kappa' N + \kappa\tau B)(0) = -\kappa_0^2 T_0 + \kappa_0' N_0 + \kappa_0\tau_0 B_0
\end{aligned}$$

Keeping only the lowest terms in the components of T, N, and B, we get the first order Frenet approximation to the curve

$$\beta(s) \doteq \beta(0) + T_0 s + \frac{1}{2}\kappa_0 N_0 s^2 + \frac{1}{6}\kappa_0\tau_0 B_0 s^3. \tag{1.34}$$

The first two terms represent the linear approximation to the curve. The first three terms approximate the curve by a parabola which lies in the osculating plane (TN-plane). If $\kappa_0 = 0$, then locally the curve looks like a straight line. If $\tau_0 = 0$, then locally the curve is a plane curve contained on the osculating plane. In this sense, the curvature measures the deviation of the curve from a straight line and the torsion (also called the second curvature) measures the deviation of the curve from a plane curve.

1.23 Theorem (Fundamental Theorem of Curves) Let $\kappa(s)$ and $\tau(s)$, $(s > 0)$ be any two analytic functions. Then there exists a unique curve (unique up to its position in $\mathbf{R}^3$) for which s is the arclength, $\kappa(s)$ the curvature and $\tau(s)$ the torsion.
Proof: Pick a point in $\mathbf{R}^3$. By an appropriate translation transformation, we may assume that this point is the origin. Pick any orthogonal frame $\{T, N, B\}$. The curve is then determined uniquely by its Taylor expansion in the Frenet frame as in equation (1.34).

1.24 Remark It is possible to prove the theorem by just assuming that $\kappa(s)$ and $\tau(s)$ are continuous. The proof, however, becomes much harder so we prefer the weaker version of the theorem based on the simplicity of Taylor's theorem.

1.25 Proposition A curve with $\kappa = 0$ is part of a straight line.
We leave the proof as an exercise.

1.26 Proposition A curve $\alpha(t)$ with $\tau = 0$ is a plane curve.
Proof: If $\tau = 0$, then $(\alpha'\alpha''\alpha''') = 0$. This means that the three vectors α', α'', and α''' are linearly dependent and hence there exist functions $a_1(s)$,$a_2(s)$ and $a_3(s)$ such that

$$a_3\alpha''' + a_2\alpha'' + a_1\alpha' = 0.$$

This linear homogeneous equation will have a solution of the form

$$\alpha = \mathbf{c}_1\alpha_1 + \mathbf{c}_2\alpha_2 + \mathbf{c}_3, \quad c_i = \text{constant vectors.}$$

This curve lies in the plane

$$(\mathbf{x} - \mathbf{c}_3) \cdot \mathbf{n} = 0, \quad \text{where} \quad \mathbf{n} = \mathbf{c}_1 \times \mathbf{c}_2$$

Chapter 2

Differential Forms

2.1 1-Forms

One of the most puzzling ideas in elementary calculus is that of the of the differential. In the usual definition, the differential of a dependent variable $y = f(x)$ is given in terms of the differential of the independent variable by $dy = f'(x)dx$. The problem is with the quantity dx. What does "dx" mean? What is the difference between Δx and dx? How much "smaller" than Δx does dx have to be? There is no trivial resolution to this question. Most introductory calculus texts evade the issue by treating dx as an arbitrarily small quantity (lacking mathematical rigor) or by simply referring to dx as an infinitesimal (a term introduced by Newton for an idea that could not otherwise be clearly defined at the time.)

In this section we introduce linear algebraic tools that will allow us to interpret the differential in terms of an linear operator.

2.1 Definition Let $\mathbf{p} \in \mathbf{R}^n$, and let $T_p(\mathbf{R}^n)$ be the tangent space at $\mathbf{p}$. A **1-form at p** is a linear map ϕ from $T_p(\mathbf{R}^n)$ into $\mathbf{R}$. We recall that such a map must satisfy the following properties:

$$\begin{aligned} &\text{a)} \quad \phi(X_p) \in \mathbf{R}, && \forall X_p \in \mathbf{R}^n \\ &\text{b)} \quad \phi(aX_p + bY_p) = a\phi(X_p) + b\phi(Y_p), && \forall a, b \in \mathbf{R},\ X_p, Y_p \in T_p(\mathbf{R}^n) \end{aligned} \tag{2.1}$$

A **1-form** is a smooth choice of a linear map ϕ as above for each point in the space.

2.2 Definition Let $f : \mathbf{R}^n \to \mathbf{R}$ be a real-valued C^∞ function. We define the differential df of the function as the 1-form such that

$$df(X) = X(f) \tag{2.2}$$

for every vector field in X in $\mathbf{R}^n$.

In other words, at any point $\mathbf{p}$, the differential df of a function is an operator that assigns to a tangent vector X_p the directional derivative of the function in the direction of that vector.

$$df(X)(p) = X_p(f) = \boldsymbol{\nabla} f(p) \cdot \mathbf{X}(p) \tag{2.3}$$

In particular, if we apply the differential of the coordinate functions x^i to the basis vector fields, we get

$$dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \tag{2.4}$$

The set of all linear functionals on a vector space is called the **dual** of the vector space. It is a standard theorem in linear algebra that the dual of a vector space is also a vector space of the same dimension. Thus, the space $T^\star_p\mathbf{R}^n$ of all 1-forms at $\mathbf{p}$ is a vector space which is the dual of the tangent space $T_p\mathbf{R}^n$. The space $T^\star_p(\mathbf{R}^n)$ is called the **cotangent space** of $\mathbf{R}^n$ at the point $\mathbf{p}$. Equation (2.4) indicates that the set of differential forms $\{(dx^1)_p,\dots,(dx^n)_p\}$ constitutes the basis of the cotangent space which is dual to the standard basis $\{(\frac{\partial}{\partial x^1})_p,\dots(\frac{\partial}{\partial x^n})_p\}$ of the tangent space. The union of all the cotangent spaces as $\mathbf{p}$ ranges over all points in $\mathbf{R}^n$ is called the **cotangent bundle** $T^*(\mathbf{R}^n)$.

2.3 Proposition Let f be any smooth function in $\mathbf{R}^n$ and let $\{x^1,\dots x^n\}$ be coordinate functions in a neighborhood U of a point $\mathbf{p}$. Then, the differential df is given locally by the expression

$$\begin{aligned} df &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \\ &= \frac{\partial f}{\partial x^i} dx^i \end{aligned} \tag{2.5}$$

Proof: The differential df is by definition a 1-form, so, at each point, it must be expressible as a linear combination of the basis elements $\{(dx^1)_p,\dots,(dx^n)_p\}$. Therefore, to prove the proposition, it suffices to show that the expression 2.5 applied to an arbitrary tangent vector coincides with definition 2.2. To see this, consider a tangent vector $X_p = a^j(\frac{\partial}{\partial x^j})_p$ and apply the expression above as follows:

$$\begin{aligned} (\frac{\partial f}{\partial x^i} dx^i)_p(X_p) &= (\frac{\partial f}{\partial x^i} dx^i)(a^j \frac{\partial}{\partial x^j})(p) \\ &= a^j(\frac{\partial f}{\partial x^i} dx^i)(\frac{\partial}{\partial x^j})(p) \\ &= a^j(\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial x^j})(p) \\ &= a^j(\frac{\partial f}{\partial x^i}\delta^i_j)(p) \\ &= (\frac{\partial f}{\partial x^i} a^i)(p) \\ &= \boldsymbol{\nabla} f(p)\cdot \mathbf{X}(p) \\ &= df(X)(p) \end{aligned} \tag{2.6}$$

The definition of differentials as linear functionals on the space of vector fields is much more satisfactory than the notion of infinitesimals, since the new definition is based on the rigorous machinery of linear algebra. If α is an arbitrary 1-form, then locally

$$\alpha = a_1(\mathbf{x})dx^1 +,\dots + a_n(\mathbf{x})dx^n, \tag{2.7}$$

where the coefficients a_i are C^∞ functions. A 1-form is also called a **covariant tensor** of rank 1, or simply a **covector**. The coefficients $(a_1,\dots,a_n)$ are called the **covariant** components of the covector. We will adopt the convention to always write the covariant components of a covector with the indices down. Physicists often refer to the covariant components of a 1-form as a covariant vector and this causes some confusion about the position of the indices. We emphasize that not all one forms are obtained by taking the differential of a function. If there exists a function f, such that $\alpha = df$, then the one form α is called **exact**. In vector calculus and elementary physics, exact forms are important in understanding the path independence of line integrals of conservative vector fields.

As we have already noted, the cotangent space $T^*_p(\mathbf{R}^n)$ of 1-forms at a point $\mathbf{p}$ has a natural vector space structure. We can easily extend the operations of addition and scalar multiplication to the space of all 1-forms by defining

$$\begin{aligned}(\alpha+\beta)(X) &= \alpha(X)+\beta(X) \\ (f\alpha)(X) &= f\alpha(X)\end{aligned} \tag{2.8}$$

for all vector fields X and all smooth functions f.

2.2 Tensors and Forms of Higher Rank

As we mentioned at the beginning of this chapter, the notion of the differential dx is not made precise in elementary treatments of calculus, so consequently, the differential of area $dxdy$ in $\mathbf{R}^2$, as well as the differential of surface area in $\mathbf{R}^3$ also need to be revisited in a more rigorous setting. For this purpose, we introduce a new type of multiplication between forms that not only captures the essence of differentials of area and volume, but also provides a rich algebraic and geometric structure generalizing cross products (which make sense only in $\mathbf{R}^3$) to Euclidean space of any dimension.

2.4 Definition A map $\phi : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ is called a **bilinear** map on the tangent space, if it is linear on each slot. That is,

$$\begin{aligned}\phi(f^1X_1+f^2X_2,Y_1) &= f^1\phi(X_1,Y_1)+f^2\phi(X_2,Y_1) \\ \phi(X_1,f^1Y_1+f^2Y_2) &= f^1\phi(X_1,Y_1)+f^2\phi(X_1,Y_2), \qquad \forall X_i,Y_i \in T(\mathbf{R}^n), \quad f^i \in C^\infty\mathbf{R}^n.\end{aligned}$$

Tensor Products

2.5 Definition Let α and β be 1-forms. The **tensor product** of α and β is defined as the bilinear map $\alpha \otimes \beta$ such that

$$(\alpha\otimes\beta)(X,Y) = \alpha(X)\beta(Y) \tag{2.9}$$

for all vector fields X and Y.

Thus, for example, if $\alpha = a_i dx^i$ and $\beta = b_j dx^j$, then,

$$\begin{aligned}(\alpha\otimes\beta)(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}) &= \alpha(\frac{\partial}{\partial x^k})\beta(\frac{\partial}{\partial x^l}) \\ &= (a_i dx^i)(\frac{\partial}{\partial x^k})(b_j dx^j)(\frac{\partial}{\partial x^l}) \\ &= a_i\delta^i_k b_j \delta^j_l \\ &= a_k b_l.\end{aligned}$$

A quantity of the form $T = T_{ij}dx^i \otimes dx^j$ is called a **covariant tensor of rank 2,** and we may think of the set $\{dx^i \otimes dx^j\}$ as a basis for all such tensors. We must caution the reader again that there is possible confusion about the location of the indices, since physicists often refer to the components T_{ij} as a covariant tensor.

In a similar fashion, one can define the tensor product of vectors X and Y as the bilinear map $X \otimes Y$ such that

$$(X\otimes Y)(f,g) = X(f)Y(g) \tag{2.10}$$

for any pair of arbitrary functions f and g.

If $X = a^i\frac{\partial}{\partial x^i}$ and $Y = b^j\frac{\partial}{\partial x^j}$, then the components of $X\otimes Y$ in the basis $\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$ are simply given by a^ib^j. Any bilinear map of the form

$$T = T^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j} \tag{2.11}$$

is called a contravariant tensor of rank 2 in $\mathbf{R}^n$.

The notion of tensor products can easily be generalized to higher rank, and in fact one can have tensors of mixed ranks. For example, a tensor of contravariant rank 2 and covariant rank 1 in $\mathbf{R}^n$ is represented in local coordinates by an expression of the form

$$T = T^{ij}{}_k \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dx^k.$$

This object is also called a tensor of type $T^{2,1}$. Thus, we may think of a tensor of type $T^{2,1}$ as map with three input slots. The map expects two functions in the first two slots and a vector in the third one. The action of the map is bilinear on the two functions and linear on the vector. The output is a real number. An assignment of a tensor to each point in $\mathbf{R}^n$ is called a tensor field.

Inner Products

Let $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$ be two vector fields and let

$$g(X, Y) = \delta_{ij} a^i b^j. \tag{2.12}$$

The quantity $g(X, Y)$ is an example of a bilinear map that the reader will recognize as the usual dot product.

2.6 Definition A bilinear map $g(X, Y)$ on the tangent space is called a vector **inner product** if

1. $g(X, Y) = g(Y, X)$,
2. $g(X, X) \geq 0, \ \ \forall X$,
3. $g(X, X) = 0$ iff $X = 0$.

Since we assume $g(X, Y)$ to be bilinear, an inner product is completely specified by its action on ordered pairs of basis vectors. The components g_{ij} of the inner product are thus given by

$$g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = g_{ij}, \tag{2.13}$$

where g_{ij} is a symmetric $n \times n$ matrix which we assume to be non-singular. By linearity, it is easy to see that if $X = a^i \frac{\partial}{\partial x^i}$ and $Y = b^j \frac{\partial}{\partial x^j}$ are two arbitrary vectors, then

$$g(X, Y) = g_{ij} a^i b^j.$$

In this sense, an inner product can be viewed as a generalization of the dot product. The standard Euclidean inner product is obtained if we take $g_{ij} = \delta_{ij}$. In this case, the quantity $g(X, X) = \| X \|^2$ gives the square of the length of the vector. For this reason g_{ij} is called a **metric** and g is called a **metric tensor**.

Another interpretation of the dot product can be seen if instead one considers a vector $X = a^i \frac{\partial}{\partial x^i}$ and a 1-form $\alpha = b_j dx^j$. The action of the 1-form on the vector gives

$$\begin{aligned} \alpha(X) &= (b_j dx^j)(a^i \frac{\partial}{\partial x^i}) \\ &= b_j a^i (dx^j)(\frac{\partial}{\partial x^i}) \\ &= b_j a^i \delta^j_i \\ &= a^i b_i. \end{aligned}$$

If we now define

$$b_i = g_{ij} b^j, \tag{2.14}$$

we see that the equation above can be rewritten as

$$a^i b_j = g_{ij} a^i b^j,$$

and we recover the expression for the inner product.

Equation (2.14) shows that the metric can be used as a mechanism to lower indices, thus transforming the contravariant components of a vector to covariant ones. If we let g^{ij} be the inverse of the matrix g_{ij}, that is

$$g^{ik} g_{kj} = \delta^i_j, \tag{2.15}$$

we can also raise covariant indices by the equation

$$b^i = g^{ij} b_j. \tag{2.16}$$

We have mentioned that the tangent and cotangent spaces of Euclidean space at a particular point are isomorphic. In view of the above discussion, we see that the metric accepts a dual interpretation; one as a bilinear pairing of two vectors

$$g : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$$

and another as a linear isomorphism

$$g : T^\star(\mathbf{R}^n) \longrightarrow T(\mathbf{R}^n)$$

that maps vectors to covectors and vice-versa.

In elementary treatments of calculus, authors often ignore the subtleties of differential 1-forms and tensor products and define the differential of arclength as

$$ds^2 \equiv g_{ij} dx^i dx^j,$$

although what is really meant by such an expression is

$$ds^2 \equiv g_{ij} dx^i \otimes dx^j. \tag{2.17}$$

2.7 Example In cylindrical coordinates, the differential of arclength is

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2. \tag{2.18}$$

In this case, the metric tensor has components

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.19}$$

2.8 Example In spherical coordinates,

$$\begin{aligned} x &= \rho \sin\theta \cos\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\theta, \end{aligned} \tag{2.20}$$

and the differential of arclength is given by

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2. \tag{2.21}$$

In this case the metric tensor has components

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin\theta^2 \end{bmatrix}. \tag{2.22}$$

Minkowski Space

An important object in mathematical physics is the so-called **Minkowski space** which is defined as the pair $(\mathcal{M}_{1,3}, g\eta)$, where

$$\mathcal{M}_{(1,3)} = \{(t, x^1, x^2, x^3) |\ t, x^i \in \mathbf{R}\} \tag{2.23}$$

and η is the bilinear map such that

$$\eta(X, X) = -t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{2.24}$$

The matrix representing Minkowski's metric η is given by

$$\eta = \text{diag}(-1, 1, 1, 1),$$

in which case, the differential of arclength is given by

$$\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^\mu \otimes dx^\nu \\ &= -dt \otimes dt + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \\ &= -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \end{aligned} \tag{2.25}$$

Note: Technically speaking, Minkowski's metric is not really a metric since $\eta(X, X) = 0$ does not imply that $X = 0$. Non-zero vectors with zero length are called **Light-like** vectors and they are associated with with particles that travel at the speed of light (which we have set equal to 1 in our system of units.)

The Minkowski metric $g_{\mu\nu}$ and its matrix inverse $g^{\mu\nu}$ are also used to raise and lower indices in the space in a manner completely analogous to $\mathbf{R}^n$. Thus, for example, if A is a covariant vector with components

$$A_\mu = (\rho, A_1, A_2, A_3),$$

then the contravariant components of A are

$$\begin{aligned} A^\mu &= \eta^{\mu\nu} A_\nu \\ &= (-\rho, A_1, A_2, A_3) \end{aligned}$$

Wedge Products and n-Forms

2.9 Definition A map $\phi : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ is called **alternating** if

$$\phi(X, Y) = -\phi(Y, X).$$

The alternating property is reminiscent of determinants of square matrices that change sign if any two column vectors are switched. In fact, the determinant function is a perfect example of an alternating bilinear map on the space $M_{2\times2}$ of two by two matrices. Of course, for the definition above to apply, one has to view $M_{2\times2}$ as the space of column vectors.

2.10 Definition A **2-form** ϕ is a map $\phi : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$ which is alternating and bilinear.

2.11 Definition Let α and β be 1-forms in $\mathbf{R}^n$ and let X and Y be any two vector fields. The **wedge product** of the two 1-forms is the map $\alpha \wedge \beta : T(\mathbf{R}^n) \times T(\mathbf{R}^n) \longrightarrow \mathbf{R}$, given by the equation

$$(\alpha \wedge \beta)(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{2.26}$$

2.12 Theorem If α and β are 1-forms, then $\alpha \wedge \beta$ is a 2-form.
Proof: We break up the proof into the following two lemmas:

2.13 Lemma The wedge product of two 1-forms is alternating.
Proof: Let α and β be 1-forms in $\mathbf{R}^n$ and let X and Y be any two vector fields. Then

$$\begin{aligned}(\alpha \wedge \beta)(X,Y) &= \alpha(X)\beta(Y) - \alpha(Y)\beta(X) \\ &= -(\alpha(Y)\beta(X) - \alpha(X)\beta(Y)) \\ &= -(\alpha \wedge \beta)(Y,X).\end{aligned}$$

2.14 Lemma The wedge product of two 1-forms is bilinear.
Proof: Consider 1-forms, α, β, vector fields X_1, X_2, Y and functions f^1, F^2. Then, since the 1-forms are linear functionals, we get

$$\begin{aligned}(\alpha \wedge \beta)(f^1X_1 + f^2X_2, Y) &= \alpha(f^1X_1 + f^2X_2)\beta(Y) - \alpha(Y)\beta(f^1X_1 + f^2X_2) \\ &= [f^1\alpha(X_1) + f^2\alpha(X_2)]\beta(Y) - \alpha(Y)[f^1\beta(X_1) + f^2\alpha(X_2)] \\ &= f^1\alpha(X_1)\beta(Y) + f^2\alpha(X_2)\beta(Y) + f^1\alpha(Y)\beta(X_1) + f^2\alpha(Y)\beta(X_2) \\ &= f^1[\alpha(X_1)\beta(Y) + \alpha(Y)\beta(X_1)] + f^2[\alpha(X_2)\beta(Y) + \alpha(Y)\beta(X_2)] \\ &= f^1(\alpha \wedge \beta)(X_1, Y) + f^2(\alpha \wedge \beta)(X_2, Y).\end{aligned}$$

The proof of linearity on the second slot is quite similar and is left to the reader.

2.15 Corollary If α and β are 1-forms, then

$$\alpha \wedge \beta = -\beta \wedge \alpha. \tag{2.27}$$

This last result tells us that wedge products have characteristics similar to cross products of vectors in the sense that both of these products anti-commute. This means that we need to be careful to introduce a minus sign every time we interchange the order of the operation. Thus, for example, we have

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

if $i \neq j$, whereas

$$dx^i \wedge dx^i = -dx^i \wedge dx^i = 0$$

since any quantity that equals the negative of itself must vanish. The similarity between wedge products is even more striking in the next proposition but we emphasize again that wedge products are much more powerful than cross products, because wedge products can be computed in any dimension.

2.16 Proposition Let $\alpha = A_i dx^i$ and $\beta = B_i dx^i$ be any two 1-forms in $\mathbf{R}^n$. Then

$$\alpha \wedge \beta = (A_iB_j)dx^i \wedge dx^j. \tag{2.28}$$

Proof: Let X and Y be arbitrary vector fields. Then

$$\begin{aligned}(\alpha \wedge \beta)((X,Y) &= (A_idx^i)(X)(B_jdx^j)(Y) - (A_idx^i)(Y)(B_jdx^j)(X) \\ &= (A_iB_j)[dx^i(X)dx^j(Y) - dx^i(Y)dx^j(X)] \\ &= (A_iB_j)(dx^i \wedge dx^j)(X,Y).\end{aligned}$$

Because of the antisymmetry of the wedge product, the last of the above equations can be written as

$$\alpha \wedge \beta = \sum_{i=1}^{n} \sum_{j<i}^{n} (A_i B_j - A_j B_i)(dx^i \wedge dx^j).$$

In particular, if $n = 3$, then the coefficients of the wedge product are the components of the cross product of $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$.

2.17 Example Let $\alpha = x^2dx - y^2dy$ and $\beta = dx + dy - 2xydz$. Then

$$\begin{aligned}
\alpha \wedge \beta &= (x^2dx - y^2dy) \wedge (dx + dy - 2xydz) \\
&= x^2dx \wedge dx + x^2dx \wedge dy - 2x^3ydx \wedge dz - y^2dy \wedge dx - y^2dy \wedge dy + 2xy^3dy \wedge dz \\
&= x^2dx \wedge dy - 2x^3ydx \wedge dz - y^2dy \wedge dx + 2xy^3dy \wedge dz \\
&= (x^2 + y^2)dx \wedge dy - 2x^3ydx \wedge dx + 2xy^3dy \wedge dz.
\end{aligned}$$

2.18 Example Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\begin{aligned}
dx \wedge dy &= (-r\sin\theta d\theta + \cos\theta dr) \wedge (r\cos\theta d\theta + \sin\theta dr) \\
&= -r\sin^2\theta d\theta \wedge dr + r\cos^2\theta dr \wedge d\theta \\
&= (r\cos^2\theta + r\sin^2\theta)(dr \wedge d\theta) \\
&= r(dr \wedge d\theta). \qquad (2.29)
\end{aligned}$$

2.19 Remark

1. The result of the last example yields the familiar differential of area in polar coordinates.
2. The differential of area in polar coordinates is a special example of the change of coordinate theorem for multiple integrals. It is easy to establish that if $x = f^1(u,v)$ and $y = f^2(u,v)$, then $dx \wedge dy = det|J|du \wedge dv$, where $det|J|$ is the determinant of the Jacobian of the transformation.
3. Quantities such as $dxdy$ and $dydz$ which often appear in calculus, are not well defined. In most cases, these entities are actually wedge products of 1-forms.
4. We state without proof that all 2-forms ϕ in $\mathbf{R}^n$ can be expressed as linear combinations of wedge products of differentials such as

$$\phi = F_{ij}dx^i \wedge dx^j. \qquad (2.30)$$

In a more elementary (ie, sloppier) treatment of this subject one could simply define 2-forms to be gadgets which look like the quantity in equation (2.30). With this motivation, we introduce the next definition.

2.20 Definition A 3-form ϕ in $\mathbf{R}^n$ is an object of the following type

$$\phi = A_{ijk}dx^i \wedge dx^j \wedge dx^k \qquad (2.31)$$

where we assume that the wedge product of three 1-forms is associative but alternating in the sense that if one switches any two differentials, then the entire expression changes by a minus sign. We challenge the reader to come up with a rigorous definition of three forms (or an n-form, for that matter) in the spirit of multilinear maps. There is nothing really wrong with using definition (2.31).

This definition however, is coordinate-dependent and differential geometers prefer coordinate-free definitions, theorems and proofs.

Now for a little combinatorics. Let us count the number of linearly independent differential forms in Euclidean space. More specifically, we want to find the vector space dimension of the space of k-forms in $\mathbf{R}^n$. We will think of 0-forms as being ordinary functions. Since functions are the "scalars", the space of 0-forms as a vector space has dimension 1.

$\mathbf{R}^2$	Forms	Dim
0-forms	f	1
1-forms	$f dx^1, g dx^2$	2
2-forms	$f dx^1 \wedge dx^2$	1

$\mathbf{R}^3$	Forms	Dim
0-forms	f	1
1-forms	$f_1 dx^1, f_2 dx^2, f_3 dx^3$	3
2-forms	$f_1 dx^2 \wedge dx^3,\ f_2 dx^3 \wedge dx^1,\ f_3 dx^1 \wedge dx^2$	3
3-forms	$f_1 dx^1 \wedge dx^2 \wedge dx^3$	1

The binomial coefficient pattern should be evident to the reader.

2.3 Exterior Derivatives

In this section we introduce a differential operator that generalizes the classical gradient, curl and divergence operators.

Denote by $\bigwedge^m_{(p)}(\mathbf{R}^n)$ the space of m-forms at $\mathbf{p} \in \mathbf{R}^n$. This vector space has dimension

$$\dim \bigwedge^m_{(p)}(\mathbf{R}^n) = \frac{n!}{m!(n-m)!}$$

for $m \leq n$ and dimension 0 for $m > n$. We will identify $\bigwedge^0_{(p)}(\mathbf{R}^n)$ with the space of $\mathcal{C}^\infty$ functions at $\mathbf{p}$. The union of all $\bigwedge^m_{(p)}(\mathbf{R}^n)$ as $\mathbf{p}$ ranges through all points in $\mathbf{R}^n$ is called the **bundle of m-forms** and will be denoted by $\bigwedge^m(\mathbf{R}^n)$.

A section α of the bundle

$$\bigwedge^m(\mathbf{R}^n) = \bigcup_p \bigwedge^m_p(\mathbf{R}^n).$$

is called an **m-form** and it can be written as:

$$\alpha = A_{i_1,\ldots i_m}(x) dx^{i_1} \wedge \ldots dx^{i_m}. \tag{2.32}$$

.

2.21 Definition Let α be an m-form (given in coordinates as in equation (2.32)). The **exterior derivative** of α is the (m+1-form) $d\alpha$ given by

$$\begin{aligned} d\alpha &= dA_{i_1,\ldots i_m} \wedge dx^{i_0} \wedge dx^{i_1} \ldots dx^{i_m} \\ &= \frac{\partial A_{i_1,\ldots i_m}}{\partial dx^{i_0}}(x) dx^{i_0} \wedge dx^{i_1} \ldots dx^{i_m}. \end{aligned} \tag{2.33}$$

In the special case where α is a 0-form, that is, a function, we write

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

2.22 Proposition

$$\begin{array}{ll} \text{a)} & d : \bigwedge^m \longrightarrow \bigwedge^{m+1} \\ \text{b)} & d^2 = d \circ d = 0 \\ \text{c)} & d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \qquad \forall \alpha \in \bigwedge^p, \beta \in \bigwedge^q \end{array} \tag{2.34}$$

Proof:
a) Obvious from equation (2.33).
b) First, we prove the proposition for $\alpha = f \in \bigwedge^0$. We have

$$\begin{aligned} d(d\alpha) &= d(\frac{\partial f}{\partial dx^i}) \\ &= \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \\ &= \frac{1}{2}[\frac{\partial^2 f}{\partial x^j \partial x^i} \frac{\partial^2 f}{\partial x^i \partial x^j}] dx^j \wedge dx^i \\ &= 0. \end{aligned}$$

Now, suppose that α is represented locally as in equation (2.32). It follows from equation 2.33 that

$$d(d\alpha) = d(dA_{i_1,\ldots i_m}) \wedge dx^{i_0} \wedge dx^{i_1} \ldots dx^{i_m} = 0.$$

c) Let $\alpha \in \bigwedge^p, \beta \in \bigwedge^q$. Then, we can write

$$\begin{aligned} \alpha &= A_{i_1,\ldots i_p}(x) dx^{i_1} \wedge \ldots dx^{i_p} \\ \beta &= B_{j_1,\ldots j_q}(x) dx^{j_1} \wedge \ldots dx^{j_q}. \end{aligned} \tag{2.35}$$

By definition,

$$\alpha \wedge \beta = A_{i_1 \ldots i_p} B_{j_1 \ldots j_q} (dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q}).$$

Now, we take take the exterior derivative of the last equation, taking into account that $d(fg) = fdg + gdf$ for any functions f and g. We get

$$\begin{aligned} d(\alpha \wedge \beta) &= [d(A_{i_1 \ldots i_p}) B_{j_1 \ldots j_q} + (A_{i_1 \ldots i_p} d(B_{j_1 \ldots j_q})](dx^{i_1} \wedge \ldots \wedge dx^{i_p}) \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q}) \\ &= [dA_{i_1 \ldots i_p} \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_p})] \wedge [B_{j_1 \ldots j_q} \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q})] + \\ &= [A_{i_1 \ldots i_p} \wedge (dx^{i_1} \wedge \ldots \wedge dx^{i_p})] \wedge (-1)^p [dB_{j_1 \ldots j_q} \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_q})] \\ &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta. \end{aligned} \tag{2.36}$$

The $(-1)^p$ factor comes into play since in order to pass the term $dB_{j_i \ldots j_p}$ through p 1-forms of the type dx^i, one must perform p transpositions.

2.23 Example Let $\alpha = P(x,y)dx + Q(x,y)d\beta$. Then,

$$\begin{aligned} d\alpha &= (\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y}) \wedge dx + (\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y}) \wedge dy \\ &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\ &= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx \wedge dy. \end{aligned} \tag{2.37}$$

This example is related to Green's theorem in $\mathbf{R}^2$.

2.24 Example Let $\alpha = M(x,y)dx + N(x,y)dy$, and suppose that $d\alpha = 0$. Then, by the previous example,

$$d\alpha = (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dx \wedge dy.$$

Thus, $d\alpha = 0$ iff $N_x = M_y$, which implies that $N = f_y$ and M_x for some function $f(x,y)$. Hence,

$$\alpha = f_x dx + f_y df = df.$$

The reader should also be familiar with this example in the context of exact differential equations of first order and conservative force fields.

2.25 Definition A differential form α is called **exact** if $d\alpha = 0$.

2.26 Definition A differential form α is called **closed** if there exists a form β such that $\alpha = d\beta$. Since $d \circ d = 0$, it is clear that an exact form is also closed. For the converse to be true, one must require a topological condition that the space be contractible. A contractible space is one that can be deformed continuously to an interior point.

2.27 Poincare's Lemma In a contractible space (such as $\mathbf{R}^n$), if a differential is closed, then it is exact.

The standard counterexample showing that the topological condition in Poincare's Lemma is needed is the form $d\theta$ where θ is the polar coordinate angle in the plane. It is not hard to prove that

$$d\theta = \frac{-ydx + xdy}{x^2 + y^2}$$

is closed, but not exact in the punctured plane $\{\mathbf{R}^2 \setminus \{0\}\}$.

2.4 The Hodge-$*$ Operator

An important lessons students learn in linear algebra is that all vector spaces of finite dimension n are isomorphic to each other. Thus, for instance, the space P_3 of all real polynomials in x of degree 3, and the space $\mathcal{M}_{2\times 2}$ of real 2 by 2 matrices are, in terms of their vector space properties, basically no different from the Euclidean vector space $\mathbf{R}^4$. A good example of this is the tangent space $T_p\mathbf{R}^3$ which has dimension 3. The process of replacing $\frac{\partial}{\partial x}$ by $\mathbf{i}$, $\frac{\partial}{\partial y}$ by $\mathbf{j}$ and $\frac{\partial}{\partial z}$ by $\mathbf{k}$ is a linear, 1-1 and onto map that sends the "vector" part of a tangent vector $a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial z}$ to regular Eucildean vector $\langle a^1, a^2, a^3\rangle$.

We have also observed that the tangent space $T_p\mathbf{R}^n$ is isomorphic to the cotangent space $T_p^\star\mathbf{R}^n$. In this case, the vector space isomorphism maps the standard basis vectors $\{\frac{\partial}{\partial x^i}\}$ to their duals $\{dx^i\}$. This isomorphism then transforms a contravariant vector to a covariant vector. In terms if components, the isomorphism is provided by the Euclidean metric that maps a the components of a contravariant vector with indices up to a covariant vector with indices down.

Another interesting example is provided by the spaces $\bigwedge_p^1(\mathbf{R}^3)$ and $\bigwedge_p^2(\mathbf{R}^3)$, both of which have dimension 3. It follows that these two spaces must be isomorphic. In this case the isomorphism is given as follows:

$$\begin{array}{ccc} dx & \longmapsto & dy \wedge dz \\ dy & \longmapsto & -dx \wedge dz \\ dz & \longmapsto & dx \wedge dy \end{array} \tag{2.38}$$

More generally, we have seen that the dimension of the space of m-forms in $\mathbf{R}^n$ is given by the binomial coefficient $(\overset{n}{m})$. Since

$$\binom{n}{m} = \binom{n}{n-m} = \frac{n!}{(n-m)!},$$

it must be true that

$$\textstyle\bigwedge_p^m(\mathbf{R}^n) \cong \bigwedge_p^m(\mathbf{R}^{n-m}). \tag{2.39}$$

To describe the isomorphism between these two spaces, we will first need to introduce the totally antisymmetric **Levi-Civita** permutation symbol defined as follows:

$$\epsilon_{i_1\ldots i_m} = \begin{cases} +1 & \text{if}(i_1,\ldots,i_m) \text{ is an even permutation of}(1,\ldots,m) \\ -1 & \text{if}(i_1,\ldots,i_m) \text{ is an odd permutation of}(1,\ldots,m) \\ 0 & \text{otherwise} \end{cases} \tag{2.40}$$

In dimension 3, there are only 3 (3!=6) nonvanishing components of $\epsilon_{i,j,k}$ in

$$\begin{aligned} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \end{aligned} \tag{2.41}$$

The permutation symbols are useful in the theory of determinants. In fact, if $A = (a^i_j)$ is a 3×3 matrix, then, using equation (2.41), the reader can easily verify that

$$\det A = |A| = \epsilon_{i_1 i_2 i_3} a_1^{i_1} a_2^{i_2} a_3^{i_3}. \tag{2.42}$$

This formula for determinants extends in an obvious manner to $n \times n$ matrices. A more thorough discussion of the Levi-Civita symbol will appear later in these notes.

In $\mathbf{R}^n$, the Levi-Civita symbol with some or all the indices up is numerically equal to the permutation symbol will all the indices down,

$$\epsilon_{i_1\ldots i_m} = \epsilon^{i_1\ldots i_m},$$

since the Euclidean metric used to raise and lower indices is the Kronecker δ_{ij}.

On the other hand, in Minkowski space, raising an index with a value of 0 costs a minus sign, because $\eta_{00} = \eta^{00} = -1$. Thus, in $\mathcal{M}_{(\infty,\ni)}$

$$\epsilon_{i_0 i_1 i_2 i_3} = -\epsilon^{i_0 i_1 i_2 i_3},$$

since any permutation of $\{0, 1, 2, 3\}$ must contain a 0.

2.28 Definition The Hodge-$*$ operator is a linear map $* : \bigwedge_p^m(\mathbf{R}^n) \longrightarrow \bigwedge_p^m(\mathbf{R}^{n-m})$ defined in standard local coordinates by the equation,

$$*(dx^{i_1} \wedge \ldots \wedge dx^{i_m}) = \frac{1}{(n-m)!}\epsilon^{i_1\ldots i_m}{}_{i_{m+1}\ldots i_n} dx^{i_{m+1}} \wedge \ldots \wedge dx^{i_m}, \tag{2.43}$$

Since the forms $dx^{i_1} \wedge \ldots \wedge dx^{i_m}$ constitute a basis of the vector space $\bigwedge_p^m(\mathbf{R}^n)$ and the $*$-operator is assumed to be a linear map, equation (2.43) completely specifies the map for all m-forms.

2.29 Example Consider the dimension n=3 case. then

$$\begin{aligned} *dx^1 &= \epsilon^1{}_{jk} dx^j \wedge dx^k \\ &= \frac{1}{2!}[\epsilon^1{}_{23} dx^2 \wedge dx^3 + \epsilon^1{}_{32} dx^3 \wedge dx^2] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2!}[dx^2 \wedge dx^3 - dx^3 \wedge dx^2] \\
&= \frac{1}{2!}[dx^2 \wedge dx^3 + dx^2 \wedge dx^3] \\
&= dx^2 \wedge dx^3.
\end{aligned}
$$

We leave it to the reader to complete the computation of the action of the $*$-operator on the other basis forms. The results are

$$
\begin{aligned}
*dx^1 &= +dx^2 \wedge dx^3 \\
*dx^2 &= -dx^1 \wedge dx^3 \\
*dx^3 &= +dx^1 \wedge dx^2,
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
*(dx^2 \wedge dx^3) &= dx^1 \\
*(-dx^3 \wedge dx^1) &= dx^2 \\
*(dx^1 \wedge dx^2) &= dx^3,
\end{aligned}
\tag{2.45}
$$

and

$$
*(dx^1 \wedge dx^2 \wedge dx^3) = 1. \tag{2.46}
$$

In particular, if $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ is any 0-form (a function), then,

$$
\begin{aligned}
*f &= f(dx^1 \wedge dx^2 \wedge dx^3) \\
&= f dV,
\end{aligned}
\tag{2.47}
$$

where dV is the differential of volume, also called the **volume form**.

2.30 Example Let $\alpha = A_1 dx^1 A_2 dx^2 + A_3 dx^3$, and $\beta = B_1 dx^1 B_2 dx^2 + B_3 dx^3$. Then,

$$
\begin{aligned}
*(\alpha \wedge \beta) &= (A_2B_3 - A_3B_2) * (dx^2 \wedge dx^3) + (A_1B_3 - A_3B_1) * (dx^1 \wedge dx^3) + \\
&\quad (A_1B_2 - A_2B_1) * (dx^1 \wedge dx^2) \\
&= (A_2B_3 - A_3B_2)dx^1 + (A_1B_3 - A_3B_1)dx^2 + (A_1B_2 - A_2B_1)dx^3 \\
&= (\vec{\mathbf{A}} \times \vec{\mathbf{B}})_i dx^i
\end{aligned}
\tag{2.48}
$$

The previous examples provide some insight on the action of the $\wedge$ and $*$ operators. If one thinks of the quantities dx^1, dx^2 and dx^3 as analogous to $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$, then it should be apparent that equations (2.44) are the differential geometry versions of the well known relations

$$
\begin{aligned}
\mathbf{i} &= \mathbf{j} \times \mathbf{k} \\
\mathbf{j} &= -\mathbf{i} \times \mathbf{k} \\
\mathbf{k} &= \mathbf{i} \times \mathbf{j}.
\end{aligned}
$$

This is even more evident upon inspection of equation (2.48), which relates the $\wedge$ operator to the Cartesian cross product.

2.31 Example In Minkowski space the collection of all 2-forms has dimension $\binom{4}{2} = 6$. The Hodge-$*$ operator in this case splits $\bigwedge^2(\mathcal{M}_{1,3})$ into two 3-dim subspaces $\bigwedge^2_{\pm}$, such that $* : \bigwedge^2_{\pm} \longrightarrow \bigwedge^2_{\mp}$. More specifically, $\bigwedge^2_{+}$ is spanned by the forms $\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3\}$, and $\bigwedge^2_{-}$ is spanned by the forms $\{dx^2 \wedge dx^3, -dx^1 \wedge dx^3, dx^1 \wedge dx^2\}$. The action of $*$ on $\bigwedge^2_{+}$ is

$$
\begin{aligned}
*(dx^0 \wedge dx^1) &= \tfrac{1}{2}\epsilon^{01}{}_{kl} dx^k \wedge dx^l = -dx^2 \wedge dx^3 \\
*(dx^0 \wedge dx^2) &= \tfrac{1}{2}\epsilon^{02}{}_{kl} dx^k \wedge dx^l = +dx^1 \wedge dx^3 \\
*(dx^0 \wedge dx^3) &= \tfrac{1}{2}\epsilon^{03}{}_{kl} dx^k \wedge dx^l = -dx^1 \wedge dx^2,
\end{aligned}
$$

and on $\bigwedge^2_-$,

$$\begin{aligned}
*(+dx^2 \wedge dx^3) = & \tfrac{1}{2}\epsilon^{23}{}_{kl}dx^k \wedge dx^l = & dx^0 \wedge dx^1 \\
*(-dx^1 \wedge dx^3) = & \tfrac{1}{2}\epsilon^{13}{}_{kl}dx^k \wedge dx^l = & dx^0 \wedge dx^2 \\
*(+dx^1 \wedge dx^2) = & \tfrac{1}{2}\epsilon^{12}{}_{kl}dx^k \wedge dx^l = & dx^0 \wedge dx^3.
\end{aligned}$$

In verifying the equations above, we recall that the Levi-Civita symbols that contain an index with value 0 in the up position have an extra minus sign as a result of raising the index with η^{00}. If $F \in \bigwedge^2(\mathcal{M})$, we will formally write $F = F_+ + F_-$, where $F_\pm \in \bigwedge^2_\pm$. We would like to note that the action of the dual operator on $\bigwedge^2(\mathcal{M})$ is such that $* \bigwedge^2(\mathcal{M}) \longrightarrow \bigwedge^2(\mathcal{M})$, and $*^2 = -1$. In a vector space a map like $*$ with the property $*^2 = -1$ is a **linear involution** of the space. In the case in question $\bigwedge^2_\pm$ are the eigenspaces corresponding to the +1 and -1 eigenvalues of this involution.

It is also worthwhile to calculate the duals of 1-forms in $\mathcal{M}_{1,3}$. The results are

$$\begin{aligned}
*dt &= -dx^1 \wedge dx^2 \wedge dx^3 \\
*dx^1 &= +dx^2 \wedge dt \wedge dx^3 \\
*dx^2 &= +dt \wedge dx^1 \wedge dx^3 \\
*dx^3 &= +dx^1 \wedge dt \wedge dx^2.
\end{aligned} \tag{2.49}$$

Gradient, Curl and Divergence

Classical differential operators that enter in Green's and Stokes' Theorems are better understood as special manifestations of the exterior differential and the Hodge-$*$ operators in $\mathbf{R}^3$. Here is precisely how this works:

1. Let $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ be a $\mathcal{C}^\infty$ function. Then

$$df = \frac{\partial f}{\partial x^j} dx^j = \nabla f \cdot \mathbf{dx} \tag{2.50}$$

2. Let $\alpha = A_i dx^i$ be a 1-form in $\mathbf{R}^3$. Then

$$\begin{aligned}
(*d)\alpha &= \frac{1}{2}(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_i}{\partial x^j}) * (dx^i \wedge dx^j) \\
&= (\nabla \times \mathbf{A}) \cdot \mathbf{dx}
\end{aligned} \tag{2.51}$$

3. Let $\alpha = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2$ be a 2-form in $\mathbf{R}^3$. Then

$$\begin{aligned}
d\alpha &= (\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3}) dx^1 \wedge dx^2 \wedge dx^3 \\
&= (\nabla \cdot \mathbf{B}) dV
\end{aligned} \tag{2.52}$$

4. Let $\alpha = B_i dx^i$, then

$$*d * \alpha = \nabla \cdot \mathbf{B} \tag{2.53}$$

It is also possible to define and manipulate formulas of classical vector calculus using the permutation symbols. For example, let $\mathbf{a} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ be any two Euclidean vectors. Then it is easy to see that

$$(\mathbf{A} \times \mathbf{B})_k = \epsilon^{ij}{}_k A_i B_j,$$

and

$$(\nabla \times \mathbf{B})_k = \epsilon^{ij}{}_k \frac{\partial A_i}{\partial x^j},$$

To derive many classical vector identities in this formalism, it is necessary to first establish the following identity:

$$\epsilon^{ijm}\epsilon_{klm} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k. \tag{2.54}$$

2.32 Example

$$\begin{aligned}
[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_l &= \epsilon_l^{mn} A\ {}_m(\mathbf{B} \times \mathbf{C})_m \\
&= \epsilon_l^{mn} A_m(\epsilon^{jk}{}_n B_j C_k) \\
&= \epsilon^{mn}{}_l \epsilon^{jk}{}_n A_m B_j C_k) \\
&= \epsilon_{mnl}\epsilon^{jkn} A^m B_j C_k) \\
&= (\delta^j_l \delta^k_m - \delta^k_l \delta^j_n) A^m B_j C_k \\
&= B_l A^m C_m - C_l A^m b_n,
\end{aligned}$$

Or, rewriting in vector form

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}). \tag{2.55}$$

Maxwell Equations

The classical equations of Maxwell describing electromagnetic phenomena are

$$\begin{array}{ll}
\nabla \cdot \mathbf{E} = 4\pi\rho & \nabla \times \mathbf{B} = 4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{B} = 0 & \nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}
\end{array} \tag{2.56}$$

We would like to formulate these equations in the language of differential forms. Let $x^\mu = (t, x^1, x^2, x^3)$ be local coordinates in Minkowski's space $\mathcal{M}_{1,3}$. Define the Maxwell 2-form F by the equation

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3), \tag{2.57}$$

where

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_y \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}. \tag{2.58}$$

Written in complete detail, Maxwell's 2-form is given by

$$\begin{aligned}
F &= -E_x dt \wedge dx^1 - E_y dt \wedge dx^2 - E_z dt \wedge dx^3 + \\
&\quad B_z dx^1 \wedge dx^2 - B_y dx^1 \wedge dx^3 + B_x dx^2 \wedge dx^3.
\end{aligned} \tag{2.59}$$

We also define the source current 1-form

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3. \tag{2.60}$$

2.33 Proposition Maxwell's Equations (2.56) are equivalent to the equations

$$\begin{aligned}
dF &= 0, \\
d * F &= 4\pi * J.
\end{aligned} \tag{2.61}$$

Proof: The proof is by direct computation using the definitions of the exterior derivative and the Hodge-$*$ operator.

$$\begin{aligned}
dF &= -\frac{\partial E_x}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^1 - \frac{\partial E_x}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^1 + \\
& -\frac{\partial E_y}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^2 - \frac{\partial E_y}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^2 + \\
& -\frac{\partial E_z}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^3 - \frac{\partial E_z}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^3 + \\
& \frac{\partial B_z}{\partial t} \wedge dt \wedge dx^1 \wedge dx^2 - \frac{\partial B_z}{\partial x^3} \wedge dx^3 \wedge dx^1 \wedge dx^2 - \\
& \frac{\partial B_y}{\partial t} \wedge dt \wedge dx^1 \wedge dx^3 - \frac{\partial B_y}{\partial x^2} \wedge dx^2 \wedge dx^1 \wedge dx^3 + \\
& \frac{\partial B_x}{\partial t} \wedge dt \wedge dx^2 \wedge dx^3 + \frac{\partial B_x}{\partial x^1} \wedge dx^1 \wedge dx^2 \wedge dx^3.
\end{aligned}$$

Collecting terms and using the antisymmetry of the wedge operator, we get

$$\begin{aligned}
dF &= (\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3}) dx^1 \wedge dx^2 \wedge dx^3 + \\
& (\frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial t}) dx^2 \wedge dt \wedge dx^3 + \\
& (\frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial dx^3} - \frac{\partial B_y}{\partial t}) dt \wedge dx^1 \wedge x^3 + \\
& (\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial t}) dx^1 \wedge dt \wedge dx^2.
\end{aligned}$$

Therefore, $dF = 0$ iff

$$\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_y}{\partial x^3} = 0,$$

which is the same as

$$\nabla \cdot \mathbf{B} = 0,$$

and

$$\begin{aligned}
\frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial x^1} &= 0, \\
\frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial x^3} - \frac{\partial B_y}{\partial x^2} &= 0, \\
\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial x^3} &= 0,
\end{aligned}$$

which means that

$$-\nabla \times \mathbf{E} - \frac{\partial B}{\partial t} = 0. \tag{2.62}$$

To verify the second set of Maxwell equations, we first compute the dual of the current density 1-form (2.60) using the results from example 2.4. We get

$$*J = -\rho dx^1 \wedge dx^2 \wedge dx^3 + J_1 dx^2 \wedge dt \wedge dx^3 + J_2 dt \wedge dx^1 \wedge dx^3 + J_3 dx^1 \wedge dt \wedge dx^2. \tag{2.63}$$

We could now proceed to compute $d*F$, but perhaps it is more elegant to notice that $F \in \bigwedge^2(\mathcal{M})$, and so, according to example (2.4), F splits into $F = F_+ + F_-$. In fact, we see from (2.58) that the components of F_+ are those of $-\mathbf{E}$ and the components of F_- constitute the magnetic field vector $\mathbf{B}$. Using the results of example (2.4), we can immediately write the components of $*F$:

$$\begin{aligned} *F &= B_x dt \wedge dx^1 + B_y dt \wedge dx^2 + B_z dt \wedge dx^3 + \\ &\quad E_z dx^1 \wedge dx^2 - E_y dx^1 \wedge dx^3 + E_x dx^2 \wedge dx^3, \end{aligned} \tag{2.64}$$

or equivalently,

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_y \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}. \tag{2.65}$$

Since the effect of the dual operator amounts to exchanging

$$\begin{aligned} \mathbf{E} &\longmapsto -\mathbf{B} \\ \mathbf{B} &\longmapsto +\mathbf{E}, \end{aligned}$$

we can infer from equations (2.62) and (2.63) that

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

and

$$\nabla \times \mathbf{B} - \frac{\partial E}{\partial t} = 4\pi\mathbf{J}.$$

Chapter 3

Connections

Connections

3.1 Frames

As we have already noted in Chapter 1, the theory of curves in $\mathbf{R}^3$ can be elegantly formulated by introducing orthonormal triplets of vectors which we called Frenet frames. The Frenet vectors are adapted to the curves in such a manner that the rate of change of the frame gives information about the curvature of the curve. In this chapter we will study the properties of arbitrary frames and their corresponding rates of change in the direction of the various vectors in the frame. These concepts will then be applied later to special frames adapted to surfaces.

3.1 Definition A coordinate **frame** in $\mathbf{R}^n$ is an n-tuple of vector fields $\{e_1, \ldots, e_n\}$ which are linearly independent at each point $\mathbf{p}$ in the space.

In local coordinates $x^1, \ldots, x^n$, we can always express the frame vectors as linear combinations of the standard basis vectors

$$e_i = \partial_j A^j{}_i, \tag{3.1}$$

where $\partial_j = \frac{\partial}{\partial x^1}$. We assume the matrix $A = (A^j{}_i)$ to be nonsingular at each point. In linear algebra, this concept is referred to as a change of basis, the difference being that in our case, the transformation matrix A depends on the position. A frame field is called **orthonormal** if at each point,

$$< e_i, e_j >= \delta_{ij}. \tag{3.2}$$

Throughout this chapter, we will assume that all frame fields are orthonormal. Whereas this restriction is not necessary, it is convenient because it it results in considerable simplification in computing the components of an arbitrary vector in the frame.

3.2 Proposition If $\{e_1, \ldots, e_n\}$ is an orthonormal frame, then the transformation matrix is orthogonal (ie, $AA^T = I$)

Proof: The proof is by direct computation. Let $e_i = \partial_j A^j{}_i$. Then

$$\begin{aligned}
\delta_{ij} &= < e_i, e_j > \\
&= < \partial_k A^k{}_i, \partial_l A^l{}_j > \\
&= A^k{}_i A^l{}_j < \partial_k, \partial_l > \\
&= A^k{}_i A^l{}_j \delta_{kl}
\end{aligned}$$

$$\begin{aligned} &= A^k{}_i A_{kj} \\ &= A^k{}_i (A^T)_{jk}. \end{aligned}$$

Hence

$$\begin{aligned} (A^T)_{jk} A^k_i &= \delta_{ij} \\ (A^T)^j{}_k A^k_i &= \delta^j{}_i \\ A^T A &= I \end{aligned}$$

Given a frame vectors e_i, we can also introduce the corresponding dual coframe forms θ_i by requiring that

$$\theta^i(e_j) = \delta^i_j. \tag{3.3}$$

Since the dual coframe is a set of 1-forms, they can also be expressed in local coordinates as linear combinations

$$\theta^i = B^i{}_k dx^k.$$

It follows from equation(3.3), that

$$\begin{aligned} \theta^i(e_j) &= B^i_k dx^k(\partial_l A^l{}_j) \\ &= B^i_k A^l{}_j dx^k(\partial_l) \\ &= B^i_k A^l{}_j \delta^k_l \\ \delta^i{}_j &= B^i_k A^k_j. \end{aligned}$$

Therefore, we conclude that $BA = I$, so $B = A^{-1} = A^T$. In other words, when the frames are orthonormal, we have

$$\begin{aligned} e_i &= \partial_k A^k_i \\ \theta^i &= A^i{}_k dx^k. \end{aligned} \tag{3.4}$$

3.3 Example Consider the transformation from Cartesian to cylindrical coordinates:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned}$$

Using the chain rule for partial derivatives, we have

$$\begin{aligned} \frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial z} \end{aligned}$$

From these equations we easily verify that the quantities

$$\begin{aligned} e_1 &= \frac{\partial}{\partial r} \\ e_2 &= \frac{1}{r}\frac{\partial}{\partial \theta} \\ e_3 &= \frac{\partial}{\partial z}, \end{aligned}$$

are a triplet of mutually orthogonal unit vectors and thus constitute an orthonormal frame.

3.4 Example For spherical coordinates(2.20)

$$\begin{aligned} x &= \rho \sin\theta \cos\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\theta, \end{aligned}$$

the chain rule leads to

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \sin\theta\cos\phi \frac{\partial}{\partial x} + \sin\theta \sin\phi \frac{\partial}{\partial y} + \cos\theta \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= \rho\cos\theta\cos\phi \frac{\partial}{\partial x} + \rho\cos\theta \sin\phi \frac{\partial}{\partial y} + -\rho\sin\theta \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \phi} &= -\rho\sin\theta\sin\phi \frac{\partial}{\partial x} + \rho\sin\theta \cos\phi \frac{\partial}{\partial y}. \end{aligned}$$

In this case, the vectors

$$\begin{aligned} e_1 &= \frac{\partial}{\partial \rho} \\ e_2 &= \frac{1}{\rho}\frac{\partial}{\partial \theta} \\ e_3 &= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \phi} \end{aligned} \tag{3.5}$$

also constitute an orthonormal frame.

The fact that the chain rule in the two situations above leads to orthonormal frames is not coincidental. The results are related to the orthogonality of the level surfaces $x^i = constant$. Since the level surfaces are orthogonal whenever they intersect, one expects the gradients of the surfaces to also be orthogonal. Transformations of this type are called triply orthogonal systems.

3.2 Curvilinear Coordinates

Orthogonal transformations such as spherical and cylindrical coordinates appear ubiquitously in mathematical physics because the geometry of a large number of problems in this area exhibit symmetry with respect to an axis or to the origin. In such situations, transformations to the appropriate coordinate system often result in considerable simplification of the field equations involved in the problem. It has been shown that the Laplace operator that appears in the the potential, heat, wave equation, and Schrdinger field equations is separable in twelve orthogonal coordinate systems. A simple and efficient method to calculate the Laplacian in orthogonal coordinates can be implemented using differential forms.

3.5 Example In spherical coordinates the differential of arc length is given by (see equation 2.21) the metric:

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2.$$

Let

$$\begin{aligned} \theta^1 &= d\rho \\ \theta^2 &= \rho d\theta \\ \theta^3 &= \rho\sin\theta d\phi. \end{aligned} \tag{3.6}$$

Note that these three 1-forms constitute the dual coframe to the orthonormal frame derived in equation(3.5). Consider a scalar field $f = f(\rho, \theta, \phi)$. We now calculate the Laplacian of f in spherical coordinates using the methods of section 2.4. To do this, we first compute the differential df and express the result in terms of the coframe.

$$\begin{aligned} df &= \frac{\partial f}{\partial \rho} d\rho + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi \\ &= \frac{\partial f}{\partial \rho}\theta^1 + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\theta^2 + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\theta^3 \end{aligned}$$

The components df in the coframe represent the gradient in spherical coordinates. Continuing with the scheme of section 2.4, we next apply the Hodge-$*$ operator. Then, we rewrite the resulting 2-form in terms of wedge products of coordinate differentials so that we can apply the definition of the exterior derivative.

$$\begin{aligned} *df &= \frac{\partial f}{\partial \rho}\theta^2 \wedge \theta^3 - \frac{1}{\rho}\frac{\partial f}{\partial \theta}\theta^1 \wedge \theta^3 + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\theta^1 \wedge \theta^2 \\ &= \rho^2 \sin\theta \frac{\partial f}{\partial \rho} d\theta \wedge d\phi - \rho \sin\theta \frac{1}{\rho}\frac{\partial f}{\partial \theta} d\rho \wedge d\phi + \rho \sin\theta \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi} d\rho \wedge d\theta \\ &= \rho^2 \sin\theta \frac{\partial f}{\partial \rho} d\theta \wedge d\phi - \sin\theta \, \frac{\partial f}{\partial \theta} d\rho \wedge d\phi + \frac{\partial f}{\partial \phi} d\rho \wedge d\theta \\ d * df &= \frac{\partial}{\partial \rho}(\rho^2 \sin\theta \frac{\partial f}{\partial \rho}) d\rho \wedge d\theta \wedge d\phi - \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) d\theta \wedge d\rho \wedge d\phi + \frac{\partial}{\partial \phi}(\frac{\partial f}{\partial \phi}) d\phi \wedge d\rho \wedge d\theta \\ &= \left[\sin\theta \frac{\partial}{\partial \rho}(\rho^2 \frac{\partial f}{\partial \rho}) + \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial^2 f}{\partial \phi^2}\right] d\rho \wedge d\theta \wedge d\phi. \end{aligned}$$

Finally, rewriting the differentials back in terms of the the coframe, we get

$$d * df = \frac{1}{\rho^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial \rho}(\rho^2 \frac{\partial f}{\partial \rho}) + \frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial^2 f}{\partial \phi^2}\right] \theta^1 \wedge \theta^2 \wedge \theta^3.$$

So, the Laplacian of f is given by

$$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left[\rho^2 \frac{\partial f}{\partial \rho}\right] + \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial^2 f}{\partial \phi^2}\right] \tag{3.7}$$

The derivation of the expression for the spherical Laplacian through the use of differential forms is elegant and leads naturally to the operator in Sturm Liouville form.

The process above can be carried out for general orthogonal transformations. A change of coordinates $x^i = x^i(u^k)$ leads to an orthogonal transformation if in the new coordinate system u^k, the line metric

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2 \tag{3.8}$$

only has diagonal entries. In this case, we choose the coframe

$$\begin{aligned} \theta^1 &= \sqrt{g_{11}}du^1 = h_1 du^1 \\ \theta^2 &= \sqrt{g_{22}}du^2 = h_2 du^2 \\ \theta^3 &= \sqrt{g_{33}}du^3 = h_3 du^3 \end{aligned}$$

The quantities $\{h_1, h_2, h_3\}$ are classically called the weights. Please note that, in the interest of connecting to classical terminology, we have exchanged two indices for one and this will cause small discrepancies with the index summation convention. We will revert to using a summation symbol when these discrepancies occur. To satisfy the duality condition $\theta^i(e_j) = \delta^i_j$, we must choose the corresponding frame vectors e_i as follows:

$$e_1 = \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial u^1} = \frac{1}{h_1}\frac{\partial}{\partial u^1}$$
$$e_2 = \frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial u^2} = \frac{1}{h_2}\frac{\partial}{\partial u^2}$$
$$e_3 = \frac{1}{\sqrt{g_{33}}}\frac{\partial}{\partial u^3} = \frac{1}{h_3}\frac{\partial}{\partial u^3}$$

Gradient. Let $f = f(x^i)$ and $x^i = x^i(u^k)$. Then

$$\begin{aligned} df &= \frac{\partial f}{\partial x^k} dx^k \\ &= \frac{\partial f}{\partial u^i}\frac{\partial u^i}{\partial x^k} dx^k \\ &= \frac{\partial f}{\partial u^i} du^i \\ &= \sum_i \frac{1}{h^i}\frac{\partial f}{\partial u^i}\theta^i \\ &= e_i(f)\theta^i. \end{aligned}$$

As expected, the components of the gradient in the coframe θ^i are the just the frame vectors.

$$\nabla = \left(\frac{1}{h_1}\frac{\partial}{\partial u^1}, \frac{1}{h_2}\frac{\partial}{\partial u^2}, \frac{1}{h_3}\frac{\partial}{\partial u^3}\right) \tag{3.9}$$

Curl. Let $F = (F_1, F_2, F_3)$ be a classical vector field. Construct the corresponding 1-form $F = F_i\theta^i$ in the coframe. We calculate the curl using the dual of the exterior derivative.

$$\begin{aligned} F &= F_1\theta^1 + F_2\theta^2 + F_3\theta^3 \\ &= (h_1F_1)du^1 + (h_2F_2)du^2 + (h_3F_3)du^3 \\ &= (hF)_i du^i, \text{ where } (hF)_i = h_iF_i \\ dF &= \frac{1}{2}\left[\frac{\partial(hF)_i}{\partial u^j} - \frac{\partial(hF)_j}{\partial u^i}\right] du^i \wedge du^j \\ &= \frac{1}{h_ih_j}\left[\frac{\partial(hF)_i}{\partial u^j} - \frac{\partial(hF)_j}{\partial u^i}\right] d\theta^i \wedge d\theta^j \\ *dF &= \epsilon^{ij}_{\ \ k}\left[\frac{1}{h_ih_j}[\frac{\partial(hF)_i}{\partial u^j} - \frac{\partial(hF)_j}{\partial u^i}]\right]\theta^k = (\nabla \times F)_k\theta^k. \end{aligned}$$

Thus, the components of the curl are

$$\left(\frac{1}{h_2h_3}[\frac{\partial(h_3F_3)}{\partial u^2} - \frac{\partial(h_2F_2)}{\partial u^3}], \frac{1}{h_1h_3}[\frac{\partial(h_3F_3)}{\partial u^1} - \frac{\partial(h_1F_1)}{\partial u^3}], \frac{1}{h_1h_2}[\frac{\partial(h_1F_1)}{\partial u^2} - \frac{\partial(h_2F_2)}{\partial u^1}].\right) \tag{3.10}$$

Divergence. As before, let $F = F_i\theta^i$ and recall that $\nabla \cdot F = *d*F$. The computation yields

$$\begin{aligned} F &= F_1\theta^1 + F_2\theta^2 + F_3\theta^3 \\ *F &= F_1\theta^2 \wedge \theta^3 + F_2\theta^3 \wedge \theta^1 + F_3\theta^1 \wedge \theta^2 \\ &= (h_2h_3F_1)du^2 \wedge du^3 + (h_1h_3F_2)du^3 \wedge du^1 + (h_1h_2F_3)du^1 \wedge du^2 \\ d*dF &= \left[\frac{\partial(h_2h_3F_1)}{\partial u^1} + \frac{\partial(h_1h_3F_2)}{\partial u^2} + \frac{\partial(h_1h_2F_3)}{\partial u^3}\right] du^1 \wedge du^2 \wedge du^3. \end{aligned}$$

Therefore,

$$\nabla \cdot F = *d*F = \frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial u^1} + \frac{\partial(h_1h_3F_2)}{\partial u^2} + \frac{\partial(h_1h_2F_3)}{\partial u^3}\right]. \tag{3.11}$$

3.3 Covariant Derivative

In this section we introduce a generalization of directional derivatives. The directional derivative measures the rate of change of a function in the direction of a vector. What we want is a quantity which measures the rate of change of a vector field in the direction of another.

3.6 Definition Let X be an arbitrary vector field in $\mathbf{R}^n$. A map $\overline{\nabla}_X : T(\mathbf{R}^n) \longrightarrow T(\mathbf{R}^n)$ is called a **Koszul connection** if it satisfies the following properties:

1. $\overline{\nabla}_{fX}(Y) = f\overline{\nabla}_X Y$,
2. $\overline{\nabla}_{(X_1+X_2)}Y = \overline{\nabla}_{X_1}Y + \overline{\nabla}_{X_2}Y$,
3. $\overline{\nabla}_X(Y_1+Y_2) = \overline{\nabla}_X Y_1 + \overline{\nabla}_X Y_2$,
4. $\overline{\nabla}_X fY = X(f)Y + f\overline{\nabla}_X Y$,

for all vector fields $X, X_1, X_2, Y, Y_1, Y_2 \in T(\mathbf{R}^n)$ and all smooth functions f. The definition states that the map $\overline{\nabla}_X$ is linear on X but behaves like a linear derivation on Y. For this reason, the quantity $\overline{\nabla}_X Y$ is called the **covariant derivative** of Y in the direction of X.

3.7 Proposition Let $Y = f^i \frac{\partial}{\partial x^i}$ be a vector field in $\mathbf{R}^n$, and let X another C^∞ vector field. Then the operator given by

$$\overline{\nabla}_X Y = X(f^i)\frac{\partial}{\partial x^i} \tag{3.12}$$

defines a Koszul connection. The proof just requires verification that the four properties above are satisfied, and it is left as an exercise. The operator defined in this proposition is called the **standard connection** compatible with the standard Euclidean metric. The action of this connection on a vector field Y yields a new vector field whose components are the directional derivatives of the components of Y.

3.8 Example Let

$$X = x\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y}, \; Y = x^2\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y}.$$

Then,

$$\begin{aligned}
\overline{\nabla}_X Y &= X(x^2)\frac{\partial}{\partial x} + X(xy^2)\frac{\partial}{\partial y} \\
&= [x\frac{\partial}{\partial x}(x^2) + xz\frac{\partial}{\partial y}(x^2)]\frac{\partial}{\partial x} + [x\frac{\partial}{\partial x}(xy^2) + xz\frac{\partial}{\partial y}(xy^2)]\frac{\partial}{\partial y} \\
&= 2x^2\frac{\partial}{\partial x} + (xy^2 + 2x^2yz)\frac{\partial}{\partial y}.
\end{aligned}$$

3.9 Definition A Koszul connection $\overline{\nabla}_X$ is compatible with the metric $g(Y,Z) = <Y,Z>$ if

$$\overline{\nabla}_X <Y,Z> = <\overline{\nabla}_X Y, Z> + <Y, \overline{\nabla}_X Z> . \tag{3.13}$$

In Euclidean space, the components of the standard frame vectors are constant, and thus their rates of change in any direction vanish. Let e_i be arbitrary frame field with dual forms θ^i. The covariant derivatives of the frame vectors in the directions of a vector X will in general yield new vectors. The new vectors must be linear combinations of the the basis vectors as follows:

$$\begin{aligned} \overline{\nabla}_X e_1 &= \omega^1{}_1(X)e_1 + \omega^2{}_1(X)e_2 + \omega^3{}_1(X)e_3 \\ \overline{\nabla}_X e_2 &= \omega^1{}_2(X)e_1 + \omega^2{}_2(X)e_2 + \omega^3{}_2(X)e_3 \\ \overline{\nabla}_X e_3 &= \omega^1{}_3(X)e_1 + \omega^2{}_3(X)e_2 + \omega^3{}_3(X)e_3 \end{aligned} \tag{3.14}$$

The coefficients can be more succinctly expressed using the compact index notation,

$$\overline{\nabla}_X e_i = e_j \omega^j{}_i(X). \tag{3.15}$$

It follows immediately that

$$\omega^j{}_i(X) = \theta^j(\overline{\nabla}_X e_i). \tag{3.16}$$

Equivalently, one can take the inner product of both sides of equation (3.15) with e_k to get

$$\begin{aligned} < \overline{\nabla}_X e_i, e_k > &= < e_j \omega^j{}_i(X), e_k > \\ &= \omega^j{}_i(X) < e_j, e_k > \\ &= \omega^j{}_i(X) g_{jk} \end{aligned}$$

Hence,

$$< \overline{\nabla}_X e_i, e_k >= \omega_{ki}(X) \tag{3.17}$$

The left hand side of the last equation is the inner product of two vectors, so the expression represents an array of functions. Consequently, the right hand side also represents an array of functions. In addition, both expressions are linear on X, since by definition $\overline{\nabla}_X$ is linear on X. We conclude that the right hand side can be interpreted as a matrix in which each entry is a 1-forms acting on the vector X to yield a function. The matrix valued quantity $\omega^i{}_j$ is called the **connection form** .

3.10 Definition Let $\overline{\nabla}_X$ be a Koszul connection and let $\{e_i\}$ be a frame. The **Christoffel** symbols associated with the connection in the given frame are the functions $\Gamma^k{}_{ij}$ given by

$$\overline{\nabla}_{e_i} e_j = \Gamma^k{}_{ij} e_k \tag{3.18}$$

The Christoffel symbols are the coefficients that give the representation of the rate of change of the frame vectors in the direction of the frame vectors themselves. Many physicists therefore refer to the Christoffel symbols as the connection once again giving rise to possible confusion. The precise relation between the Christoffel symbols and the connection 1-forms is captured by the equations,

$$\omega^k{}_i(e_j) = \Gamma^k{}_{ij}, \tag{3.19}$$

or equivalently

$$\omega^k{}_i = \Gamma^k{}_{ij} \theta^j. \tag{3.20}$$

In a general frame in $\mathbf{R}^n$ there are n^2 entries in the connection 1-form and n^3 Christoffel symbols. The number of independent components is reduced if one assumes that the frame is orthonormal.

3.11 Proposition Let and e_i be an orthonormal frame and $\overline{\nabla}_X$ be a Koszul connection compatible with the metric . Then

$$\omega_{ji} = -\omega_{ij} \tag{3.21}$$

Proof: Since it is given that $< e_i, e_j >= \delta_{ij}$, we have

$$\begin{aligned} 0 &= \overline{\nabla}_X < e_i, e_j > \\ &= < \overline{\nabla}_X e_i, e_j > + < e_i, \overline{\nabla}_X e_j > \\ &= < \omega^k_i e_k, e_j > + < e_i, \omega^k_j e_k > \\ &= \omega^k_i < e_k, e_j > + \omega^k_j < e_i, e_k > \\ &= \omega^k_i g_{kj} + \omega^k_j g_{ik} \\ &= \omega_{ji} + \omega_{ij}. \end{aligned}$$

thus proving that ω is indeed antisymmetric.

3.12 Corollary The Christoffel symbols of a Koszul connection in an orthonormal frame are antisymmetric on the lower indices; that is,

$$\Gamma^k_{ji} = -\Gamma^k_{ij}. \tag{3.22}$$

We conclude that in an orthonormal frame in $\mathbf{R}^n$, the number of independent coefficients of the connection 1-form is $(1/2)n(n-1)$ since by antisymmetry, the diagonal entries are zero, and one only needs to count the number of entries in the upper triangular part of the $n \times n$ matrix ω_{ij} Similarly, the number of independent Christoffel symbols gets reduced to $(1/2)n^2(n-1)$. In the case of an orthonormal frame in $\mathbf{R}^3$, where g_{ij} is diagonal, $\omega^i{}_j$ is also antisymmetric, so the connection equations become

$$\overline{\nabla}_X \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega^1{}_2(X) & \omega^1{}_3(X) \\ -\omega^1{}_2(X) & 0 & \omega^2{}_3(X) \\ -\omega^1{}_3(X) & -\omega^2{}_3(X) & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{3.23}$$

Comparing the Frenet frame equation(1.27), we notice the obvious similarity to the general frame equations above. Clearly, the Frenet frame is a special case in which the basis vectors have been adapted to a curve resulting in a simpler connection in which some of the coefficients vanish. A further simplification occurs in the Frenet frame since here the equations represent the rate of change of the frame only along the direction of the curve rather than an arbitrary direction vector X.

3.4 Cartan Equations

Perhaps the most important contribution to the development of Differential Geometry is the work of Cartan culminating into famous equations of structure discussed in this chapter.

First Structure Equation

3.13 Theorem Let $\{e_i\}$ be a frame with connection ω_j^i and dual coframe θ^i. Then

$$\Theta^i \equiv d\theta^i + \omega^i{}_j \wedge \theta^j = 0 \tag{3.24}$$

Proof: Let

$$e_i = \partial_j A^j{}_i.$$

be a frame, and let θ^i be the corresponding coframe. Since $\theta^i(e_j)$, we have

$$\theta^i = (A^{-1})^i{}_j dx^j.$$

Let X be an arbitrary vector field. Then

$$\begin{aligned}
\overline{\nabla}_X e_i &= \overline{\nabla}_X(\partial_j A^j{}_i) \\
e_j \omega^j{}_i(X) &= \partial_j X(A^j{}_i) \\
&= \partial_j d(A^j{}_i)(X) \\
&= e_k (A^{-1})^k{}_j d(A^j{}_i)(X) \\
\omega^k{}_i(X) &= (A^{-1})^k{}_j d(A^j{}_i)(X).
\end{aligned}$$

Hence,

$$\omega^k_i = (A^{-1})^k{}_j d(A^j{}_i),$$

or, in matrix notation,

$$\omega = A^{-1} dA. \tag{3.25}$$

On the other hand, taking the exterior derivative of θ^i, we find that

$$\begin{aligned}
d\theta^i &= d(A^{-1})^i{}_j \wedge dx^j \\
&= d(A^{-1})^i{}_j \wedge A^j{}_k \theta^k \\
d\theta &= d(A^{-1})A \wedge \theta.
\end{aligned}$$

However, since $A^{-1}A = I$, we have $d(A^{-1})A = -A^{-1}dA = -\omega$, hence

$$d\theta = -\omega \wedge \theta. \tag{3.26}$$

In other words

$$d\theta^i + \omega^i{}_j \wedge \theta^j = 0.$$

Second Structure Equation

Let θ^i be a coframe in $\mathbf{R}^n$ with connection $\omega^i{}_j$. Taking the exterior derivative of the first equation of structure and recalling the properties (2.34), we get

$$\begin{aligned}
d(d\theta^i) + d(\omega^i{}_j \wedge \theta^j) &= 0 \\
d\omega^i{}_j \wedge \theta^j - \omega^i{}_j \wedge d\theta^j &= 0.
\end{aligned}$$

Substituting recursively from the first equation of structure, we get

$$\begin{aligned}
d\omega^i{}_j \wedge \theta^j - \omega^i{}_j \wedge (-\omega^j{}_k \wedge \theta^k) &= 0 \\
d\omega^i{}_j \wedge \theta^j + \omega^i{}_k \wedge \omega^k{}_j \wedge \theta^j &= 0 \\
(d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j) \wedge \theta^j &= 0 \\
d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j &= 0.
\end{aligned}$$

3.14 Definition The curvature Ω of a connection ω is the matrix valued 2-form,

$$\Omega^i{}_j \equiv d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \tag{3.27}$$

3.15 Theorem Let θ be a coframe with connection ω in $\mathbf{R}^n$. Then the curvature form vanishes:

$$\Omega = d\omega + \omega \wedge \omega = 0. \tag{3.28}$$

Proof: Given that there is a non-singular matrix A such that $\theta = A^{-1}dx$ and $\omega = A^{-1}dA$, we have

$$d\omega = d(A^{-1}) \wedge dA.$$

On the other hand,

$$\begin{aligned} \omega \wedge \omega &= (A^{-1}dA) \wedge (A^{-1}dA) \\ &= -d(A^{-1})A \wedge A^{-1}dA \\ &= -d(A^{-1})(AA^{-1}) \wedge dA \\ &= -d(A^{-1}) \wedge dA. \end{aligned}$$

Therefore, $d\Omega = -\omega \wedge \omega$.

Change of Basis

We briefly explore the behavior of the quantities Θ^i and $\Omega^i{}_j$ under a change of basis.

Let e_i be frame with dual forms θ^i, and let $\overline{e}_i$ be another frame related to the first frame by an invertible transformation.

$$\overline{e}_i = e_j B^j{}_i, \tag{3.29}$$

which we will write in matrix notation as $\overline{e} = eB$. Referring back to the definition of connections (3.15), we introduce the **covariant differential** $\overline{\nabla}$ by the formula

$$\begin{aligned} \overline{\nabla} e_i &= e_j \otimes \omega^j{}_i \\ &= e_j \omega^j{}_i \\ \overline{\nabla} e &= e\,\omega \end{aligned} \tag{3.30}$$

where, once again, we have simplified the equation by using matrix notation. This definition is elegant because it does not explicitly show the dependence on X in the connection (3.15). The idea of switching from derivatives to differentials is familiar from basic calculus, but we should point out that in the present context, the situation is more subtle. The operator $\overline{\nabla}$ here maps a vector field to a matrix-valued tensor of rank $T^{1,1}$. Another way to view the covariant differential is to think of $\overline{\nabla}$ as an operator such that if e is a frame, and X a vector field, then $\overline{\nabla} e(X) = \overline{\nabla}_X e$. If f is a function, then $\overline{\nabla} f(X) = \overline{\nabla}_X f = df(X)$, so that $\overline{\nabla} f = df$. In other words, $\overline{\nabla}$ behaves like a covariant derivative on vectors, but like a differential on functions. We require $\overline{\nabla}$ to behave like a derivation on tensor products:

$$\overline{\nabla}(T_1 \otimes T_2) = \overline{\nabla} T_1 \otimes T_2 + T_1 \otimes \overline{\nabla} T_2. \tag{3.31}$$

Taking the exterior differential of (3.29) and using (3.30) recursively, we get

$$\begin{aligned} \overline{\nabla}\,\overline{e} &= \overline{\nabla} e \otimes B + e \otimes \overline{\nabla} B \\ &= (\overline{\nabla} e)B + e(dB) \\ &= e\,\omega B + e(dB) \\ &= \overline{e}B^{-1}\omega B + \overline{e}B^{-1}dB \\ &= \overline{e}[B^{-1}\omega B + B^{-1}dB] \\ &= \overline{e}\,\overline{\omega} \end{aligned}$$

provided that the connection $\overline{\omega}$ in the new frame $\overline{e}$ is related to the connection ω by the transformation law,

$$\overline{\omega} = B^{-1}\omega B + B^{-1}dB. \tag{3.32}$$

It should be noted than if e is the standard frame $e_i = \partial_i$ in $\mathbf{R}^n$, then $\overline{\nabla} e = 0$, so that $\omega = 0$. In this case, the formula above reduces to $\overline{\omega} = B^{-1}dB$, showing that the transformation rule is consistent with equation (3.25).

Chapter 4

Theory of Surfaces

Surfaces in $\mathbf{R}^3$

4.1 Manifolds

4.1 Definition A C^∞ coordinate chart is a C^∞ map x from an open subset of $\mathbf{R}^2$ into $\mathbf{R}^3$.

$$\begin{array}{rcl} \mathbf{x} : U \in \mathbf{R}^2 & \longrightarrow & \mathbf{R}^3 \\ (u,v) & \stackrel{\mathbf{x}}{\longmapsto} & (x(u,v), y(u,v), z(u,v)) \end{array} \tag{4.1}$$

We will always assume that the Jacobian of the map has maximal rank. In local coordinates, a coordinate chart is represented by three equations in two variables:

$$x^i = f^i(u^\alpha), \text{ where } i = 1,2,3, \ \ \alpha = 1,2. \tag{4.2}$$

The local coordinate representation allows us to use the tensor index formalism introduced in earlier chapters. The assumption that the Jacobian $J = (\partial x^i/\partial u^\alpha)$ be of maximal rank allows one to invoke the Implicit Function Theorem. Thus, in principle, one can locally solve for one of the coordinates, say x^3, in terms of the other two, like so:

$$x^3 = f(x^1, x^2). \tag{4.3}$$

The locus of points in $\mathbf{R}^3$ satisfying the equations $x^i = f^i(u^\alpha)$ can also be locally represented by an expression of the form

$$F(x^1, x^2, x^3) = 0 \tag{4.4}$$

4.2 Definition Let $\mathbf{x}(u^1,u^2) : U \longrightarrow \mathbf{R}^3$ and $\mathbf{y}(v^1,v^2) : V \longrightarrow \mathbf{R}^3$ be two coordinate charts with a non-empty intersection $\mathbf{x}(U) \cap \mathbf{y}(V) \neq \emptyset$. The two charts are said to be C^∞ equivalent if the map $\phi = \mathbf{y}^{-1}\mathbf{x}$ and its inverse ϕ^{-1} (see fig.(4.1))are infinitely differentiable.

The definition simply states that two equivalent charts $\mathbf{x}(u^\alpha)$ and $\mathbf{y}(v^\beta)$ represent different reparametrizations for the same set of points in $\mathbf{R}^3$.

4.3 Definition A differentiably smooth surface in $\mathbf{R}^3$ is a set of points $\mathcal{M}$ in $\mathbf{R}^3$ satisfying the following properties:

1. If $\mathbf{p} \in \mathcal{M}$, then $\mathbf{p}$ belongs to some C^∞ chart.

(Figure not yet available)

Figure 4.1: Chart Equivalence

2. If $\mathbf{p} \in \mathcal{M}$ belongs to two different charts $\mathbf{x}$ and $\mathbf{y}$, then the two charts are C^∞ equivalent.

Intuitively, we may think of a surface as consisting locally of number of patches "sewn" to each other so as to form a quilt from a global perspective.

The first condition in the definition states that each local patch looks like a piece of $\mathbf{R}^2$, whereas the second differentiability condition indicates that the patches are joined together smoothly. Another way to state this idea is to say that a surface a space that is locally Euclidean and it has a differentiable structure so that the notion of differentiation makes sense. If the Euclidean space is of dimension n, then the "surface" is called an n-dimensional **manifold**.

4.4 Example Consider the local coordinate chart

$$\mathbf{x}(u, v) = (\sin u \cos v, \sin u \sin v, \cos v).$$

The vector equation is equivalent to three scalar functions in two variables:

$$\begin{aligned} x &= \sin u \cos v, \\ y &= \sin u \sin v, \\ z &= \cos u. \end{aligned} \tag{4.5}$$

Clearly, the surface represented by this chart is part of the sphere $x^2+y^2+z^2 = 1$. The chart cannot possibly represent the whole sphere because, although a sphere is locally Euclidean, (the earth is locally flat) there is certainly a topological difference between a sphere and a plane. Indeed, if one analyzes the coordinate chart carefully, one will note that at the North pole ($u = 0,\ z = 1$, the coordinates become singular. This happens because $u = 0$ implies that $x = y = 0$ regardless of the value of v, so that the North pole has an infinite number of labels. The fact that it is required to have two parameters to describe a patch on a surface in $\mathbf{R}^3$ is a manifestation of the 2-dimensional nature of of the surfaces. If one holds one of the parameters constant while varying the other, then the resulting 1-parameter equations describe a curve on the surface. Thus, for example, letting $u = \textit{constant}$ in equation (4.5), we get the equation of a meridian great circle.

4.5 Notation Given a parametrization of a surface in a local chart $\mathbf{x}(u, v) = \mathbf{x}(u^1, u^2) = \mathbf{x}(u^\alpha)$, we will denote the partial derivatives by any of the following notations:

$$\begin{aligned} \mathbf{x}_u = \mathbf{x}_1 &= \frac{\partial \mathbf{x}}{\partial u}, & \mathbf{x}_{uu} = \mathbf{x}_{11} &= \frac{\partial^2 \mathbf{x}}{\partial u^2} \\ \mathbf{x}_v = \mathbf{x}_v &= \frac{\partial \mathbf{x}}{\partial v}, & \mathbf{x}_{vv} = \mathbf{x}_{22} &= \frac{\partial^2 \mathbf{x}}{\partial v^2} \\ \mathbf{x}_\alpha &= \frac{\partial \mathbf{x}}{\partial u^\alpha} & \mathbf{x}_{\alpha\beta} &= \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial v^\beta} \end{aligned}$$

4.2 The First Fundamental Form

Let $x^i(u^\alpha)$ be a local parametrization of a surface. Then, the Euclidean inner product in $\mathbf{R}^3$ induces an inner product in the space of tangent vectors at each point in the surface. This metric on the surface is obtained as follows:

$$\begin{aligned} dx^i &= \frac{\partial x^i}{\partial u^\alpha} du^\alpha \\ ds^2 &= \delta_{ij} dx^i dx^j \\ &= \delta_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} du^\alpha du^\beta. \end{aligned}$$

Thus,

$$ds^2 = g_{\alpha\beta} du^\alpha du^\beta, \tag{4.6}$$

where

$$g_{\alpha\beta} = \delta_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}. \tag{4.7}$$

We conclude that the surface, by virtue of being embedded in $\mathbf{R}^3$, inherits a natural metric (4.6) which we will call the **induced metric**. A pair $\{\mathcal{M}, g\}$, where $\mathcal{M}$ is a manifold and $g = g_{\alpha\beta} du^\alpha \otimes du^\beta$ is a metric is called a **Riemannian manifold** if considered as an entity in itself, and a Riemannian submanifold of $\mathbf{R}^n$ if viewed as an object embedded in Euclidean space. An equivalent version of the metric (4.6) can be obtained by using a more traditional calculus notation:

$$\begin{aligned} d\mathbf{x} &= \mathbf{x}_u du + \mathbf{x}_v dv \\ ds^2 &= d\mathbf{x} \cdot d\mathbf{x} \\ &= (\mathbf{x}_u du + \mathbf{x}_v dv) \cdot (\mathbf{x}_u du + \mathbf{x}_v dv) \\ &= (\mathbf{x}_u \cdot \mathbf{x}_u) du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v) du dv + (\mathbf{x}_v \cdot \mathbf{x}_v) dv^2. \end{aligned}$$

We can rewrite the last result as

$$ds^2 = E du^2 + 2F du dv + G dv^2, \tag{4.8}$$

where

$$\begin{aligned} E &= g_{11} = \mathbf{x}_u \cdot \mathbf{x}_u \\ F &= g_{12} = \mathbf{x}_u \cdot \mathbf{x}_v \\ &= g_{21} = \mathbf{x}_v \cdot \mathbf{x}_u \\ G &= g_{22} = \mathbf{x}_v \cdot \mathbf{x}_v. \end{aligned}$$

That is

$$g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta = < \mathbf{x}_\alpha, \mathbf{x}_\beta > .$$

4.6 Definition The element of arclength,

$$ds^2 = g_{\alpha\beta} du^\alpha \otimes du^\beta, \tag{4.9}$$

is also called the **first fundamental form**. We must caution the reader that this quantity is not a form in the sense of differential geometry since ds^2 involves the symmetric tensor product rather than the wedge product.

The first fundamental form plays such a crucial role in the theory of surfaces that we will find it convenient to introduce a third, more modern version. Following the same development as in the theory of curves, consider a surface $\mathcal{M}$ defined locally by a function $(u^1, u^2) \longmapsto \alpha(u^1, u^2)$. We say that a quantity X_p is a tangent vector at a point $\mathbf{p} \in \mathcal{M}$ if X_p is a linear derivation on the space of C^∞ real-valued functions $\{f | f : \mathcal{M} \longrightarrow \mathbf{R}\}$ on the surface. The set of all tangent vectors at a point $\mathbf{p} \in \mathcal{M}$ is called the **tangent space** $T_p\mathcal{M}$. As before, a vector field X on the surface is a smooth choice of a tangent vector at each point on the surface and the union of all tangent spaces is called the **tangent bundle** $T\mathcal{M}$.

The coordinate chart map

$$\alpha : \mathbf{R}^2 \longrightarrow \mathcal{M} \in \mathbf{R}^3$$

induces a push-forward map

$$\alpha_* : T\mathbf{R}^2 \longrightarrow T\mathcal{M}$$

defined by

$$\alpha_*(V)(f) \mid_{\alpha(u^\alpha)} = V(\alpha \circ f) \mid_{u^\alpha} .$$

Just as in the case of curves, when we revert back to classical notation to describe a surface as $x^i(u^\alpha)$, what we really mean is $(x^i \circ \alpha)(u^\alpha)$, where x^1 are the coordinate functions in $\mathbf{R}^3$. Particular examples of tangent vectors on $\mathcal{M}$ are given by the push-forward of the standard basis of $T\mathbf{R}^2$. These tangent vectors which earlier we called $\mathbf{x}_\alpha$ are defined by

$$\alpha_*(\frac{\partial}{\partial u^\alpha})(f) \mid_{\alpha(u^\alpha)} = \frac{\partial}{\partial u^\alpha}(\alpha \circ f) \mid_{u^\alpha}$$

In this formalism, the first fundamental form I is just the symmetric bilinear tensor defined by the induced metric,

$$I(X, Y) = g(X, Y) = < X, Y >, \tag{4.10}$$

where X and Y are any pair of vector fields in $T\mathcal{M}$.

Orthogonal Parametric Curves

Let V and W be vectors tangent to a surface $\mathcal{M}$ defined locally by a chart $\mathbf{x}(u^\alpha)$. Since the vectors $\mathbf{x}_\alpha$ span the tangent space of $\mathcal{M}$ at each point, the vectors V and W can be written as linear combinations,

$$\begin{aligned} V &= V^\alpha \mathbf{x}_\alpha \\ W &= W^\alpha \mathbf{x}_\alpha. \end{aligned}$$

The functions V^α and W^α are called the **curvilinear coordinates** of the vectors. We can calculate the length and the inner product of the vectors using the induced Riemannian metric as follows:

$$\begin{aligned} \|V\|^2 &= < V, V > = < V^\alpha \mathbf{x}_\alpha, V^\beta \mathbf{x}_\beta > = V^\alpha V^\beta < \mathbf{x}_\alpha, \mathbf{x}_\beta > \\ \|V\|^2 &= g_{\alpha\beta} V^\alpha V^\beta \\ \|W\|^2 &= g_{\alpha\beta} W^\alpha W^\beta, \end{aligned}$$

and

$$\begin{aligned} < V, W > &= < V^\alpha \mathbf{x}_\alpha, W^\beta \mathbf{x}_\beta > = V^\alpha W^\beta < \mathbf{x}_\alpha, \mathbf{x}_\beta > \\ &= g_{\alpha\beta} V^\alpha W^\beta. \end{aligned}$$

The angle θ subtended by the the vectors V and W is the given by the equation

$$\begin{aligned} \cos\theta &= \frac{< V, W >}{\|V\| \cdot \|W\|} \\ &= \frac{I(V, W)}{\sqrt{I(V, V)}\sqrt{I(W, W)}} \\ &= \frac{g_{\alpha\beta} V^\alpha W^\beta}{g_{\alpha\beta} V^\alpha V^\beta \cdot g_{\alpha\beta} W^\alpha W^\beta}. \end{aligned} \tag{4.11}$$

Let $u^\alpha = \phi^\alpha(t)$ and $u^\alpha = \psi^\alpha(t)$ be two curves on the surface. Then the total differentials

$$du^\alpha = \frac{d\phi^\alpha}{dt}dt, \quad \text{and} \quad \delta u^\alpha = \frac{d\psi^\alpha}{dt}\delta t$$

represent infinitesimal tangent vectors (1.12) to the curves. Thus, the angle between two infinitesimal vectors tangent to two intersecting curves on the surface satisfies the equation:

$$\cos\theta = \frac{g_{\alpha\beta}du^\alpha \delta u^\beta}{\sqrt{g_{\alpha\beta}du^\alpha du^\beta}\sqrt{g_{\alpha\beta}\delta u^\alpha \delta u^\beta}} \tag{4.12}$$

In particular, if the two curves happen to be the parametric curves, $u^1 = constant$ and $u^2 = constant$, then along one curve we have $du^1 = 0$, with du^2 arbitrary, and along the second δu^1 is arbitrary and $\delta u^2 = 0$. In this case, the cosine of the angle subtended by the infinitesimal tangent vectors reduces to:

$$\cos\theta = \frac{g_{12}\delta u^1 du^2}{\sqrt{g_{11}(\delta u^1)^2}\sqrt{g_{22}(du^2)^2}} = \frac{g_{12}}{g_{11}g_{22}} = \frac{F}{\sqrt{EG}}. \tag{4.13}$$

I follows immediately from the equation above that:

4.7 Proposition The parametric lines are orthogonal if $F = 0$.

4.8 Examples
a) Sphere

$$\begin{aligned}
\mathbf{x} &= (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)\\
\mathbf{x}_\theta &= (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta)\\
\mathbf{x}_\phi &= (-a\sin\theta\sin\phi, a\sin\theta\cos\phi, 0)\\
E &= \mathbf{x}_\theta\cdot\mathbf{x}_\theta = a^2\\
F &= \mathbf{x}_\theta\cdot\mathbf{x}_\phi = 0\\
G &= \mathbf{x}_\phi\cdot\mathbf{x}_\phi = a^2\sin^2\theta\\
ds^2 &= a^2d\theta^2 + a^2\sin^2\theta d\phi^2
\end{aligned}$$

b) Surface of Revolution

$$\begin{aligned}
\mathbf{x} &= (r\cos\theta, r\sin\theta, f(r))\\
\mathbf{x}_r &= (\cos\theta, \sin\theta, f'(r))\\
\mathbf{x}_\theta &= (-r\sin\theta, r\cos\theta, 0)\\
E &= \mathbf{x}_r\cdot\mathbf{x}_r = 1 + f'^2(r)\\
F &= \mathbf{x}_r\cdot\mathbf{x}_\theta = 0\\
G &= \mathbf{x}_\theta\cdot\mathbf{x}_\theta = r^2\\
ds^2 &= [1 + f'^2(r)]dr^2 + r^2d\theta^2
\end{aligned}$$

c) Pseudosphere

$$\begin{aligned}
\mathbf{x} &= (a\sin u\cos v, a\sin u\sin v, a(\cos u\ln tan\frac{u}{2}))\\
E &= a^2\cot^2 u\\
F &= = 0\\
G &= a^2\sin^2 u\\
ds^2 &= a^2\cot^2 u du^2 + a^2\sin^2 u dv^2
\end{aligned}$$

d) Torus

$$\begin{aligned}
\mathbf{x} &= ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u) \\
E &= a^2 \\
F &= 0 \\
G &= (b + a\cos u)^2 \\
ds^2 &= a^2 du^2 + (b + a\cos u)^2 dv^2
\end{aligned}$$

e) Helicoid

$$\begin{aligned}
\mathbf{x} &= (u\cos v, u\sin v, av) \\
E &= 1 \\
F &= 0 \\
G &= u^2 + a^2 \\
ds^2 &= du^2 + (u^2 + a^2)dv^2
\end{aligned}$$

f) Catenoid

$$\begin{aligned}
\mathbf{x} &= (u\cos v, u\sin v, c\cosh^{-1}\frac{u}{c}) \\
E &= \frac{u^2}{u^2 - c^2} \\
F &= 0 \\
G &= u^2 \\
ds^2 &= \frac{u^2}{u^2 - c^2}du^2 + u^2 dv^2
\end{aligned}$$

4.3 The Second Fundamental Form

Let $\mathbf{x} = \mathbf{x}(u^\alpha)$ be a coordinate patch on a surface $\mathcal{M}$. Since $\mathbf{x}_u$ and $\mathbf{x}_v$ are tangential to the surface, we can construct a unit normal $\mathbf{n}$ to the surface by taking

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \tag{4.14}$$

Now, consider a curve on the surface given by $u^\alpha = u^\alpha(s)$. Without loss of generality, we assume that the curve is parametrized by arclength s so that the curve has unit speed. Using the chain rule, we se that the unit tangent vector T to the curve is given by

$$T = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{du^\alpha}\frac{du^\alpha}{ds} = \mathbf{x}_\alpha \frac{du^\alpha}{ds} \tag{4.15}$$

Since the curve lives on the surface and the the vector T is tangent to the curve, it is clear that T is also tangent to the surface. On the other hand, the vector $T' = dT/ds$ does not in general have this property, so we decompose T' into its normal and tangential components (see fig. (4.2))

$$\begin{aligned}
T' &= K_n + K_g \\
&= \kappa_n \mathbf{n} + K_g,
\end{aligned} \tag{4.16}$$

where $\kappa_n = \|K_n\| =< T', \mathbf{n} >$.

The scalar quantity κ_n is called the **normal curvature** of the curve and K_g is called the **geodesic curvature** vector. The normal curvature measures the the curvature of $\mathbf{x}(u^\alpha(s))$ resulting

Figure not yet available.

Figure 4.2: Normal Curvature

from the constraint of the curve to lie on a surface. The geodesic curvature vector , measures the "sideward" component of the curvature in the tangent plane to the surface. Thus, if one draws a straight line on a flat piece of paper and then smoothly bends the paper into a surface, then the straight line would now acquire some curvature. Since the line was originally straight, there is no sideward component of curvature so $K_g = 0$ in this case. This means that the entire contribution to the curvature comes from the normal component, reflecting the fact that the only reason there is curvature here is due to the bend in the surface itself.

Similarly, if one specifies a point $\mathbf{p} \in \mathcal{M}$ and a direction vector $X_p \in T_p\mathcal{M}$, one can geometrically envision the normal curvature by considering the equivalence class of all unit speed curves in $\mathcal{M}$ that contain the point $\mathbf{p}$ and whose tangent vectors line up with the direction of X. Of course, there are infinitely many such curves, but at an infinitesimal level, all these curves can be obtained by intersecting the surface with a "vertical" plane containing the vector X and the normal to $\mathcal{M}$. All curves in this equivalence class have the same normal curvature and their geodesic curvatures vanish. In this sense, the normal curvature is more of a property pertaining to a direction on the surface at a point, whereas the geodesic curvature really depends on the curve itself. It might be impossible for a hiker walking on the undulating hills of the Ozarks to find a straight line trail, since the rolling hills of the terrain extend in all directions. It might be possible, however, for the hiker to walk on a path with zero geodesic curvature as long the same compass direction is maintained.

To find an explicit formula for the normal curvature we first differentiate equation (4.15)

$$\begin{aligned} T' &= \frac{dT}{ds} \\ &= \frac{d}{ds}(\mathbf{x}_\alpha \frac{du^\alpha}{ds}) \\ &= \frac{d}{ds}(\mathbf{x}_\alpha)\frac{du^\alpha}{ds} + \mathbf{x}_\alpha \frac{d^2u^\alpha}{ds^2} \\ &= (\frac{d\mathbf{x}_\alpha}{du^\beta}\frac{du^\beta}{ds})\frac{du^\alpha}{ds} + \mathbf{x}_\alpha \frac{d^2u^\alpha}{ds^2} \\ &= \mathbf{x}_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + \mathbf{x}_\alpha \frac{d^2u^\alpha}{ds^2}. \end{aligned}$$

Taking the inner product of the last equation with the normal and noticing that $< \mathbf{x}_\alpha, \mathbf{n} >= 0$, we get

$$\begin{aligned} \kappa_n &= < T', \mathbf{n} >=< \mathbf{x}_{\alpha\beta}, \mathbf{n} > \frac{du^\alpha}{ds}\frac{du^\beta}{ds} \\ &= \frac{b_{\alpha\beta}du^\alpha du^\beta}{g_{\alpha\beta}du^\alpha du^\beta}, \end{aligned} \tag{4.17}$$

where

$$b_{\alpha\beta} =< \mathbf{x}_{\alpha\beta}, \mathbf{n} > \tag{4.18}$$

4.9 Definition The expression

$$II = b_{\alpha\beta}du^\alpha \otimes du^\beta \tag{4.19}$$

is called the **second fundamental form** .

4.10 Proposition The second fundamental form is symmetric.
Proof: In the classical formulation of the second fundamental form, the proof is trivial. We have $b_{\alpha\beta} = b_{\beta\alpha}$, since for a C^∞ patch $\mathbf{x}(u^\alpha)$, we have $\mathbf{x}_{\alpha\beta} = \mathbf{x}_{\beta\alpha}$, because the partial derivatives commute.

We will denote the coefficients of the second fundamental form as follows:

$$\begin{aligned} e &= b_{11} =< \mathbf{x}_{uu}, \mathbf{n} > \\ f &= b_{12} =< \mathbf{x}_{uv}, \mathbf{n} > \\ &= b_{21} =< \mathbf{x}_{vu}, \mathbf{n} > \\ g &= b_{22} =< \mathbf{x}_{vv}, \mathbf{n} >, \end{aligned}$$

so that equation (4.19) can be written as

$$II = edu^2 + 2fdudv + gdv^2, \tag{4.20}$$

and equation (4.17) as

$$\kappa_n = \frac{II}{I} = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2}. \tag{4.21}$$

We would also like to point out that just as the first fundamental form can be represented as

$$I =< d\mathbf{x}, d\mathbf{x} >,$$

we can represent the second fundamental form as

$$II = - < d\mathbf{x}, d\mathbf{n} >$$

To see this, it suffices to note that differentiation of the identity, $< \mathbf{x}_\alpha, \mathbf{n} >= 0$, implies that

$$< \mathbf{x}_{\alpha\beta}, \mathbf{n} >= - < \mathbf{x}_\alpha, \mathbf{n}_\beta > .$$

Therefore,

$$\begin{aligned} < d\mathbf{x}, d\mathbf{n} > &= < \mathbf{x}_\alpha du^\alpha, \mathbf{n}_\beta du^\beta > \\ &= < \mathbf{x}_\alpha du^\alpha, \mathbf{n}_\beta du^\beta > \\ &= < \mathbf{x}_\alpha, \mathbf{n}_\beta > du^\alpha du^\beta \\ &= - < \mathbf{x}_{\alpha\beta}, \mathbf{n} > du^\alpha du^\beta \\ &= -II \end{aligned}$$

From a computational point a view, a more useful formula for the coefficients of the second fundamental formula can be derived by first applying the classical vector identity

$$(A \times B) \cdot (C \times D) = \begin{vmatrix} A \cdot C & A \cdot D \\ B \cdot C & B \cdot D \end{vmatrix} \tag{4.22}$$

to compute

$$\begin{aligned} \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= (\mathbf{x}_u \times \mathbf{x}_v) \cdot (\mathbf{x}_u \times \mathbf{x}_v) \\ &= det \begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix} \\ &= EG - F^2. \end{aligned} \tag{4.23}$$

Consequently, the normal vector can be written as

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{EG - F^2}}$$

Thus, we can write the coefficients $b_{\alpha\beta}$ directly as triple products involving derivatives of $(\mathbf{x})$. The expressions for these coefficients are

$$\begin{aligned} e &= \frac{(\mathbf{x}_u \mathbf{x}_u \mathbf{x}_{uu})}{\sqrt{EG - F^2}} \\ f &= \frac{(\mathbf{x}_u \mathbf{x}_v \mathbf{x}_{uv})}{\sqrt{EG - F^2}} \\ g &= \frac{(\mathbf{x}_v \mathbf{x}_v \mathbf{x}_{vv})}{\sqrt{EG - F^2}} \end{aligned} \tag{4.24}$$

The first fundamental form on a surface measures the square of the distance between two infinitesimally separated points. There is a similar interpretation of the second fundamental form as we show below. The second fundamental form measures the distance from a point on the surface to the tangent plane at a second infinitesimally separated point. To see this simple geometrical interpretation, consider a point $\mathbf{x}_0 = \mathbf{x}(u_0^\alpha) \in \mathcal{M}$ and a nearby point $\mathbf{x}(u_0^\alpha + du^\alpha)$. Expanding on a Taylor series, we get

$$\mathbf{x}(u_0^\alpha + du^\alpha) = \mathbf{x}_0 + (\mathbf{x}_0)_\alpha du^\alpha + \frac{1}{2}(\mathbf{x}_0)_{\alpha\beta} du^\alpha du^\beta + \dots.$$

We recall that the distance formula from a point $\mathbf{x}$ to a plane which contains $\mathbf{x}_0$ is just the scalar projection of $(\mathbf{x} - \mathbf{x}_0)$ onto the normal. Since the normal to the plane at $\mathbf{x}_0$ is the same as the unit normal to the surface and $< \mathbf{x}_\alpha, \mathbf{n} >= 0$, we find that the distance D is

$$\begin{aligned} D &= < \mathbf{x} - \mathbf{x}_0, \mathbf{n} > \\ &= \frac{1}{2} < (\mathbf{x}_0)_{\alpha\beta}, \mathbf{n} > du^\alpha du^\beta \\ &= \frac{1}{2} II_0 \end{aligned}$$

The first fundamental form (or, rather, its determinant) also appears in calculus in the context of calculating the area of a parametrized surface. If one considers an infinitesimal parallelogram subtended by the vectors $\mathbf{x}_u du$ and $\mathbf{x}_v dv$, then the differential of surface area is given by the length of the cross product of these two infinitesimal tangent vectors. That is,

$$\begin{aligned} dS &= \|\mathbf{x}_u \times \mathbf{x}_v\| \, dudv \\ S &= \int\int \sqrt{EG - F^2} \, dudv \end{aligned}$$

The second fundamental form contains information about the shape of the surface at a point. For example, the discussion above indicates that if $b = |b_{\alpha\beta}| = eg - f^2 > 0$ then all the neighboring points lie on the same side of the tangent plane, and hence, the surface is concave in one direction. If at a point on a surface $b > 0$, the point is called an elliptic point, if $b < 0$, the point is called hyperbolic or a saddle point, and if $b = 0$, the point is called parabolic.

4.4 Curvature

The concept of **Curvature** and its related, constitute the central object of study in differential geometry. One would like to be able to answer questions such as "what quantities remain invariant as one surface is smoothly changed into another?" There is certainly something intrinsically different between a cone, which we can construct from a flat piece of paper, and a sphere, which we cannot. What is it that makes these two surfaces so different? How does one calculate the shortest path between two objects when the path is constrained to lie on a surface?

These and questions of similar type can be quantitatively answered through the study of curvature. We cannot overstate the importance of this subject; perhaps it suffices to say that, without a clear understanding of curvature, there would be no general theory of relativity, no concept of black holes, and even more disastrous, no Star Trek.

Studying the curvature of a hypersurface in $\mathbf{R}^n$ (a surface of dimension $n-1$) begins by trying to understand the covariant derivative of the normal to the surface. If the normal to a surface is constant, then the surface is a flat hyperplane. Variations in the normal are what indicates the presence of curvature. For simplicity, we constrain our discussion to surfaces in $\mathbf{R}^3$, but the formalism we use is applicable to any dimension. We will also introduce in the modern version of the second fundamental form

4.11 Definition Let X be a vector field on a surface M in $\mathbf{R}^3$, and let N be the normal vector. The map L, given by

$$LX = -\overline{\nabla}_X N \tag{4.25}$$

is called the **Weingarten map**.

In this definition we will be careful to differentiate operators that live on the surface from operators that live in the ambient space. We will adopt the convention of overlining objects that live in the ambient space, the operator $\overline{\nabla}$ being an example of one such object. The Weingarten map is a good place to start, since it represents the rate of change of the normal in an arbitrary direction we wish to quantify.

4.12 Definition The **Lie bracket** $[X, Y]$ of two vector fields X and Y on a surface $\mathcal{M}$ is defined as the commutator,

$$[X, Y] = XY - YX, \tag{4.26}$$

meaning that if f is a function on $\mathcal{M}$, then $[X, Y](f) = X(Y(f)) - Y(X(f))$.

4.13 Proposition The Lie bracket of two vectors $X, Y \in T(\mathcal{M})$ is another vector in $T(\mathcal{M})$.
Proof: If suffices to prove that the bracket is a linear derivation on the space of C^∞ functions. Consider vectors $X, Y \in T(\mathcal{M})$ and smooth functions f, g in $\mathcal{M}$. Then,

$$\begin{aligned}
[X, Y](f+g) &= X(Y(f+g)) - Y(X(f+g)) \\
&= X(Y(f) + Y(g)) - Y(X(f) + X(g)) \\
&= X(Y(f)) - Y(X(f)) + X(Y(g)) - Y(X(g)) \\
&= [X, Y](f) + [X, Y](g),
\end{aligned}$$

and

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X[fY(g) + gY(f)] - Y[fX(g) + gX(f)] \\
&= X(f)Y(g) + fX(Y(g)) + X(g)Y(f) + gX(Y(f)) \\
&\quad -Y(f)X(g) - f(Y(X(g)) - Y(g)X(f) - gY(X(f)) \\
&= f[X(Y(g)) - (Y(X(g))] + g[X(Y(f)) - Y(X(f))] \\
&= f[X, Y](g) + g[X, Y](f).
\end{aligned}$$

4.14 Proposition The Weingarten map is a linear transformation on $T(\mathcal{M})$.
Proof: Linearity follows from the linearity of $\overline{\nabla}$, so it suffices to show that $L : X \longrightarrow LX$ maps $X \in T(\mathcal{M})$ to a vector $LX \in T(\mathcal{M})$. Since N is the unit normal to the surface, $< N, N >= 1$, so any derivative of $< N, N >$ is 0. Assuming that the connection is compatible with the metric,

$$\begin{aligned} \overline{\nabla}_X < N, N > &= < \overline{\nabla}_X N, > + < N, \overline{\nabla}_X N > \\ &= 2 < \overline{\nabla}_X N, N > \\ &= 2 < LX, N >= 0. \end{aligned}$$

Therefore, LX is orthogonal to N; hence, it lies in $T(\mathcal{M})$.

In the previous section, we gave two equivalent definitions $< d\mathbf{x}, d\mathbf{x} >$, and $< X, Y >$) of the first fundamental form. We will now do the same for the second fundamental form.

4.15 Definition The **second fundamental form** is the bilinear map

$$II(X, Y) =< LX, Y > . \tag{4.27}$$

4.16 Remark It should be noted that the two definitions of the second fundamental form are consistent. This is easy to see if one chooses X to have components $\mathbf{x}_\alpha$ and Y to have components $\mathbf{x}_\beta$. With these choices, LX has components $-\mathbf{n}_a$ and $II(X, Y)$ becomes $b_{\alpha\beta} = - < \mathbf{x}_\alpha, \mathbf{n}_\beta >$.

We also note that there is a third fundamental form defined by

$$III(X, Y) =< LX, LY >=< L^2X, Y > . \tag{4.28}$$

In classical notation, the third fundamental form would be denoted by $< d\mathbf{n}, d\mathbf{n} >$. As one would expect, the third fundamental form contains third order Taylor series information about the surface. We will not treat $III(X, Y)$ in much detail in this work.

4.17 Definition The **torsion** of a connection $\overline{\nabla}$ is the operator T such that $\forall X, Y$,

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y]. \tag{4.29}$$

A connection is called **torsion-free** if $T(X, Y) = 0$. In this case,

$$\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y].$$

We will say more later about the torsion and the importance of torsion-free connections. For the time being, it suffices to assume that for the rest of this section, all connections are torsion-free. Using this assumption, it is possible to prove the following important theorem.

4.18 Theorem The Weingarten map is a self-adjoint operator on $T\mathcal{M}$.
Proof: We have already shown that $L : T\mathcal{M} \longrightarrow T\mathcal{M}$ is a linear map. Recall that an operator L on a linear space is self adjoint if $< LX, Y >=< X, LY >$, so the theorem is equivalent to proving that the second fundamental form is symmetric ($II[X, Y] = II[Y, X]$). Computing the difference of these two quantities, we get

$$\begin{aligned} II[X, Y] - II[Y, X] &= < LX, Y > - < LY, X > \\ &= < \overline{\nabla}_X N, Y > - < \overline{\nabla}_Y N, X > . \end{aligned}$$

Since $< X, N >=< Y, N >= 0$ and the connection is compatible with the metric, we know that

$$\begin{aligned} < \overline{\nabla}_X N, Y > &= - < N, \overline{\nabla}_X Y > \\ < \overline{\nabla}_Y N, X > &= - < N, \overline{\nabla}_Y X >, \end{aligned}$$

hence,

$$\begin{aligned} II[X,Y] - II[Y,X] &= < N, \overline{\nabla}_Y X > - < N, \overline{\nabla}_X Y >, \\ &= < N, \overline{\nabla}_Y X - \overline{\nabla}_X Y > \\ &= < N, [X,Y] > \\ &= 0 \qquad \text{(iff } [X,Y] \in T(\mathcal{M})). \end{aligned}$$

The central theorem of linear algebra is the Spectral Theorem. In the case of self-adjoint operators, the Spectral Theorem states that given the eigenvalue equation for a symmetric operator on a vector space with an inner product,

$$LX = \kappa X, \tag{4.30}$$

the eigenvalues are always real and eigenvectors corresponding to different eigenvalues are orthogonal. Here, the vector spaces in question are the tangent spaces at each point of a surface in $\mathbf{R}^3$, so the dimension is 2. Hence, we expect two eigenvalues and two eigenvectors:

$$LX_1 = \kappa_1 X_1 \tag{4.31}$$

$$LX_2 = \kappa_1 X_2. \tag{4.32}$$

4.19 Definition The eigenvalues κ_1 and κ_2 of the Weingarten map L are called the **principal curvatures** and the eigenvectors X_1 and X_2 are called the **principal directions**.

Several possible situations may occur, depending on the classification of the eigenvalues at each point **p** on a given surface:

1. If $\kappa_1 \neq \kappa_2$ and both eigenvalues are positive, then **p** is called an **elliptic point**.
2. If $\kappa_1\kappa_2 < 0$, then **p** is called a **hyperbolic point**.
3. If $\kappa_1 = \kappa_2 \neq 0$, then **p** is called an **umbilic point**.
4. If $\kappa_1\kappa_2 = 0$, then **p** is called a **parabolic point**.

It is also known from linear algebra, that the the determinant and the trace of a self-adjoint operator in a vector space of dimension two are the only invariants under a adjoint (similarity) transformation. Clearly, these invariants are important in the case of the operator L, and they deserve special names.

4.20 Definition The determinant $K = \det(L)$ is called the **Gaussian curvature** of $\mathcal{M}$ and $H = (1/2)\mathrm{Tr}(L)$ is called the **mean curvature**.

Since any self-adjoint operator is diagonalizable and in a diagonal basis the matrix representing L is $\mathrm{diag}(\kappa_1, \kappa_2)$, if follows immediately that

$$\begin{aligned} K &= \kappa_1\kappa_2 \\ H &= \frac{1}{2}(\kappa_1 + \kappa_2). \end{aligned} \tag{4.33}$$

4.21 Proposition Let X and Y be any linearly independent vectors in $T(\mathcal{M})$. Then

$$\begin{aligned} LX \times LY &= K(X \times Y) \\ (LX \times Y) + (X \times LY) &= 2H(X \times Y) \end{aligned} \tag{4.34}$$

Proof: Since $LX, LY \in T(\mathcal{M})$, they can be expressed as linear combinations of the basis vectors X and Y.

$$\begin{aligned} LX &= a_1X + b_1Y \\ LY &= a_2X + b_2Y. \end{aligned}$$

computing the cross product, we get

$$\begin{aligned} LX \times LY &= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} X \times Y \\ &= \det(L)(X \times Y). \end{aligned}$$

Similarly

$$\begin{aligned} (LX \times Y) + (X \times LY) &= (a_1 + b_2)(X \times Y) \\ &= \mathrm{Tr}(L)(X \times Y) \\ &= (2H)(X \times Y). \end{aligned}$$

4.22 Proposition

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ H &= \frac{1}{2}\frac{Eg - 2Ff + eG}{EG - F^2} \end{aligned} \tag{4.35}$$

Proof: Starting with equations (4.34), take the inner product of both sides with $X \times Y$ and use the vector identity (4.22). We immediately get

$$K = \frac{\begin{vmatrix} < LX, X > & < LX, Y > \\ < LY, X > & < LX, X > \end{vmatrix}}{\begin{vmatrix} < X, X > & < X, Y > \\ < Y, X > & < Y, Y > \end{vmatrix}}$$

$$2H = \frac{\begin{vmatrix} < LX, X > & < LX, Y > \\ < Y, X > & < Y, Y > \end{vmatrix} + \begin{vmatrix} < X, X > & < X, Y > \\ < LY, X > & < LY, Y > \end{vmatrix}}{\begin{vmatrix} < X, X > & < X, Y > \\ < Y, X > & < Y, Y > \end{vmatrix}}$$

The result follows by taking $X = \mathbf{x}_u$ and $Y = \mathbf{x}_v$

4.23 Theorem (Euler) Let X_1 and X_2 be unit eigenvectors of L and let $X = (\cos\theta)X_1 + (\sin\theta)X_2$. Then

$$II(X, X) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta. \tag{4.36}$$

Proof: Simply compute $II(X, X) =< LX, X >$, using the fact the $LX_1 = \kappa_1 X_1$, $LX_2 = \kappa_2 X_2$, and noting that the eigenvectors are orthogonal. We get

$$\begin{aligned} < LX, X > &= < (\cos\theta)\kappa_1 X_1 + (\sin\theta)\kappa_2 X_2, (\cos\theta)X_1 + (\sin\theta)X_2 > \\ &= \kappa_1 \cos^2\theta < X_1, X_1 > + \kappa_2 \sin^2\theta < X_2, X_2 > \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta. \end{aligned}$$